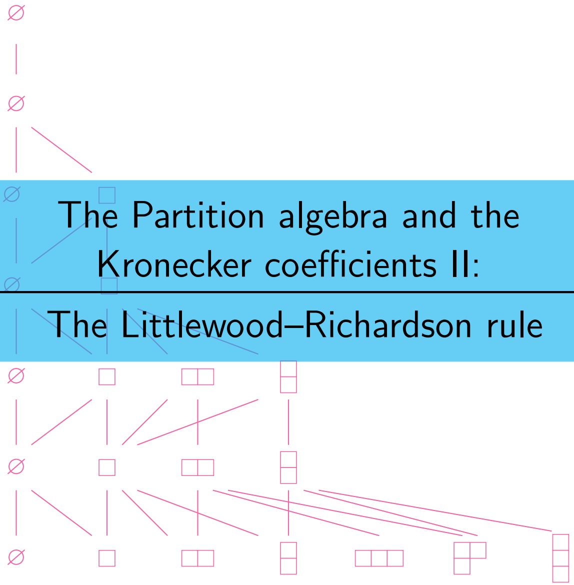
The Partition algebra and the Kronecker coefficients II:

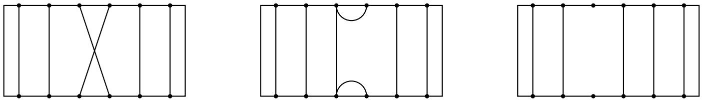
The Littlewood–Richardson rule



Section 1

The partition algebra and the stable Kronecker coefficients

The partition algebra, $P_s(n)$, is the $\mathbb{C}$-algebra generated by elements of the form



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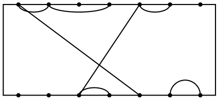

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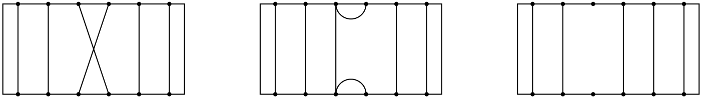

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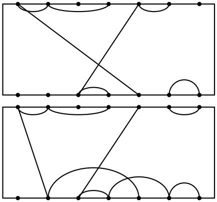

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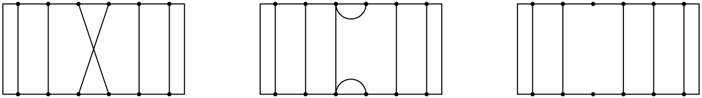

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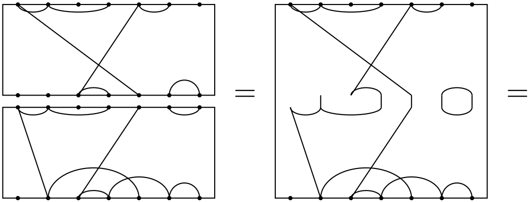

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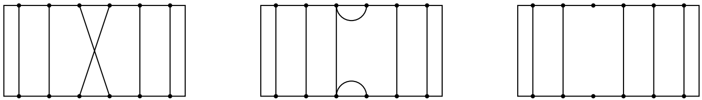

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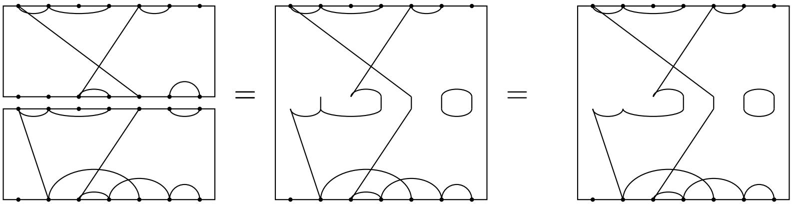

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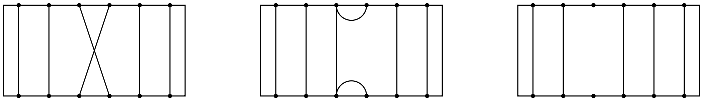

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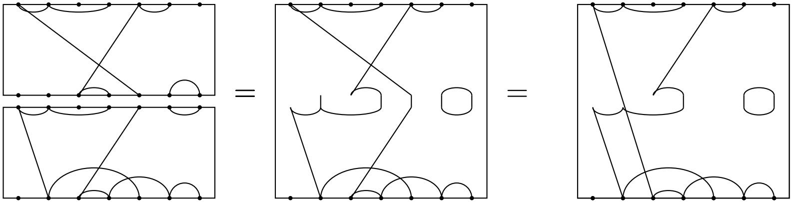

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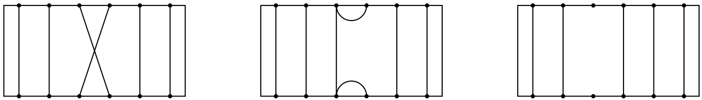

