Self-Enforcing Insurance Arrangements.

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The General Problem

- We build the model that we study on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ satisfying the usual conditions and supporting a standard one-dimensional $(\mathcal{F}_t)$-Brownian motion W. We denote by $\mathcal{A}$ the family of all càglàd $(\mathcal{F}_t)$-adapted processes A with increasing sample paths and such that $A_0 \geq 0$. We assume that all processes considered satisfy suitable integrability conditions, which we are not going to discuss any further below.

- We consider two agents indexed by $i = 1, 2$. The two agents receive exogenous endowments in a non-storable good that are given by $(\mathcal{F}_t)$-adapted continuous processes $Y_i \geq 0$ such that $Y_1 + Y_2 > 0$.

  Under autarky, the agents consume their own endowments. Their expected utility processes are given by

$$\underline{U}_{i,t} = \mathbb{E}\left[\int_t^\infty u_i(s, Y_{i,s})\, ds \;\middle|\; \mathcal{F}_t\right], \quad \text{for } i = 1, 2, \tag{1}$$

  where u_i is the utility function of agent i. We assume that the functions $u_i(t, \cdot)$, $t \geq 0$, are C^2 and satisfy the Inada conditions.

- The two agents may decide to pull their endowments and agree on a consumption allocation scheme (C_1, C_2) with a view to risk sharing. Such a consumption allocation should respect the *resource constraint*

$$C_1 + C_2 = Y_1 + Y_2 =: Y. \tag{2}$$

Since the good is non-storable, relaxing this constraint to $C_1 + C_2 \leq Y$ would not improve the agents' optimal utilities.

We denote by $\mathcal{C}_{\mathrm{f}}$ the family of all *feasible* consumption allocations, namely, all positive $(\mathcal{F}_t)$-adapted continuous pairs (C_1, C_2) respecting the resource constraint (2).

- We assume that both of the agents have limited commitment. This assumption gives rise to the *participation* or *sustainability constraints*

$$U_{i,t}(C_i) \geq \underline{U}_{i,t} \quad \text{for all } t \geq 0, \tag{3}$$

where

$$U_{i,t}(C_i) = \mathbb{E}\left[\int_t^\infty u_i(s, C_{i,s})\, ds \;\middle|\; \mathcal{F}_t\right]. \tag{4}$$

We denote by $\mathcal{C}_{\mathrm{a}} \subseteq \mathcal{C}_{\mathrm{f}}$ the family of all *admissible* consumption allocations, namely, all $(C_1, C_2) \in \mathcal{C}_{\mathrm{f}}$ that satisfy the participation constraints (3).

- The objective is to characterise the constrained-efficient allocations, namely, the solutions to the social planner's problem that is defined by

$$V = \sup_{(C_1, C_2) \in \mathcal{C}_{\mathrm{a}}} \big\{ \zeta_1 U_{1,0}(C_1) + \zeta_2 U_{2,0}(C_2) \big\}. \tag{5}$$

  In this problem, the constants $\zeta_1 > 0$ and $\zeta_2 > 0$ are the Pareto weights on Agent 1 and Agent 2, respectively.

- To solve the constrained optimisation problem defined by (5), we consider the Lagrangian

$$\begin{aligned} \mathcal{L}(C_1, C_2, \lambda_1, \lambda_2) = {} & \zeta_1 U_{1,0}(C_1) + \zeta_2 U_{2,0}(C_2) \\ & + \sum_{i=1,2} \mathbb{E} \left[\int_{[0,\infty[} \big(U_{i,t}(C_i) - \underline{U}_{i,t} \big) \, d\lambda_{i,t} \right]. \end{aligned} \tag{6}$$

  The processes $\lambda_i \in \mathcal{A}$ are such that $\lambda_{i,0} = 0$. They play the roles of (cumulative) Kuhn-Tucker multipliers associated with the participation constraints (3).

