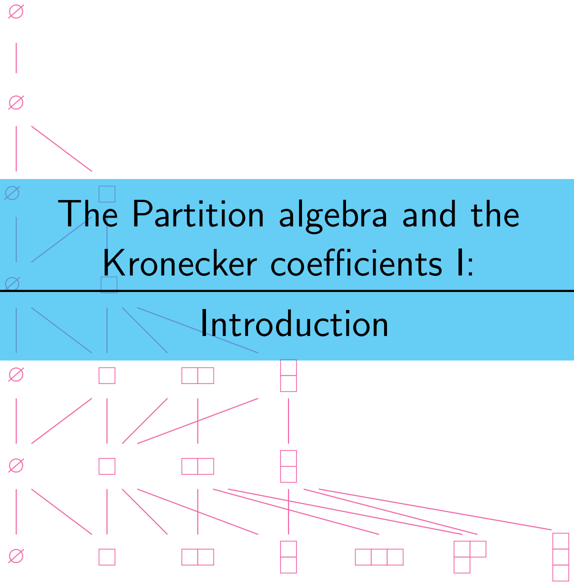


The Partition algebra and the Kronecker coefficients I:

Introduction

Section 1

Schur–Weyl duality, tensor products, and induction and restriction

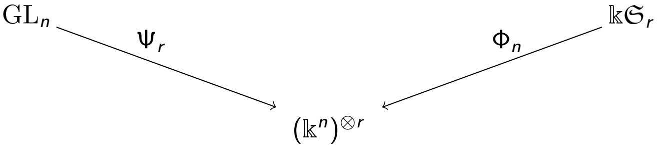
$$\mathrm{GL}_n \xrightarrow{\Psi_r} (\Bbbk^n)^{\otimes r} \xleftarrow{\Phi_n} \Bbbk\mathfrak{S}_r$$

$$\mathrm{GL}_n \xrightarrow{\Psi_r} (\mathbb{k}^n)^{\otimes r} \xleftarrow{\Phi_n} \mathbb{k}\mathfrak{S}_r$$

$$\Phi_n(\mathfrak{S}_r) = \mathrm{End}_{\mathrm{GL}_n}((\mathbb{k}^n)^{\otimes r}) \qquad \Psi_r(\mathrm{GL}_n) = \mathrm{End}_{\mathfrak{S}_r}((\mathbb{k}^n)^{\otimes r})$$

A fixed group G_n

Tower of algebras D_r



$$\Phi_n(D_r) = \mathrm{End}_{G_n}((\Bbbk^n)^{\otimes r}) \qquad \Psi_r(G_n) = \mathrm{End}_{D_r}((\Bbbk^n)^{\otimes r})$$

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$$\begin{array}{ccccc} \mathrm{GL}_n & \xrightarrow{\Psi_r} & & \xleftarrow{\Phi_n} & \Bbbk\mathfrak{S}_r \\ \cup & & (\Bbbk^n)^{\otimes r} & & \cap \\ \mathrm{Sp}_{-2n} / \ \mathrm{O}_n & & & & \end{array}$$

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$\mathrm{GL}_n \xrightarrow{\Psi_r} (\Bbbk^n)^{\otimes r} \xleftarrow{\Phi_n} \Bbbk\mathfrak{S}_r$

$\cup$ $\qquad\qquad$ $\cap$

$\mathrm{Sp}_{-2n}/\ \mathrm{O}_n$ $\qquad$ $(\Bbbk^n)^{\otimes r}$ $\qquad$ $B_r^{\Bbbk}(n)$

$$\Phi_n(D_r) = \mathrm{End}_{G_n}((\Bbbk^n)^{\otimes r}) \qquad \Psi_r(G_n) = \mathrm{End}_{D_r}((\Bbbk^n)^{\otimes r})$$

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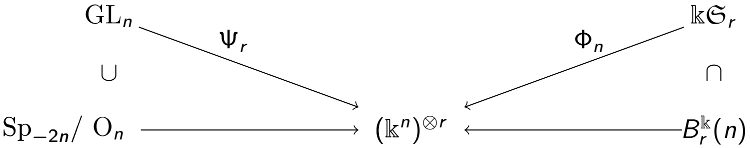

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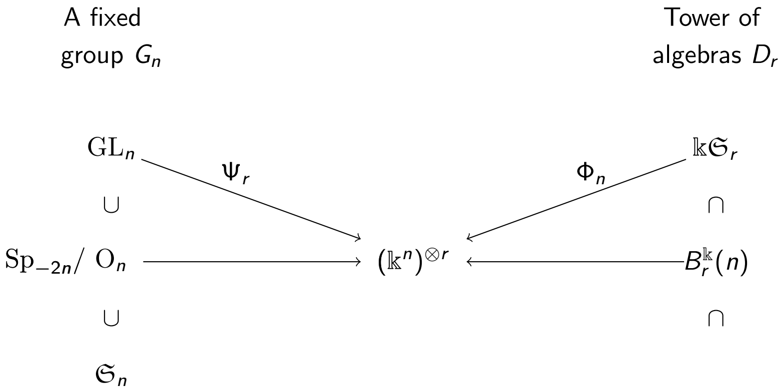