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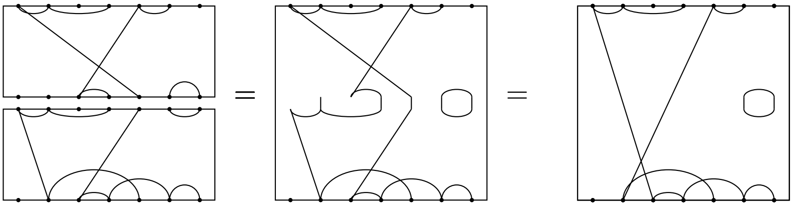

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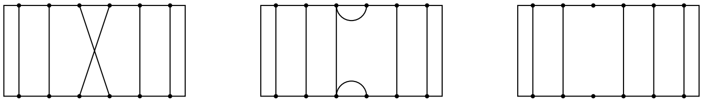

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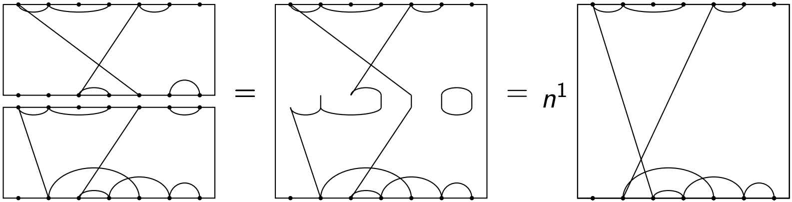

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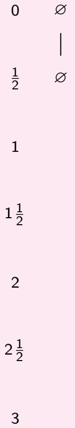

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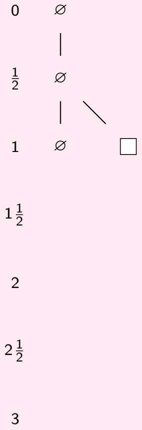

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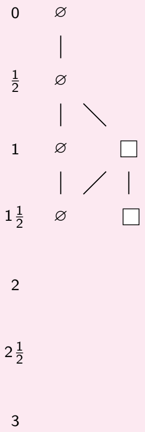

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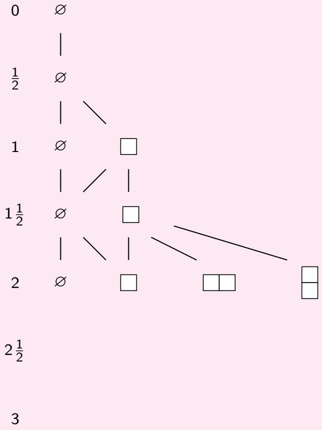

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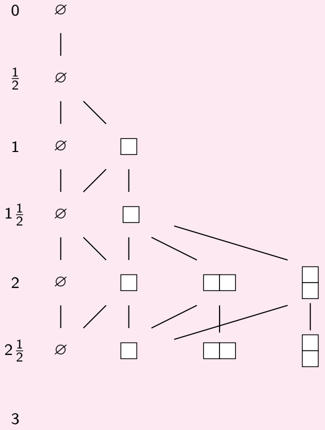

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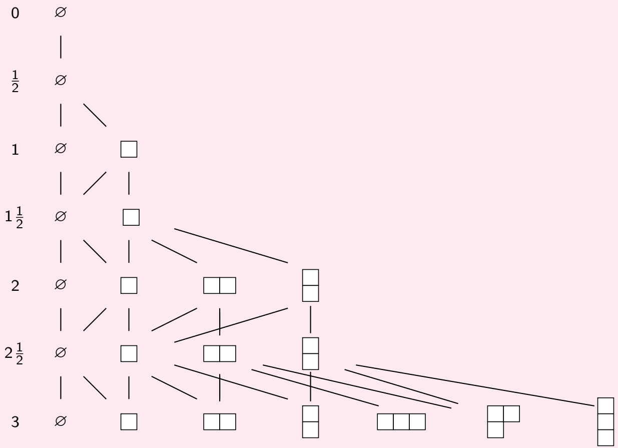

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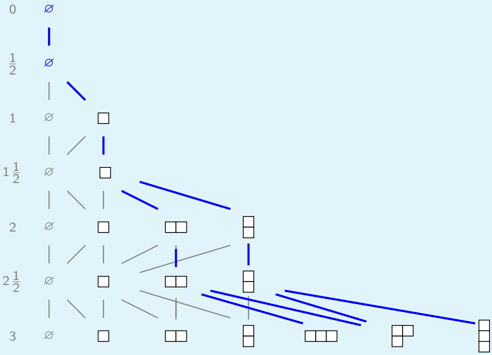