  In view of the participation constraints (3), an inspection of (5) and (6) reveals that

$$V = \inf_{\lambda_1, \lambda_2 \in \mathcal{A}} \sup_{(C_1, C_2) \in \mathcal{C}_{\mathrm{a}}} \mathcal{L}(C_1, C_2, \lambda_1, \lambda_2) \leq \inf_{\lambda_1, \lambda_2 \in \mathcal{A}} \sup_{(C_1, C_2) \in \mathcal{C}_{\mathrm{f}}} \mathcal{L}(C_1, C_2, \lambda_1, \lambda_2). \tag{7}$$

- In view of the expression

$$\mathcal{L}(C_1, C_2, \lambda_1, \lambda_2) = \zeta_1 U_{1,0}(C_1) + \zeta_2 U_{2,0}(C_2) \\ + \sum_{i=1,2} \mathbb{E}\left[\int_{[0,\infty[} \big(U_{i,t}(C_i) - \underline{U}_{i,t}\big)\, d\lambda_{i,t}\right] \tag{8}$$

and the inequalities

$$V = \inf_{\lambda_1,\lambda_2 \in \mathcal{A}} \sup_{(C_1,C_2)\in\mathcal{C}_{\rm a}} \mathcal{L}(C_1, C_2, \lambda_1, \lambda_2) \le \inf_{\lambda_1,\lambda_2 \in \mathcal{A}} \sup_{(C_1,C_2)\in\mathcal{C}_{\rm f}} \mathcal{L}(C_1, C_2, \lambda_1, \lambda_2), \tag{9}$$

we can see that, if we find $(C_1^\star, C_2^\star) \in \mathcal{C}_{\rm f}$ and $\lambda_1^\star, \lambda_2^\star \in \mathcal{A}$ such that

$$\mathcal{L}(C_1^\star, C_2^\star, \lambda_1^\star, \lambda_2^\star) = \inf_{\lambda_1,\lambda_2 \in \mathcal{A}} \sup_{(C_1,C_2)\in\mathcal{C}_{\rm f}} \mathcal{L}(C_1, C_2, \lambda_1, \lambda_2), \tag{10}$$

$$(C_1^\star, C_2^\star) \in \mathcal{C}_{\rm a} \quad \text{and} \quad \sum_{i=1,2} \mathbb{E}\left[\int_{[0,\infty[} \big(U_{i,t}(C_i^\star) - \underline{U}_{i,t}\big)\, d\lambda_{i,t}^\star\right] = 0, \tag{11}$$

then $(C_1^\star, C_2^\star)$ provides the solution to the constrained optimisation problem defined by (5).

- Using the integration by parts formula, we calculate

$$\mathbb{E}\left[\int_{[0,\infty[} U_{i,t}(C_i)\, d\lambda_{i,t}\right] = \mathbb{E}\left[\int_{[0,\infty[} \left(\int_t^\infty u_i(s, C_{i,s})\, ds\right) d\lambda_{i,t}\right]$$
$$= \mathbb{E}\left[\int_0^\infty u_i(t, C_{i,t})\lambda_{i,t}\, dt\right]. \quad (12)$$

We thus rewrite the Lagrangian as

$$\mathcal{L}(C_1, C_2, \Lambda_1, \Lambda_2) = \mathbb{E}\left[\int_0^\infty (\Lambda_{1,t} + \Lambda_{2,t})\Big[Z_t u_1(t, C_{1,t}) + (1 - Z_t)u_2(t, C_{2,t})\Big]\, dt\right]$$
$$- \sum_{i=1,2} \mathbb{E}\left[\int_{[0,\infty[} \underline{U}_{i,t}\, d\Lambda_{i,t}\right], \quad (13)$$

where

$$\Lambda_{i,t} = \zeta_i + \lambda_{i,t} \quad \text{and} \quad Z_t = \frac{\Lambda_{1,t}}{\Lambda_{1,t} + \Lambda_{2,t}}. \quad (14)$$

- Given $y, z > 0$, we define

$$\widetilde{u}(t,y,z) = \sup_{0<c<y} \big\{ z u_1(t,c) + (1-z) u_2(t, y-c) \big\}. \tag{15}$$

The unique value $\alpha(t,y,z)$ of c that achieves the supremum is such that

$$u_1\big(t, \alpha(t,y,z)\big) = \widetilde{u}(t,y,z) + (1-z)\widetilde{u}_z(t,y,z) \tag{16}$$

$$\text{and} \quad u_2\big(t, y - \alpha(t,y,z)\big) = \widetilde{u}(t,y,z) - z\widetilde{u}_z(t,y,z). \tag{17}$$

