A Boolean-valued models approach to random convex analysis and duality theory of conditional risk measures

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Partially based on joint work with Antonio Avilés (University of Murcia).

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- Every single module or conditional analogue of a classical theorem needs an adaptation of a classical proof.

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Some preliminaries on Conditional Analysis

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In this case, we write

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Stable weak topologies:

$$\sigma_{\mathfrak{s}}(E, E^*), \quad \sigma_{\mathfrak{s}}(E^*, E).$$

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"We must ask whether there is any interest in these nonstandard models aside from the independence proof; that is, do they have any mathematical interest? The answer must be yes, but we cannot yet give a really good argument."

Dana Scott, 1969.

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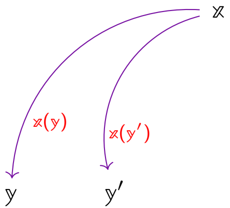

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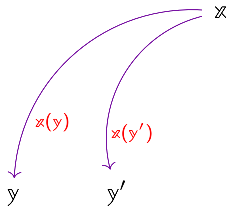

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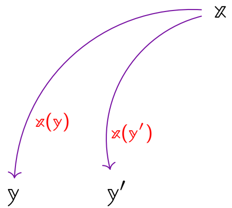

- If $\varphi(u_1, \ldots, u_n)$ is a logic formula (with $u_1, \ldots, u_n$ free variables) and $\mathbb{x}_1, \ldots, \mathbb{x}_n \in V^{(\mathcal{F})}$ we define the Boolean truth value $[\![\varphi(\mathbb{x}_1, \ldots, \mathbb{x}_n)]\!] \in \mathcal{F}$.
- A full set-theoretic reasoning is possible.

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This technique was first time applied to analysis by Gordon (1977) and Takeuti (1978) and has been fruitfully exploited by Kusraev, Kutateladze and Osawa, fulfilling the prediction of D. Scott.

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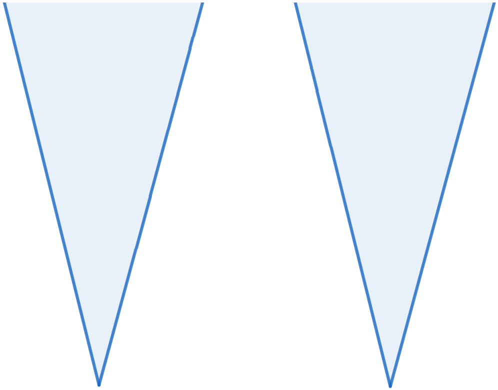

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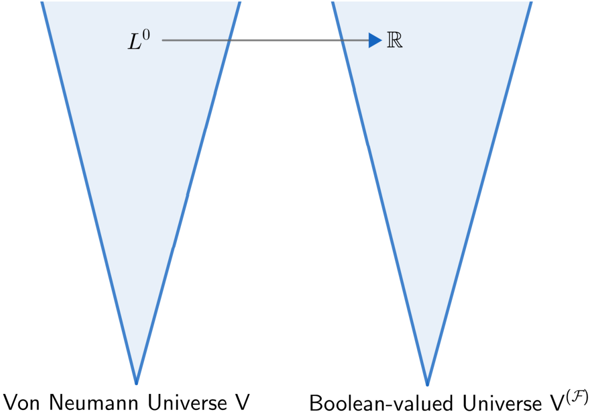

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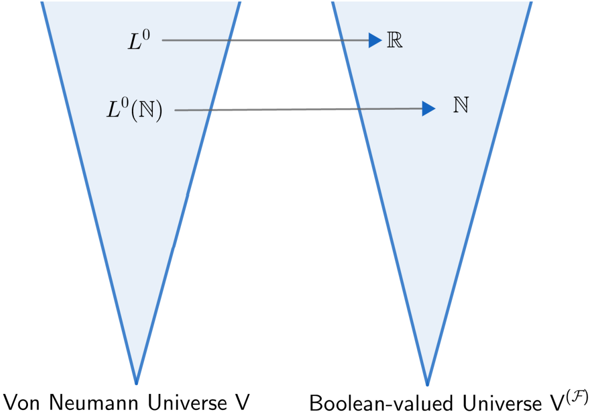

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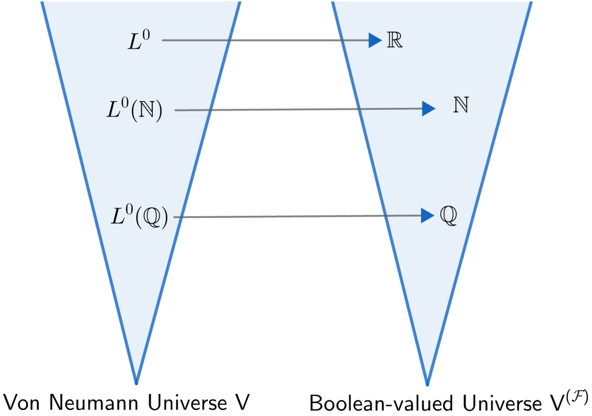