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$$
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\cup & & & & \cap \\
\mathrm{Sp}_{-2n}/\ \mathrm{O}_n & \longrightarrow & (\Bbbk^n)^{\otimes r} & \longleftarrow & B_r^{\Bbbk}(n) \\
\cup & & & & \cap \\
\mathfrak{S}_n & & & & P_r^{\Bbbk}(n)
\end{array}
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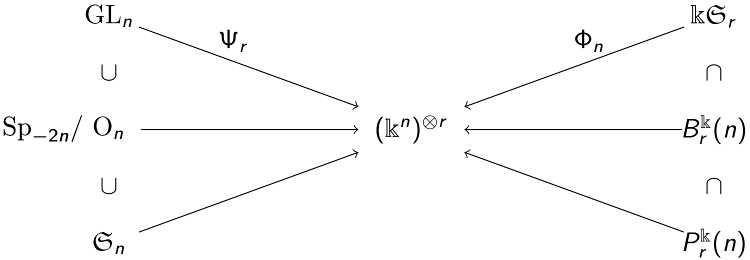

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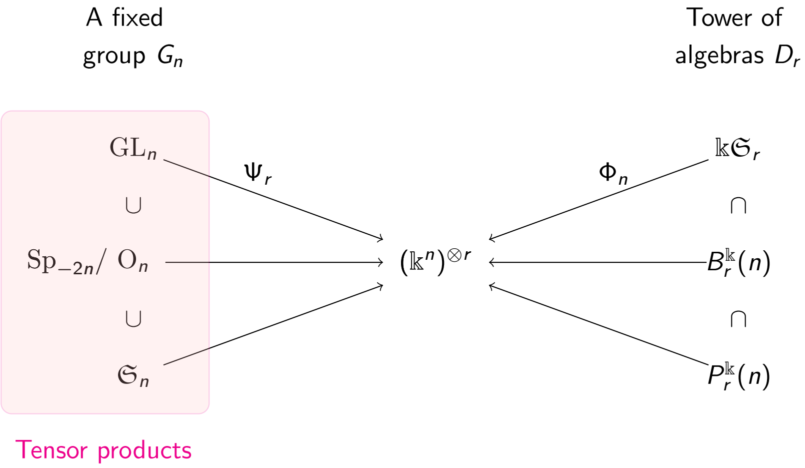
A fixed
group G_n
Tower of
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GL_n
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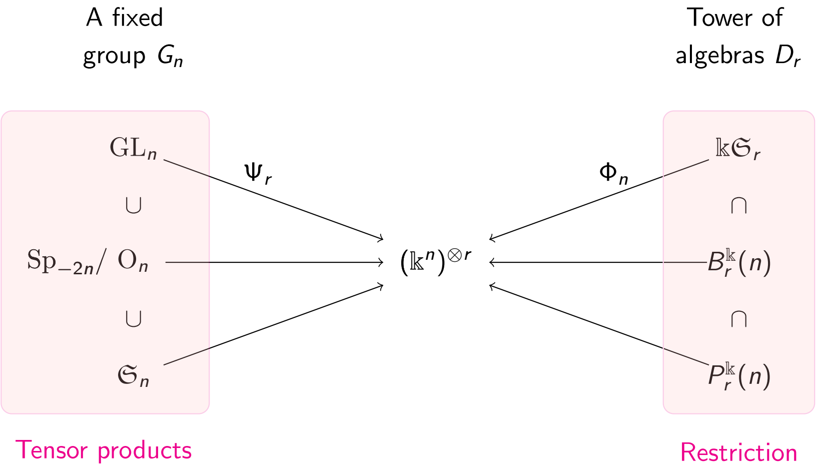
Tensor products



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Tensor products
Restriction

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$\mathfrak{S}_n$	$P_r^{\Bbbk}(n)$
Tensor products	Restriction

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GL_n ✓	Littlewood–Richardson rule $c(\lambda, \mu, \nu)$	✓ $\Bbbk\mathfrak{S}_r$
$\cup$		$\cap$
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$\mathfrak{S}_n$	×	Kronecker coefficients $g(\lambda,\mu,\nu)$	♣	$P_r^{\Bbbk}(n)$
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Provide a positive combinatorial algorithm (in NP) for calculating the value of a given Kronecker coefficient.