Given $(\Lambda_1, \Lambda_2) \in \mathcal{A}$ such that $\Lambda_{i,0} = \zeta_i$, the consumption allocation given by

$$c_1^\star(t, Y_t, \Lambda_{1,t}, \Lambda_{2,t}) = \alpha\left(t, Y_t, \frac{\Lambda_{1,t}}{\Lambda_{1,t} + \Lambda_{2,t}} \right) \tag{18}$$

$$\text{and} \quad c_2^\star(t, Y_t, \Lambda_{1,t}, \Lambda_{2,t}) = Y_t - c_1^\star(t, Y_t, \Lambda_{1,t}, \Lambda_{2,t}), \tag{19}$$

is feasible and maximises the Lagrangian given by (13). We define

$$\begin{aligned} J(\Lambda_1, \Lambda_2) &:= \sup_{(C_1, C_2) \in \mathcal{C}_{\mathrm{f}}} \mathcal{L}(C_1, C_2, \Lambda_1, \Lambda_2) \\ &= \mathbb{E}\left[\int_0^\infty (\Lambda_{1,t} + \Lambda_{2,t}) \widetilde{u}(t, Y_t, Z_t)\, dt \right] - \sum_{i=1,2} \mathbb{E}\left[\int_{[0,\infty[} \underline{U}_{i,t}\, d\Lambda_{i,t} \right]. \end{aligned} \tag{20}$$

- Given $\xi \in \mathcal{A}$, we define

$$\nabla_{1,\xi} J(\Lambda_1, \Lambda_2) = \lim_{\delta \downarrow 0} \frac{1}{\delta}\Big[J(\Lambda_1 + \delta\xi, \Lambda_2) - J(\Lambda_1, \Lambda_2)\Big] \tag{21}$$

$$\text{and} \quad \nabla_{2,\xi} J(\Lambda_1, \Lambda_2) = \lim_{\delta \downarrow 0} \frac{1}{\delta}\Big[J(\Lambda_1, \Lambda_2 + \delta\xi) - J(\Lambda_1, \Lambda_2)\Big]. \tag{22}$$

Using the expressions (16)–(17) and the integration by parts, we calculate

$$\begin{aligned}\nabla_{i,\xi} J(\Lambda_1, \Lambda_2) &= \mathbb{E}\left[\int_0^\infty \xi_t u_i\big(t, c_i^\star(t, Y_t, \Lambda_{1,t}, \Lambda_{2,t})\big)\, dt\right] - \mathbb{E}\left[\int_{[0,\infty[} \underline{U}_{i,t}\, d\xi_t\right] \\ &= \mathbb{E}\left[\int_{[0,\infty[} \Big[U_{i,t}\big(c_i^\star(\cdot, Y, \Lambda_1, \Lambda_2)\big) - \underline{U}_{i,t}\Big]\, d\xi_t\right]. \end{aligned} \tag{23}$$

- Now, suppose that there exists a pair $(\Lambda_1^\star, \Lambda_2^\star)$ that minimises the index J given by (20). This pair is such that

$$\nabla_{i,\xi} J(\Lambda_1^\star, \Lambda_2^\star) \geq 0 \quad \text{for all } \xi \in \mathcal{A} \text{ and } i = 1, 2. \tag{24}$$

This inequality and the expression (23) imply that

$$\Big(c_1^\star(\cdot, Y, \Lambda_1^\star, \Lambda_2^\star),\; c_2^\star(\cdot, Y, \Lambda_1^\star, \Lambda_2^\star)\Big) \in \mathcal{C}_{\mathrm{a}} \tag{25}$$

because this consumption allocation satisfies the resource constraint (2) as well as the participation constraints (3). Furthermore,

$$\int_{[0,\infty[} \Big[U_{i,t}\big(c_i^\star(\cdot, Y, \Lambda_1^\star, \Lambda_2^\star)\big) - \underline{U}_{i,t}\Big]\, d\Lambda_i^\star = 0, \tag{26}$$

i.e., the optimal Lagrange multipliers increase only when the constraints are binding.

It follows that the consumption allocation in (25) provides the solution to the constrained optimisation problem defined by (5).

