# Exponential functionals of PII and Mathematical Finance

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(Partially based on common work with P. Salminen)

# Motivations

Exponential functionals arise in many areas

- in self-similar Markov processes via Lamperti transform,
- in study of random processes in Random Environment,
- in mathematical statistics in the study of Pitman estimators,
- in Mathematical Finance for evaluation of perpetuities, for evaluation of Asian options, in the insurance for evaluation of ruin probability.

# Notations

For stochastic process $X = (X_t)_{t \geq 0}$ we introduce two exponential functionals

$$I_t = \int_0^t \exp(-X_s)ds,, \ \ t \geq 0$$

and also

$$I_\infty = \int_0^\infty \exp(-X_s)ds$$

when it exists.

# Previous results

When the process $X$ is Levy process, the properties of exponential functionals $I_\infty$ was studied by

- Bertoin, Yor (2005)
- Carmona, Petit, Yor (2004)
- Erikson, Maller (2004)
- Bertoin, Maller, Lindner (2008)
- Behme, Lindner (2015)
- Behme (2015)
- Kuznetsov, Prado, Savov (2012)
- Prado, Rivero, Van Schaik (2013)
- Patie, Savov (2016)

In the case of diffusions the exponential functionals see

- Salminen, Yor (2005)
- Kardaras, Robertson (2014)

# PII processes

Note that in the mathematical finance context $X$ plays a role of log of the price of the risky asset. This process is not always homogeneous.This is the explanation why it is important to generalize the results for PII processes.

As important examples we mention

- Compound Poisson process,
- Levy process subjected to deterministic time change,
- Integrals of Levy processes with deterministic integrands,
- Hitting times for diffusions.

Moreover, the time horizon is finite in real life, so that it is important to consider the exponential functionals as stochastic processes depending on $t$.

# Our aim

We suppose that the process $X = (X_t)_{t \geq 0}$ is a process with independent increments which is a semi-martingale with absolutely continuous characteristics $(B, C, \nu)$ such that

$$B_t = \int_0^t b_s ds, \quad C_t = \int_0^t c_s ds, \quad \nu(dt, dx) = K_t(dx)\, dt.$$

- In the first part of the talk we give such important characteristics of exponential functionals as moments, Laplace transform, Mellin transform.
- In the second part of the talk we present the integro-differential equations for the density of $I_t$ and $I_\infty$.
- As a special case, Levy processes are of course included.

# Impotrant observation

The process $(I_t)_{t\geq 0}$ is not a Markov process for the filtration generated by the process $X$ and this is an obstacle to use the stochastic calculus in efficient way. But if we do time reversal, we can introduce a family of Markov processes indexed by $t$ and then we can make the link between this family and the initial problem.

# Time reversion

For fixed $t > 0$ introduce a new process $Y^{(t)} = (Y_s^{(t)})_{0 \le s \le t}$ with

$$Y_s^{(t)} = X_t - X_{(t-s)-}$$

We will omit the index $(t)$ for simplicity of the notations.

**LEMMA 1** We have for all $t > 0$, that

$$I_t = e^{-Y_t} \int_0^t e^{Y_u} du$$

# Generalised OU processes

Consider generalized OU process

$$dV_t = V_{s-}\, d\hat{Y}_s + d\eta_s$$

where $(\hat{Y}_s)_{s\geq 0}$ and $(\eta_s)_{s\geq 0}$ are stochastic processes. Supposing that these two processes are independent and that the jumps of $\hat{Y}$ are bigger then -1, the solution of this equation is

$$V_t = \mathcal{E}(\hat{Y})_t \left( V_0 + \int_0^t \frac{d\eta_s}{\mathcal{E}(\hat{Y})_s} \right)$$

Observation :Put $\hat{Y}$ for the process such that $\mathcal{E}(\hat{Y})_t = \exp(-Y_t)$, $\eta_t = t$ and $V_0 = 0$ to see that the solution of this equation is exactly the process $V$ defined by time reversal.

# Mellin transform

For $\alpha > 0$ and $t \geq 0$ we introduce Mellin transform:

$$m_t^{(\alpha)} = \mathbf{E}(I_t^{\alpha})$$

and also Mellin transform for shifted process:

$$m_{s,t}^{(\alpha)} = \mathbf{E}\left[\left(\int_s^t e^{-(X_u - X_s)} du\right)^{\alpha}\right]$$

Under some integrability assumptions we introduce Laplace transform of parameter $\alpha > 0$ of $X_t$ : $\mathbf{E}(e^{-\alpha X_t}) = e^{-\Phi(\alpha,t)}$ with Laplace exponent equal to:

$$\Phi(t,\alpha) = \alpha B_t - \frac{\alpha^2}{2} C_t - \int_0^t \int_{\mathbb{R}} (e^{-\alpha x} - 1 + \alpha x)\, \nu(du, dx).$$

In the case of Lévy process, $\Phi(t,\alpha) = t\, \Phi(\alpha)$.