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How does $S^{\mathbb{C}}(2^2, 1) \otimes S(4,1)$ decompose?

	(5)	$(4,1)$	$(3,2)$	$(3,1^2)$	$(2^2,1)$	$(2,1^3)$	(1^5)
(5)	1	1	1	1	1	1	1
$(4,1)$	−1	0	1	−1	0	2	4
$(3,2)$	0	1	−1	1	−1	1	5
$(3,1^2)$	1	−2	0	0	0	0	6
$(2^2,1)$	0	1	−1	−1	1	−1	5
$(2,1^3)$	−1	0	1	1	0	−2	4
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$$g((2^2,1),(4,1),\nu) = \begin{cases} 1 & \text{for } \nu = (3,2), (3,1^2), (2^2,1) \text{ or } (2,1^3) \\ 0 & \text{otherwise.} \end{cases}$$

The following conjecture, posed by Saxl, serves as a bench-mark for our complete lack of understanding.....

Saxl's Conjecture

Let $\rho = (k, k-1, \dots, 2, 1)$, then

$$S^{\mathbb{C}}(\rho) \otimes S^{\mathbb{C}}(\rho)$$

contains all $\mathbb{C}\mathfrak{S}_n$-simples with non-zero multiplicity.

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- 2-adic staircases and other decomposable Specht modules (B.–Bessenrodt–Sutton)

Section 2

Stabilities of Kronecker coefficients

Theorem (Manivel 2011)

The non-zero Kronecker coefficients form a semigroup

$$\mathsf{Kron} = \{g(\lambda, \nu, \mu) \mid g(\lambda, \nu, \mu) > 0\}$$

under addition of partitions,

$$g(\lambda + \alpha, \nu + \beta, \mu + \gamma) \geq \max\{g(\lambda, \nu, \mu), g(\alpha, \beta, \gamma)\}$$

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- classify multiplicity-free Kronecker products (B., Bessenrodt)

Clearly $g(\alpha, \beta, \gamma) = 1 > 0$ for $\alpha = \beta = \gamma = \square$. Therefore by the semigroup property

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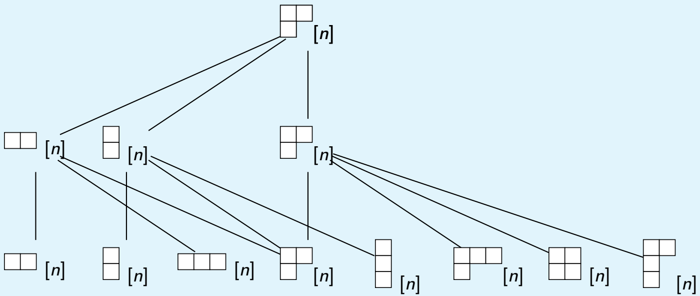
7+	$(2)_{[n]} \oplus (1,1)_{[n]} \oplus 2\,(2,1)_{[n]} \oplus (1,1,1)_{[n]} \oplus (3)_{[n]} \oplus (2,2)_{[n]} \oplus (2,1,1)_{[n]} \oplus (3,1)_{[n]}$

Example

These stable coefficients can be thought of as the number of paths in a special graph which we will come back to later....



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- $P_s(n)$ controls the representation theory of $\mathfrak{S}_n$ as $n \to \infty$

If we can add and multiply partitions, can we divide them?

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Theorem (Knutson–Tao 1999)

The sub-semigroup

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The Kronecker semigroup is not closed under scaling. The holes in this scaling property correspond to the various degrees of difficulty of the positivity problem (P $\subset$ NP $\cap$ coNP $\subset$ NP).

If we can add and multiply partitions, can we divide them?

Theorem (Knutson–Tao 1999)

The sub-semigroup

$$\{\overline{g}(\lambda, \mu, \nu) \mid |\lambda| + |\mu| = |\nu|\} \subseteq \mathsf{Kron}$$

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Klyachko's Conjecture

The sub-semigroup of **stable** Kronecker coefficients is closed under scaling.

Section 3

The modular approach to Kronecker positivity and Saxl's conjecture

How does modular representation theory help?

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E.g, Every block has at least one projective summand.