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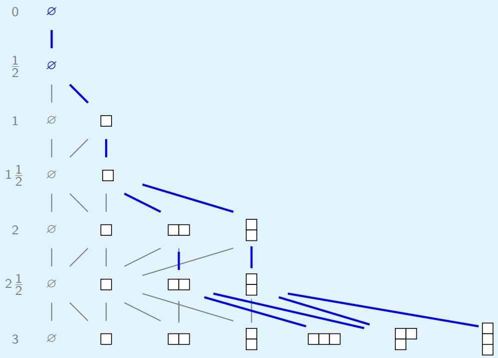

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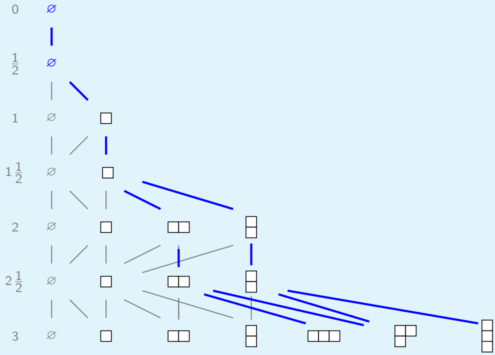

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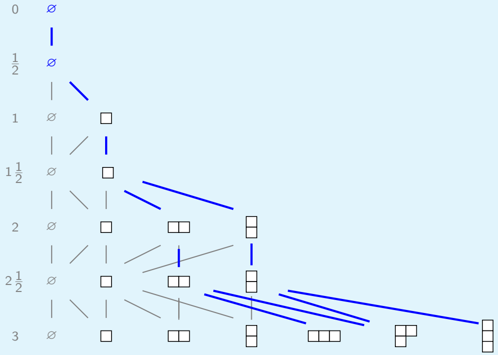

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The stable Kronecker coefficients equal dimensions of homomorphism spaces for path-theoretic $P_s(n)$-modules:

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- this is done via combinatorial resolutions.
- we do this in a language which is generalisable to the whole $P_s(n)$ branching graph.

The Littlewood–Richardson rule

Let $\lambda \vdash r - s$, $\mu \vdash s$ and $\nu \vdash r$. The multiplicities,

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- the Young tableau is semistandard;
- the μ-reverse reading word of the Young tableau is a lattice permutation.

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Section 2

Standard tableaux and representations of symmetric groups

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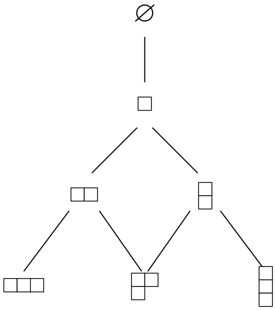

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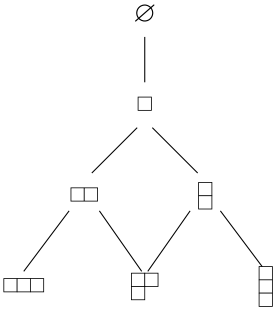

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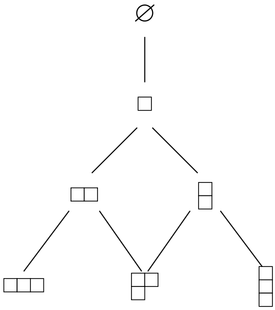

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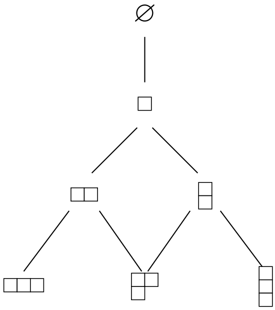

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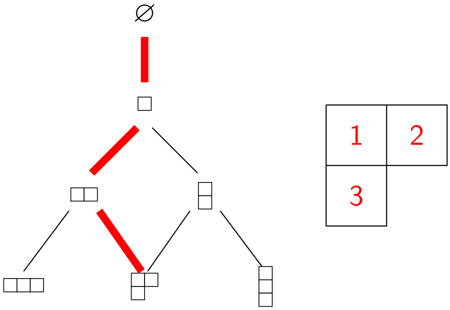

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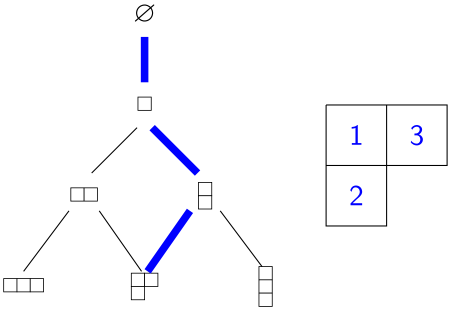

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Each point $\lambda \vdash s$ in the graph labels one left and one 2-sided ideal of the algebra. Fix any $\mathsf{t}^\lambda \in \mathrm{Std}(\lambda)$, we define

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We define the Specht module $S(\lambda)$ to be the quotient

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$$S(\nu \setminus \lambda) = \mathbb{C}\{m_{\mathsf{s}} \mid \mathsf{s} \in \mathrm{Std}(\nu \setminus \lambda)\}$$