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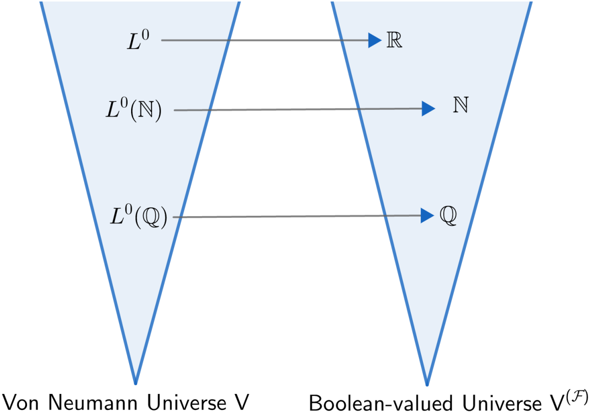

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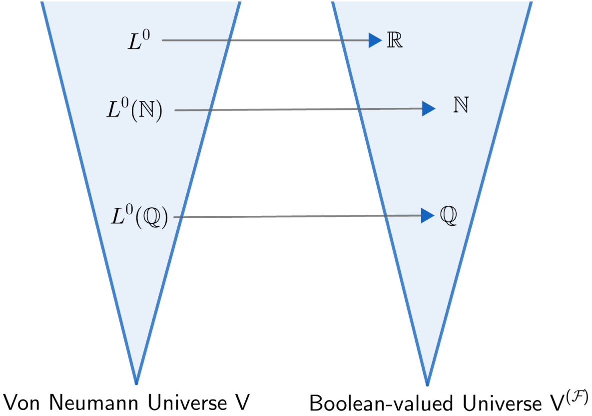

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If we manage to interpret a theorem on real numbers as a statement on L^0, we will have proved a new theorem on L^0.

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For any locally L^0-convex module $E := E[\mathscr{T}]$ there exists a locally convex space $E{\uparrow}$ within $V^{(\mathcal{F})}$ such that there is a bijection

$$\iota : E \longrightarrow \left\{ \mathbb{x} \in V^{(\mathcal{F})} \colon [\![\mathbb{x} \in E{\uparrow}]\!] = \Omega \right\}.$$

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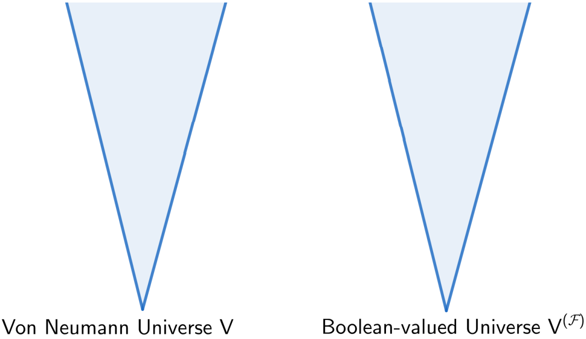

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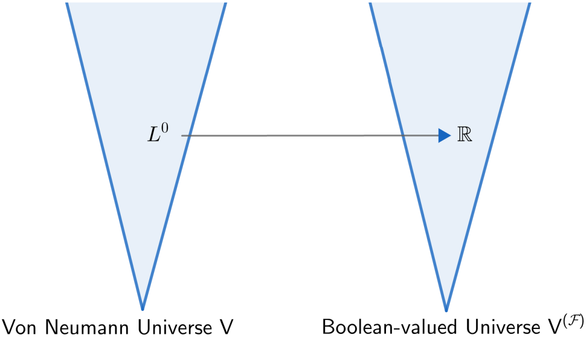

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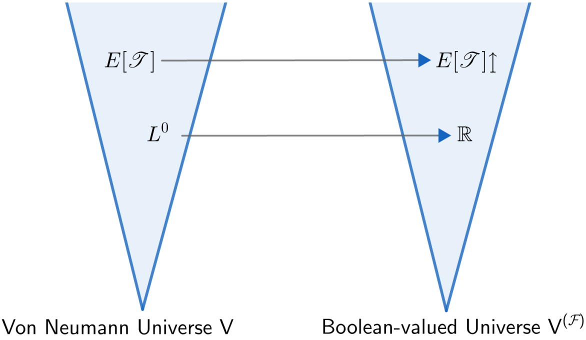

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- Cyclic compactness [A. Kusraev, 1982].
- Conditional compactness [S. Drapeau, A. Jamneshan, M. Karliczek, and M. Kupper, 2016].

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Let $E[\mathscr{T}]$ be a locally L^0-convex module and S an L^0-convex stably compact subset of E, then for any stable continuous function $f : S \to S$ there exists $x \in S$ such that $f(x) = x$.