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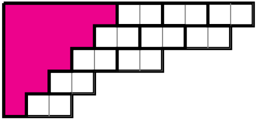
The block $B(\rho_5)$ of $\mathcal{H}_2^{\Bbbk}(\mathfrak{S}_{35})$ contains the simple module labelled by:



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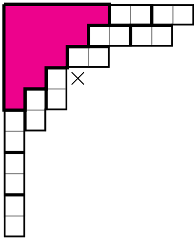

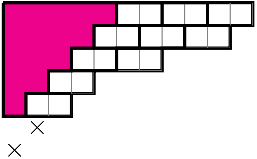

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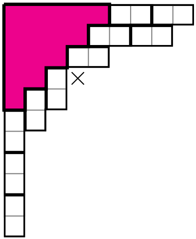

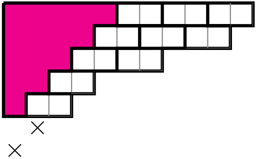

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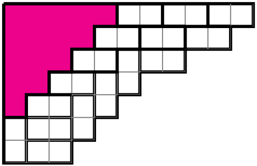

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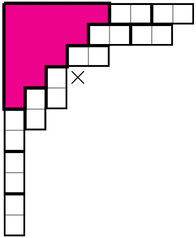

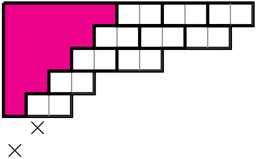

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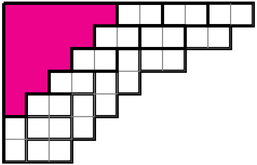
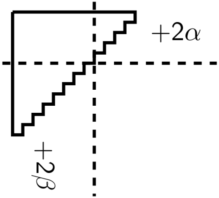


A partition, λ, in the block ρ_k is 2-q-s if $(k+3-a, a) \notin \lambda$ for some $1 \leq a \leq k+1$.

Any 2-q-s can be written in the form

$\rho_\beta^\alpha =$ 

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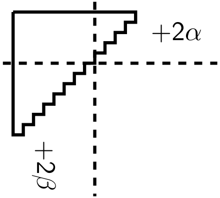
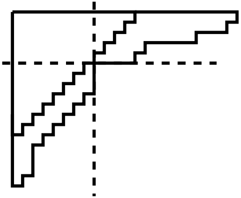


$=$



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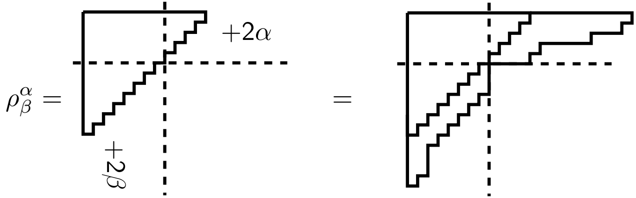

Theorem (B. Bessenrodt Sutton)

The $\mathcal{H}^{\mathbb{C}}_{-1}(\mathfrak{S}_n)$-module $S^{\mathbb{C}}_{-1}(\rho^\alpha_\beta)$ is semisimple and decomposes as follows

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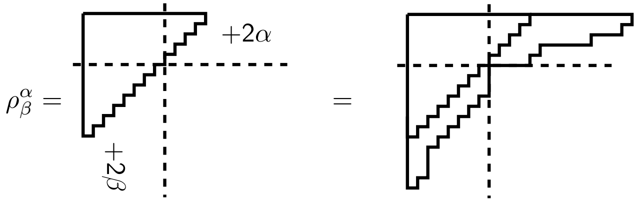

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Conjecture: classification of decomposable Spechts (BBS)

Over $\mathbb{C}$, a Specht module $S(\lambda)$ is decomposable semisimple if and only if $e = 2$ and λ is 2-q-s or a “near square”.

Kronecker positivity and Saxl's Conjecture (BBS)

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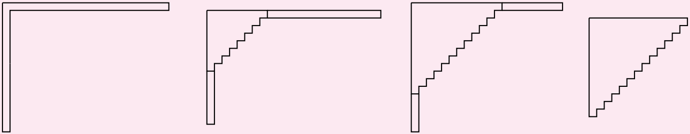

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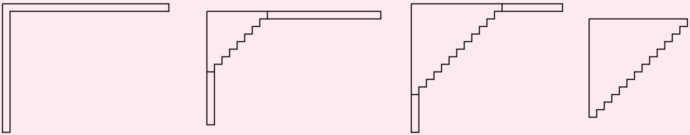

and many others coming from other choices of columns in the decomposition matrix.

Unbounding coefficients in Saxl's tensor-square (BBS)

As $k \to \infty$, the multiplicities appearing in the decomposition of

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Section 4

Classical Schur–Weyl duality: Classifying multiplicity-free Kronecker products

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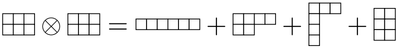

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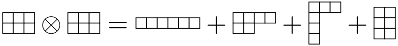

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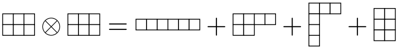

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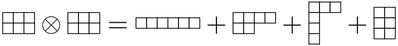

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- certain pairs of 2-line partitions
- $\{\text{rectangle}, (n-2, 1^2)\}$ or $\{\text{rectangle}, (n-2, 2)\}$