Action on skew modules

Given $\mathsf{t} \in \mathrm{Std}_s(\nu \setminus \lambda)$, say we let $\mathsf{t}_{k \leftrightarrow k+1}$ denote the tableau with k and $k+1$ swapped. We have that

$$s_k(\mathsf{t}) = \begin{cases} \mathsf{t}_{k \leftrightarrow k+1} & \text{if } k \leftrightarrow k+1 \text{ is a standard tableau} \\ \mathsf{t} & \text{if } k \ \& \ k+1 \text{ are in same row} \\ -\mathsf{t} + \sum_{\mathsf{s} \rhd \mathsf{t}} a_{\mathsf{s}} \mathsf{s} & \text{if } k \ \& \ k+1 \text{ are in same column} \end{cases}$$

Example

$$S((2,2) \setminus (1)) = \left\{ \begin{array}{|c|c|} \hline & 1 \\ \hline 2 & 3 \\ \hline \end{array}, \begin{array}{|c|c|} \hline & 2 \\ \hline 1 & 3 \\ \hline \end{array} \right\}$$

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$$s_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad s_2 = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$$

Section 3

Semistandard Young tableaux and homomorphisms

Question

Can we give a basis of

$$\mathrm{Hom}_{\mathbb{C}\mathfrak{S}_s}(M^\mu, \Delta_s(\nu \setminus \lambda))$$

and then count how many of these homomorphisms factor through the projection

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Question

- The homomorphisms are indexed by and constructed from SEMISTANDARD tableaux.
- Those which factor through π are counted by the semistandard tableaux satisfying the LATTICE PERMUTATION condition.

A semistandard tableau of shape $\nu \setminus \lambda$ and weight μ is a filling of the boxes of $\nu \setminus \lambda$ with the entries

$$\underbrace{1, \dots, 1}_{\mu_1}, \underbrace{2, \dots, 2}_{\mu_2}, \dots, \underbrace{\ell, \dots, \ell}_{\mu_\ell}$$

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For example the elements of $\mathsf{SStd}((3^2,2)\setminus(2,1),(2^2,1))$ are

		1
	1	2
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What really **IS** a semistandard tableaux?

We think of it as an orbit under the $\mathfrak{S}_{\mu_1} \times \mathfrak{S}_{\mu_2} \times \dots$ permutation action on $\mathrm{Std}(\nu \setminus \lambda)$.

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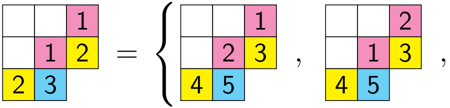

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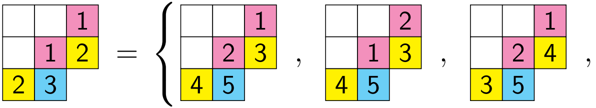

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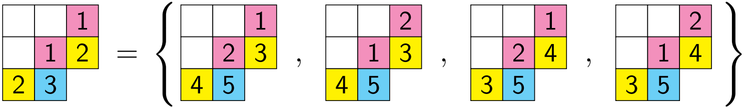

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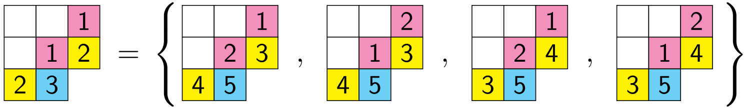

So mapping the generator of $M(\mu)$ to the sum over these four tableaux is a $\mathfrak{S}_5$-homomorphism by Frobenius reciprocity.

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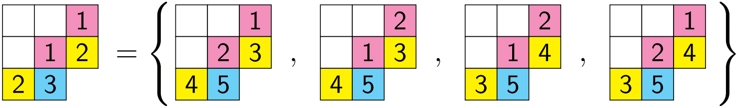

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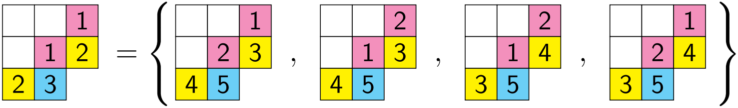

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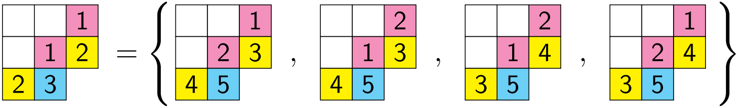

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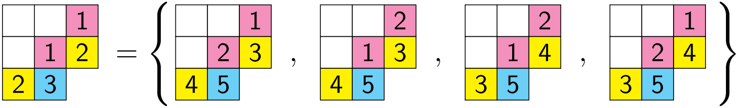

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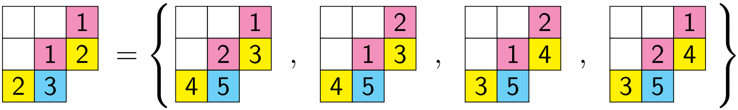