A Canonical (Symmetric) Application

- We now assume that

$$Y_1 = X \quad \text{and} \quad Y_2 = 1 - X, \tag{27}$$

where X satisfies the SDE

$$dX_t = \mu(X_t)\, dt + \sigma(X_t)\, dW_t, \quad X_0 = x \in]0, 1[. \tag{28}$$

We assume that μ, σ are such that the SDE (28) has a unique non-explosive strong solution with values in $]0, 1[$,

$$\mu(x) = -\mu(1 - x) \quad \text{and} \quad \sigma(x) = \sigma(1 - x) \quad \text{for all } x \in]0, 1[. \tag{29}$$

We denote by φ (resp., ψ) the minimal strictly decreasing (resp., strictly increasing) ϱ-excessive functions of the diffusion associated with the SDE for X. These are solutions to the ODE

$$\frac{1}{2}\sigma^2(x) f''(x) + \mu(x) f'(x) - \varrho f(x) = 0. \tag{30}$$

Also, we denote by p the scale function of the diffusion associated with the SDE for X (note that $\varphi\psi' - \varphi'\psi = Kp'$, for some constant K).

- We assume that

$$u_1(t,c) = u_2(t,c) = e^{-\varrho t} u(c), \tag{31}$$

for some C^2 utility function u that satisfies the Inada conditions and a constant $\varrho > 0$. The agents' autarky expected utility processes are then given by

$$\underline{U}_{1,t} = \mathbb{E}\left[\int_t^\infty e^{-\varrho s} u(X_s)\, ds \;\middle|\; \mathcal{F}_t\right] = e^{-\varrho t}\Gamma(X_t) \tag{32}$$

$$\text{and} \quad \underline{U}_{2,t} = \mathbb{E}\left[\int_t^\infty e^{-\varrho s} u(1 - X_s)\, ds \;\middle|\; \mathcal{F}_t\right] = e^{-\varrho t}\Gamma(1 - X_t), \tag{33}$$

where

$$\Gamma(x) = \mathbb{E}\left[\int_0^\infty e^{-\varrho t} u(X_t)\, dt\right] = \varphi(x)\int_0^x \Psi(s) u(s)\, ds + \psi(x)\int_x^1 \Phi(s) u(s)\, ds, \tag{34}$$

with

$$\Phi(s) = \frac{2\varphi(s)}{K\sigma^2(s)p'(s)} \quad \text{and} \quad \Psi(s) = \frac{2\psi(s)}{K\sigma^2(s)p'(s)}. \tag{35}$$

- Given constants $\zeta_1 > 0$ and $\zeta_2 > 0$, the objective is to solve the problem

$$V = \sup_{C \in \mathcal{C}} \left\{ \zeta_1 \, \mathbb{E}\left[\int_0^\infty e^{-\varrho s} u(C_s) \, ds \right] + \zeta_2 \, \mathbb{E}\left[\int_0^\infty e^{-\varrho s} u(1 - C_s) \, ds \right] \right\}, \tag{36}$$

where $\mathcal{C}$ is the family of all $]0,1[$-valued $(\mathcal{F}_t)$-adapted continuous processes such that

$$\mathbb{E}\left[\int_t^\infty e^{-\varrho s} u(C_s) \, ds \;\middle|\; \mathcal{F}_t \right] \geq e^{-\varrho t} \Gamma(X_t) \tag{37}$$

$$\text{and} \quad \mathbb{E}\left[\int_t^\infty e^{-\varrho s} u(1 - C_s) \, ds \;\middle|\; \mathcal{F}_t \right] \geq e^{-\varrho t} \Gamma(1 - X_t). \tag{38}$$

- In this case, the Lagrangian takes the form

$$\begin{aligned} \mathcal{L}(C, \Lambda_1, \Lambda_2) = {} & \mathbb{E}\left[\int_0^\infty e^{-\varrho t} \big(\Lambda_{1,t} + \Lambda_{2,t} \big) \Big[Z_t u(C_t) + (1 - Z_t) u(1 - C_t) \Big] \, dt \right] \\ & - \mathbb{E}\left[\int_{[0,\infty[} e^{-\varrho t} \Gamma(X_t) \, d\Lambda_{1,t} \right] - \mathbb{E}\left[\int_{[0,\infty[} e^{-\varrho t} \Gamma(1 - X_t) \, d\Lambda_{2,t} \right], \end{aligned} \tag{39}$$

where

$$\Lambda_{i,0} = \zeta_i \quad \text{and} \quad Z_t = \frac{\Lambda_{1,t}}{\Lambda_{1,t} + \Lambda_{2,t}}. \tag{40}$$