A Boolean-valued approach to conditional risk

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$$\mathscr{X}^{\#} := \left\{y \in L^0(\mathcal{E}) \colon \mathbb{E}[|xy| \mid \mathcal{F}] < \infty \text{ for all } x \in \mathscr{X}\right\}.$$

- The Fenchel transform of ρ is defined to be

$$\rho^{\#}(y) := \operatorname{ess.sup}\{\mathbb{E}[xy|\mathcal{F}] - \rho(x) \colon x \in \mathscr{X}\} \quad \text{for } y \in \mathscr{X}^{\#}.$$

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- $\rho^{\#}$ is stably inf compact w.r.t. $\sigma_s(\mathcal{X}^{\#}, \mathcal{X})$ if:

  For each $\eta \in L^0(\mathcal{F})$, $\{\rho^{\#} \leq \eta\}$ is either empty or stably compact w.r.t. $\sigma_s(\mathcal{X}^{\#}, \mathcal{X})$.

Interpretation of a conditional risk measure as a convex risk measure

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5. *ρ has the Lebesgue property if and only if $[\![\rho\!\uparrow$ has the Lebesgue property$]\!] = \Omega$.*
6. *ρ is conditional law invariant if and only if $[\![\rho\!\uparrow$ is law invariant$]\!] = \Omega$.*

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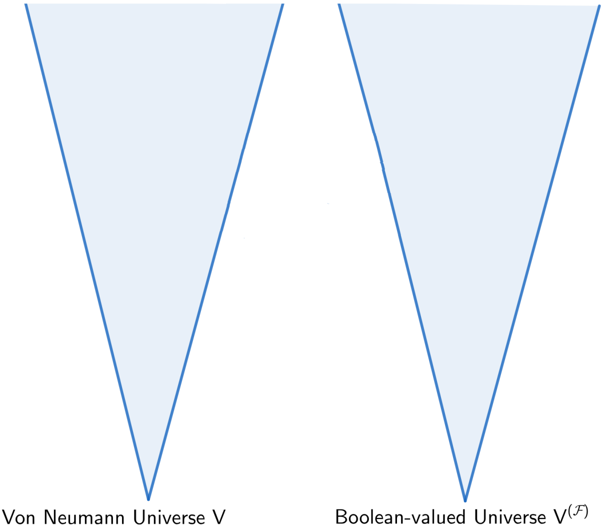

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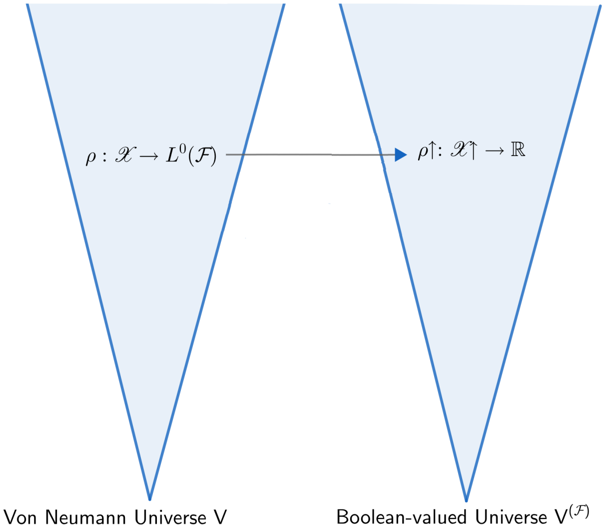

Robust representation of conditional risk measures

The following robust representation theorem was first time proved for $\mathcal{X} = L^\infty$ by [Jouini, Schachermayer, Touzi, 2006]:

Theorem (K. Owari, 2014)

Let $\rho : \mathcal{X} \to \mathbb{R}$ be a convex risk measure. Then ρ is lower semi-continuous w.r.t. $\sigma(\mathcal{X}, \mathcal{X}^\#)$ if and only if ρ admits a representation

$$\rho(x) = \sup\{\mathbb{E}[xy] - \rho^\#(y) \colon y \in \mathcal{X}^\#\} \quad \forall x \in \mathcal{X}.$$

In that case, the following conditions are equivalent:

1. *ρ attains the representation for each $x \in \mathcal{X}$;*
2. *ρ has the Lebesgue property, i.e.*

$$\lim_n x_n = x \text{ a.s.}, \ |x_n| \leq y, \ y \in \mathcal{X} \text{ implies } \lim_n \rho(x_n) = \rho(x);$$

3. *$\rho^\#$ is inf-compact w.r.t. $\sigma(\mathcal{X}^\#, \mathcal{X})$.*

Robust representation of conditional risk measures

Theorem

Let $\rho : \mathscr{X} \to L^0(\mathcal{F})$ be a conditional risk measure. Then ρ is stably lower semi-continuous w.r.t. $\sigma_{\mathfrak{s}}(\mathscr{X}, \mathscr{X}^{\#})$ if and only if ρ admits a representation

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$$\lim_n x_n = x \text{ a.s.}, |x_n| \leq y,\ y \in \mathscr{X} \text{ implies } \lim_n \rho(x_n) = \rho(x) \text{ a.s.};$$

3. *$\rho^{\#}$ is stably inf-compact w.r.t. $\sigma_{\mathfrak{s}}(\mathscr{X}^{\#}, \mathscr{X})$.*

References

References

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Thank you for your attention!