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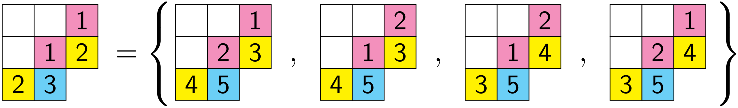
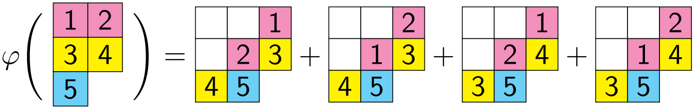
- We define a tableau, T, of weight μ to be an **equivalence class of standard tableaux** under $\overset{\mu}{\sim}$.
- T is **semistandard** if $\mathsf{t}_{k \leftrightarrow k+1} \in \mathrm{Std}(\nu \setminus \lambda)$ for all $\mathsf{t} \in \mathrm{T}$ and all $k_1, \ldots, k_{d-1} \notin \{\mu_1, \mu_1 + \mu_2, \ldots\}$.





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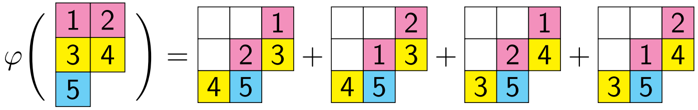


Theorem

For $\mathsf{T} \in \mathrm{SStd}_s(\nu \setminus \lambda, \mu)$, define

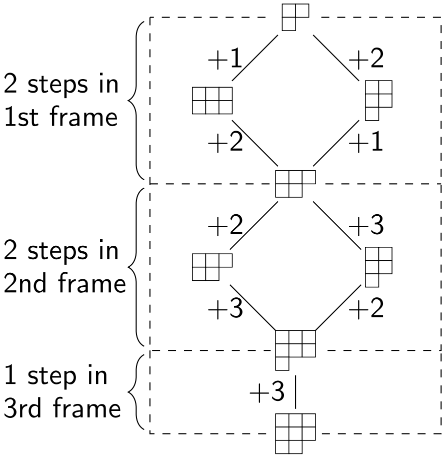
$$\varphi_{\mathsf{T}}(\mathsf{t}^{\mu}) = \sum_{\varphi(\mathsf{t})=\mathsf{T}} \mathsf{t}.$$

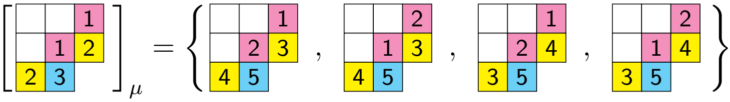
We have that

$$\{\varphi_{\mathsf{T}} \mid \mathsf{T} \in \mathrm{SStd}_s(\nu \setminus \lambda, \mu)\}$$

is a $\mathbb{Z}$-basis for $\mathrm{Hom}_{\mathfrak{S}_s}(M(\mu), \Delta_s(\nu \setminus \lambda))$.

How can we picture this in a way that generalises to $P_s(n)$?





Section 4

The lattice permutation condition

Reverse reading word

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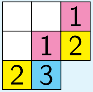
$\mathrm{read}\Big($  $\Big) = 1, 2, 1, 3, 2$

- What meaning does the ordering defining $\mathrm{read}(\mathsf{S})$ have?

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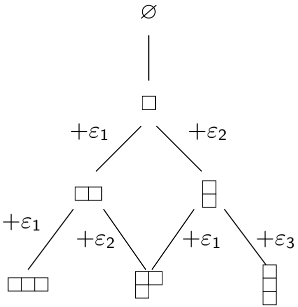
- We decorate the edges of the branching graph with the corresponding row in which a node is added.



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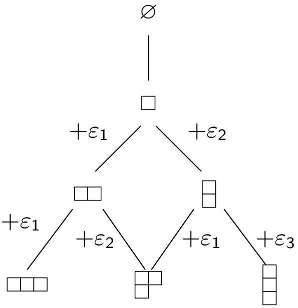