- As in the general case, the first order condition for pointwise maximisation inside the first expectation yields

$$C_t^\star = \alpha(Z_t), \tag{41}$$

where

$$\alpha(z) = \operatorname*{arg\,max}_{0<c<1} \big\{ zu(c) + (1-z)u(1-c) \big\}. \tag{42}$$

Furthermore, we define

$$\widetilde{u}(z) = \max_{0<c<1} \big\{ zu(c) + (1-z)u(1-c) \big\}, \quad \text{for } z > 0. \tag{43}$$

In view of the general analysis, the problem reduces to minimising the performance criterion

$$\begin{aligned} J(\Lambda_1, \Lambda_2) = \mathbb{E}\left[\int_0^\infty e^{-\varrho t} \big(\Lambda_{1,t} + \Lambda_{2,t} \big) \widetilde{u}(Z_t)\, dt \right] \\ - \mathbb{E}\left[\int_{[0,\infty[} e^{-\varrho t} \Gamma(X_t)\, d\Lambda_{1,t} \right] - \mathbb{E}\left[\int_{[0,\infty[} e^{-\varrho t} \Gamma(1 - X_t)\, d\Lambda_{2,t} \right] \end{aligned} \tag{44}$$

over (Λ_1, Λ_2). We denote by

$$]0,1[\times]0,\infty[^{\,2} \ni (x, \zeta_1, \zeta_2) \mapsto v(x, \zeta_1, \zeta_2) \tag{45}$$

the value function of this singular stochastic control problem.

- A suitable solution to the HJB equation

$$\min\Big\{\frac{1}{2}\sigma^2(x)w_{xx}(x,z) + \mu(x)w_x(x,z) - \varrho w(x,z) + \widetilde{u}(z),$$
$$w(x,z) + (1-z)w_z(x,z) - \Gamma(x),\; w(x,z) - zw_z(x,z) - \Gamma(1-x)\Big\} = 0 \tag{46}$$

identifies with the value function in the sense that

$$v(x,\zeta_1,\zeta_2) = (\zeta_1+\zeta_2)w\left(x, \frac{\zeta_1}{\zeta_1+\zeta_2}\right). \tag{47}$$

The idea for the relevant verification theorem is to start with an application of Itô's formula to the process $\big(e^{-\varrho t}(\Lambda_{1,t}+\Lambda_{2,t})w(X_t, \frac{\Lambda_{1,t}}{\Lambda_{1,t}+\Lambda_{2,t}})\big)$.

- The structure of the HJB equation

$$\min\left\{\frac{1}{2}\sigma^2(x)w_{xx}(x,z) + \mu(x)w_x(x,z) - \varrho w(x,z) + \widetilde{u}(z),\right.$$
$$\left. w(x,z) + (1-z)w_z(x,z) - \Gamma(x),\ w(x,z) - zw_z(x,z) - \Gamma(1-x)\right\} = 0 \qquad (48)$$

suggests that the state space $\mathcal{S} = \,]0,1[^{\,2}$ is partitioned into three regions $\mathcal{S}^1$, $\mathcal{S}^{\mathrm{c}}$ and $\mathcal{S}^2$. In the open region $\mathcal{S}^{\mathrm{c}}$, the function w satisfies the ODE

$$\frac{1}{2}\sigma^2(x)w_{xx}(x,z) + \mu(x)w_x(x,z) - \varrho w(x,z) + \widetilde{u}(z) = 0. \qquad (49)$$

Inside this region, the function w is of the form

$$w(x,z) = A(z)\varphi(x) + B(z)\psi(x) + \frac{\widetilde{u}(z)}{\varrho}. \qquad (50)$$

In the region $\mathcal{S}^1$, the first agent's participation constraint is binding and

$$w(x,z) + (1-z)w_z(x,z) = \Gamma(x), \qquad (51)$$

while, in the region $\mathcal{S}^2$, the second agent's participation constraint is binding and

$$w(x,z) - zw_z(x,z) = \Gamma(1-x). \qquad (52)$$

- (*When perfect risk-sharing is sustainable.*) We look for a point $z_\dagger \in \,]0, \frac{1}{2}[$ and a function $h : \,]0, z_\dagger[$ such that

$$\mathcal{S}^1 = \big\{(x,z) \in \mathcal{S} \mid \; z \leq z_\dagger \text{ and } x \geq h(z)\big\} \tag{53}$$

$$\text{and} \quad \mathcal{S}^2 = \big\{(x,z) \in \mathcal{S} \mid \; z \geq 1 - z_\dagger \text{ and } x \leq 1 - h(1-z)\big\}. \tag{54}$$