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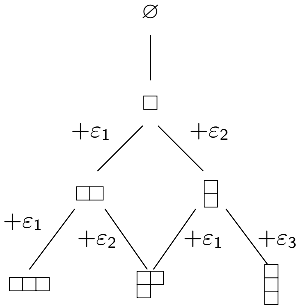
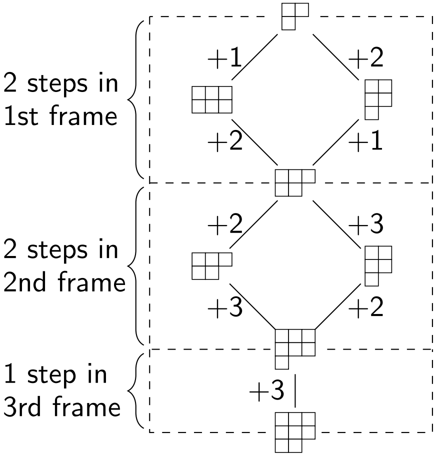
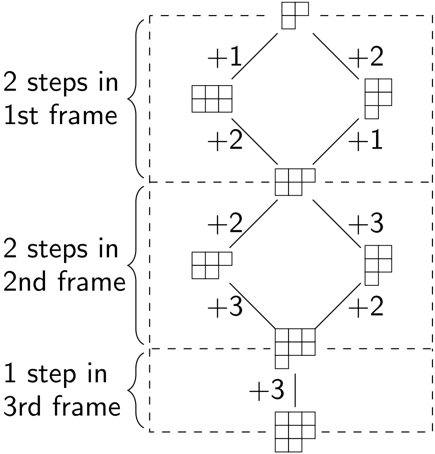


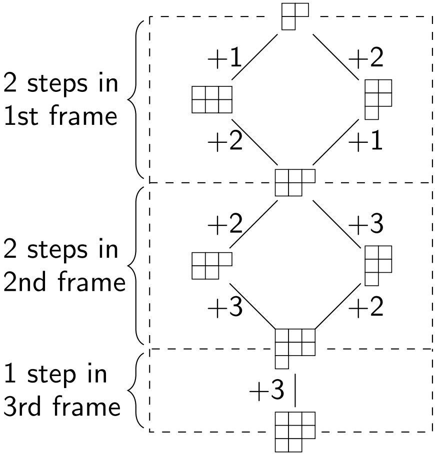
- The dominance ordering on steps is $+\varepsilon_1 < +\varepsilon_2 < +\varepsilon_3 < \dots$
- We read steps in S according to the dominance ordering and then we refine this by recording the frames in which these steps occur in decreasing fashion.



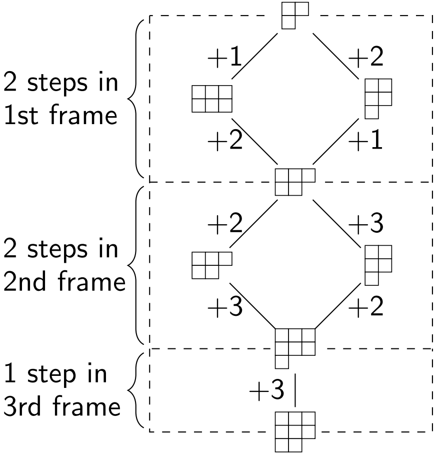
$$\left(\begin{array}{c|cc|cc} +\varepsilon_1 & +\varepsilon_2 & +\varepsilon_2 & +\varepsilon_3 & +\varepsilon_3 \\ & & & & \end{array} \right)$$



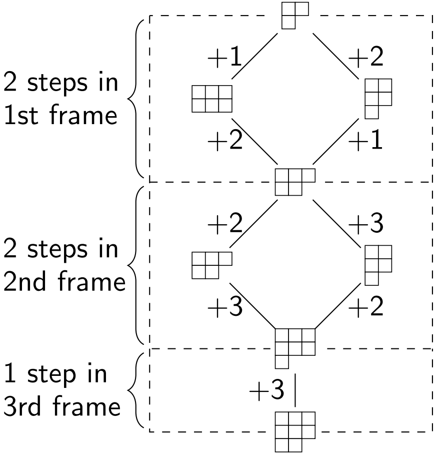
$$\left(\begin{array}{c|cc|cc} +\varepsilon_1 & +\varepsilon_2 & +\varepsilon_2 & +\varepsilon_3 & +\varepsilon_3 \\ 1 & & & & \end{array}\right)$$



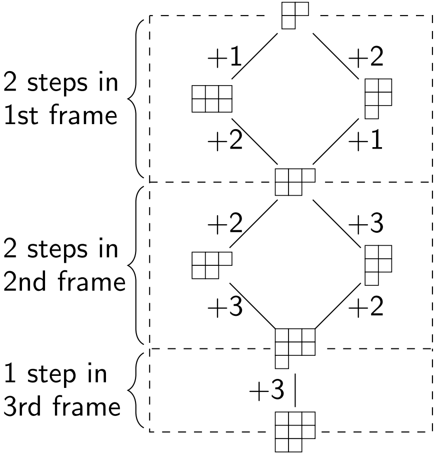
$$\left(\begin{array}{c|cc|cc} +\varepsilon_1 & +\varepsilon_2 & +\varepsilon_2 & +\varepsilon_3 & +\varepsilon_3 \\ 1 & & & & \end{array}\right)$$