To determine the free-boundary function h, we use the value matching and the smooth pasting conditions

$$w(x,z) + (1-z)w_z(x,z) = \Gamma(x) \quad \text{and} \quad w_x(x,z) + (1-z)w_{zx}(x,z) = \Gamma'(x), \tag{55}$$

for $x = h(z)$, which imply that

$$\big[B(z) + (1-z)B'(z)\big]\psi\big(h(z)\big) + \frac{u\big(\alpha(z)\big)}{\varrho} = \Gamma\big(h(z)\big) \tag{56}$$

$$\text{and} \quad B(z) + (1-z)B'(z) = \frac{\Gamma'\big(h(z)\big)}{\psi'\big(h(z)\big)}, \tag{57}$$

respectively. These identities imply that h should satisfy the equation $F\big(h(z), z\big) = 0$, where

$$F(x,z) = \int_0^x \frac{\psi(s)}{\sigma^2(s)p'(s)}\big[u(s) - u\big(\alpha(z)\big)\big]\, ds. \tag{58}$$

There exists $z_\dagger \in \,]0, \frac{1}{2}[$ such that this equation has a unique solution $h(z) \in \,]0,1[$ for all $z \in \,]0, z_\dagger[$ if and only if $F(1, \frac{1}{2}) < 0$.

- (*When perfect risk-sharing is not sustainable.*) We look for a point $z_\dagger \in]\frac{1}{2}, 1[$ and a function $h :]0, z_\dagger[$ such that

$$\mathcal{S}^1 = \big\{(x,z) \in \mathcal{S} \mid \ z \leq z_\dagger \text{ and } x \geq h(z)\big\} \tag{59}$$

$$\text{and} \quad \mathcal{S}^2 = \big\{(x,z) \in \mathcal{S} \mid \ z \geq 1 - z_\dagger \text{ and } x \leq 1 - h(1-z)\big\}. \tag{60}$$

For $z \in]0, 1 - z_\dagger[$, the free-boundary point $h(z)$ is as in the previous case, namely, $F\big(h(z), z\big) = 0$, where F is defined by (58). Note that this satisfies the ODE

$$h'(z) = H\big(z, h(z)\big), \tag{61}$$

where

$$H\big(z, \bar{x}\big) = \frac{\alpha'(z)u'\big(\alpha(z)\big)}{u(\bar{x}) - u\big(\alpha(z)\big)} \frac{\int_0^{\bar{x}} \Psi(s)\, ds}{\Psi(\bar{x})}. \tag{62}$$

For $z \in \,]1 - z_\dagger, z_\dagger[$, we define $\ell(z) = 1 - h(1 - z)$. Using the value matching and the smooth pasting conditions along the free-boundary points $h(z)$ and $\ell(z)$, we derive the system of ODEs

$$h'(z) = H\big(z, \ell(z), h(z)\big) \quad \text{and} \quad \ell'(z) = L\big(z, \ell(z), h(z)\big), \tag{63}$$

where

$$H(z, \underline{x}, \bar{x}) = \frac{\alpha'(z) u'\big(\alpha(z)\big)}{u(\bar{x}) - u\big(\alpha(z)\big)} \frac{\varphi(\underline{x}) \int_{\underline{x}}^{\bar{x}} \Psi(s)\, ds - \psi(\underline{x}) \int_{\underline{x}}^{\bar{x}} \Phi(s)\, ds}{\varphi(\underline{x})\Psi(\bar{x}) - \psi(\underline{x})\Phi(\bar{x})} \tag{64}$$

and

$$L(z, \underline{x}, \bar{x}) = \frac{\alpha'(z) u'\big(1 - \alpha(z)\big)}{u(1 - \underline{x}) - u(1 - \alpha(z))} \frac{\psi(\bar{x}) \int_{\underline{x}}^{\bar{x}} \Phi(s)\, ds - \varphi(\bar{x}) \int_{\underline{x}}^{\bar{x}} \Psi(s)\, ds}{\psi(\bar{x})\Phi(\underline{x}) - \varphi(\bar{x})\Psi(\underline{x})}. \tag{65}$$

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