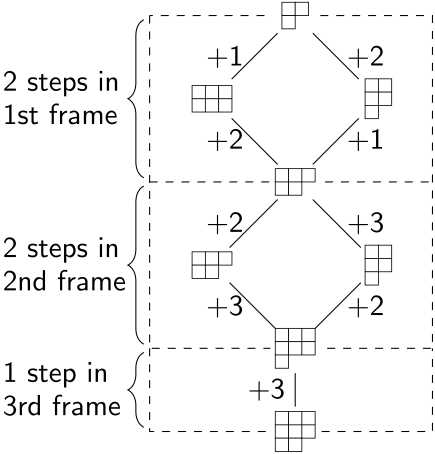
$$\left(\begin{array}{c|cc|cc} +\varepsilon_1 & +\varepsilon_2 & +\varepsilon_2 & +\varepsilon_3 & +\varepsilon_3 \\ 1 & 2 & & & \end{array}\right)$$



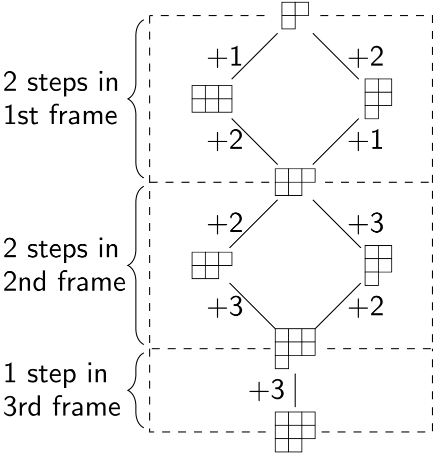
$$\left(\begin{array}{c|cc|cc} +\varepsilon_1 & +\varepsilon_2 & +\varepsilon_2 & +\varepsilon_3 & +\varepsilon_3 \\ 1 & 2 & 1 & & \end{array}\right)$$

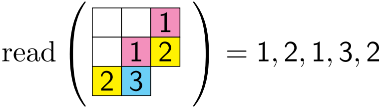


$$\left(\begin{array}{c|cc|cc} +\varepsilon_1 & +\varepsilon_2 & +\varepsilon_2 & +\varepsilon_3 & +\varepsilon_3 \\ 1 & 2 & 1 & 3 & \end{array}\right)$$



$$\left(\begin{array}{c|cc|cc} +\varepsilon_1 & +\varepsilon_2 & +\varepsilon_2 & +\varepsilon_3 & +\varepsilon_3 \\ 1 & 2 & 1 & 3 & 2 \end{array} \right)$$





$$\left(\begin{array}{c|cc|cc} +\varepsilon_1 & +\varepsilon_2 & +\varepsilon_2 & +\varepsilon_3 & +\varepsilon_3 \\ 1 & 2 & 1 & 3 & 2 \end{array}\right)$$

Lattice permutation

Now that we have defined the reading word of a tableau, we are ready to define the quality (good/bad) of each term in $\mathrm{read}(\mathsf{S})$ as follows.

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A sequence of positive integers is called a lattice permutation if every term in the sequence is good.

		1
	1	2
2	3	

$$\left(\begin{array}{c|cc|cc} +\varepsilon_1 & +\varepsilon_2 & +\varepsilon_2 & +\varepsilon_3 & +\varepsilon_3 \\ 1 & 2 & 1 & 3 & 2 \end{array} \right)$$

		1
	1	2
2	3	

$$\left(\begin{array}{c|cc|cc} +\varepsilon_1 & +\varepsilon_2 & +\varepsilon_2 & +\varepsilon_3 & +\varepsilon_3 \\ 1 & 2 & 1 & 3 & 2 \end{array}\right)$$

		1
	1	3
2	2	

		1
	1	2
2	3	

$$\left(\begin{array}{c|cc|cc} +\varepsilon_1 & +\varepsilon_2 & +\varepsilon_2 & +\varepsilon_3 & +\varepsilon_3 \\ 1 & 2 & 1 & 3 & 2 \end{array}\right)$$

		1
	1	3
2	2	

$$\left(\begin{array}{c|cc|cc} +\varepsilon_1 & +\varepsilon_2 & +\varepsilon_2 & +\varepsilon_3 & +\varepsilon_3 \\ 1 & \color{red}{3} & 1 & 2 & 2 \end{array}\right)$$

		1
	1	2
2	3	

$$\left(\begin{array}{c|cc|cc} +\varepsilon_1 & +\varepsilon_2 & +\varepsilon_2 & +\varepsilon_3 & +\varepsilon_3 \\ 1 & 2 & 1 & 3 & 2 \end{array}\right)$$

		1
	1	3
2	2	

$$\left(\begin{array}{c|cc|cc} +\varepsilon_1 & +\varepsilon_2 & +\varepsilon_2 & +\varepsilon_3 & +\varepsilon_3 \\ 1 & 3 & 1 & 2 & 2 \end{array}\right)$$

		1
	2	2
1	3	

		1
	1	2
2	3	

$$\left(\begin{array}{c|cc|cc} +\varepsilon_1 & +\varepsilon_2 & +\varepsilon_2 & +\varepsilon_3 & +\varepsilon_3 \\ 1 & 2 & 1 & 3 & 2 \end{array}\right)$$

		1
	1	3
2	2	

$$\left(\begin{array}{c|cc|cc} +\varepsilon_1 & +\varepsilon_2 & +\varepsilon_2 & +\varepsilon_3 & +\varepsilon_3 \\ 1 & 3 & 1 & 2 & 2 \end{array}\right)$$

		1
	2	2
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$$\left(\begin{array}{c|cc|cc} +\varepsilon_1 & +\varepsilon_2 & +\varepsilon_2 & +\varepsilon_3 & +\varepsilon_3 \\ 1 & 2 & 2 & 3 & 1 \end{array}\right)$$

		1
	1	2
2	3	

$$\left(\begin{array}{c|cc|cc} +\varepsilon_1 & +\varepsilon_2 & +\varepsilon_2 & +\varepsilon_3 & +\varepsilon_3 \\ 1 & 2 & 1 & 3 & 2 \end{array}\right)$$

		1
	1	3
2	2	

$$\left(\begin{array}{c|cc|cc} +\varepsilon_1 & +\varepsilon_2 & +\varepsilon_2 & +\varepsilon_3 & +\varepsilon_3 \\ 1 & 3 & 1 & 2 & 2 \end{array}\right)$$

		1
	2	2
1	3	

$$\left(\begin{array}{c|cc|cc} +\varepsilon_1 & +\varepsilon_2 & +\varepsilon_2 & +\varepsilon_3 & +\varepsilon_3 \\ 1 & 2 & 2 & 3 & 1 \end{array}\right)$$

		2
	1	3
1	2	

		1
	1	2
2	3	

$$\left(\begin{array}{c|cc|cc} +\varepsilon_1 & +\varepsilon_2 & +\varepsilon_2 & +\varepsilon_3 & +\varepsilon_3 \\ 1 & 2 & 1 & 3 & 2 \end{array}\right)$$

		1
	1	3
2	2	

$$\left(\begin{array}{c|cc|cc} +\varepsilon_1 & +\varepsilon_2 & +\varepsilon_2 & +\varepsilon_3 & +\varepsilon_3 \\ 1 & 3 & 1 & 2 & 2 \end{array}\right)$$

		1
	2	2
1	3	

$$\left(\begin{array}{c|cc|cc} +\varepsilon_1 & +\varepsilon_2 & +\varepsilon_2 & +\varepsilon_3 & +\varepsilon_3 \\ 1 & 2 & 2 & 3 & 1 \end{array}\right)$$

		2
	1	3
1	2	

$$\left(\begin{array}{c|cc|cc} +\varepsilon_1 & +\varepsilon_2 & +\varepsilon_2 & +\varepsilon_3 & +\varepsilon_3 \\ 2 & 3 & 1 & 2 & 1 \end{array}\right)$$

$$\begin{array}{|c|c|c|}\hline & & 1 \\ \hline & 1 & 2 \\ \hline 2 & 3 \\ \cline{1-2}\end{array} \qquad \left(\begin{array}{c|cc|cc} +\varepsilon_1 & +\varepsilon_2 & +\varepsilon_2 & +\varepsilon_3 & +\varepsilon_3 \\ 1 & 2 & 1 & 3 & 2 \end{array}\right)$$

$$\begin{array}{|c|c|c|}\hline & & 1 \\ \hline & 1 & 3 \\ \hline 2 & 2 \\ \cline{1-2}\end{array} \qquad \left(\begin{array}{c|cc|cc} +\varepsilon_1 & +\varepsilon_2 & +\varepsilon_2 & +\varepsilon_3 & +\varepsilon_3 \\ 1 & 3 & 1 & 2 & 2 \end{array}\right)$$

$$\begin{array}{|c|c|c|}\hline & & 1 \\ \hline & 2 & 2 \\ \hline 1 & 3 \\ \cline{1-2}\end{array} \qquad \left(\begin{array}{c|cc|cc} +\varepsilon_1 & +\varepsilon_2 & +\varepsilon_2 & +\varepsilon_3 & +\varepsilon_3 \\ 1 & 2 & 2 & 3 & 1 \end{array}\right)$$

$$\begin{array}{|c|c|c|}\hline & & 2 \\ \hline & 1 & 3 \\ \hline 1 & 2 \\ \cline{1-2}\end{array} \qquad \left(\begin{array}{c|cc|cc} +\varepsilon_1 & +\varepsilon_2 & +\varepsilon_2 & +\varepsilon_3 & +\varepsilon_3 \\ 2 & 3 & 1 & 2 & 1 \end{array}\right)$$

And so

$$c((3^2, 2), (2, 1), (2^2, 1)) = 1.$$