**THEOREM 1** Let $\alpha \geq 1$ and there exists $\delta > 0$ such that the following condition holds:

$$\int_0^t \int_{x<-1} e^{-(\alpha+\delta)x} K_s(dx)\, ds < \infty.$$

Then, $m_t^{(\alpha)}$ is well-defined and satisfy the following recurrent integral equation:

$$m_t^{(\alpha)} = \alpha \int_0^t m_{s,t}^{(\alpha-1)}\; e^{-\Phi(s,\alpha)} ds$$

If $X$ is Levy process, then for all $t > 0$

$$m_t^{(\alpha)} = \alpha e^{-\Phi(\alpha)\, t} \int_0^t m_s^{(\alpha-1)}\; e^{\Phi(\alpha)\, s} ds$$

**COROLLARY 1.1** Let $n \geq 1$ and $\Phi$ is bijective on $[0, n] \cap \mathbb{N}$. Then

$$\mathbf{E}(I_t^n) = n! \sum_{k=0}^{n} \frac{e^{-\Phi(k)t} - e^{-\Phi(n)t}}{\prod_{0 \leq i \leq n, i \neq k} (\Phi(i) - \Phi(k))}$$

**EXAMPLE** Let us consider Geometric Brownian Motion with drift $\mu > 0$ and diffusion coefficient $\sigma > 0$, i.e. $X_t = \mu t + \sigma W_t$ where $W = (W_t)_{t \geq 0}$ is standard Brownian motion. Then, $\Phi(\alpha) = \alpha\mu - \frac{\alpha^2 \sigma^2}{2}$ and under one of the conditions $\frac{\mu}{\sigma^2} < 1$ or $n < \frac{\mu}{\sigma^2}$ we get:

$$\mathbf{E}(I_t^n) = n! \sum_{k=0}^{n} \frac{e^{-(k\mu - k^2\sigma^2/2)t} - e^{-(n\mu - n^2\sigma^2/2)t}}{\prod\limits_{0 \leq i \leq n,\, i \neq k} (i-k)(\mu - (i+k)\sigma^2/2)}$$

**COROLLARY 1.2** Let $X$ be a Levy process such that for fixed $n \geq 1$, $\mathbf{E}(I_\infty^n) < \infty$. Then

$$\mathbf{E}(I_\infty^n) = \frac{n!}{\prod_{k=1}^n \Phi(k)}$$

Moreover, if all positive moments of $I_\infty$ exist and the above series is converging, then its Laplace transform of parameter $\beta > 0$ is given by:

$$\mathbf{E}(e^{-\beta I_\infty}) = \sum_{n=0}^{\infty} \frac{(-1)^n \beta^n}{\prod_{k=1}^n \Phi(k)}$$

**Remark :** Similar results was obtained by Bertoin, Yor (2005) for subordinators and under stronger integrability conditions.

**COROLLARY 1.3** Let $\alpha_0 = \inf\{\alpha > 0 \,|\, \Phi(\alpha) \leq 0\}$ with $\inf\{\emptyset\} = +\infty$. Then, $\mathbf{E}(I_\infty^\alpha) < \infty$ if and only if $1 \leq \alpha < \alpha_0$. In particular, in continuous case with $c_0 \neq 0$,

$$\Phi(\alpha) = \alpha b_0 - \frac{1}{2}\alpha^2 c_0$$

and the moment of $I_\infty$ of order $\alpha \geq 1$ will exist if $\alpha < \frac{2b_0}{c_0}$.
If $X$ be a subordinator with Levy measure which has a density $K_0$ w.r.t. Lebesgue measure, then under the condition

$$b_0 - \int_{\mathbb{R}^+} x K_0(dx) \geq 0,$$

all moments of $I_\infty$ exist.

**Remark :** Similar results as Theorem 1 and Corollaries 1.1, 1.2 and 1.3 are obtained for $\alpha < 0$.

# A family of processes $V$ indexed by $t$

Let $t > 0$ be fixed. We consider again the process $V = (V_s)_{0 \leq s \leq t}$ with

$$V_s = e^{-Y_s} \int_0^s e^{Y_u} du.$$

According to Lemma 1 $I_t = V_t$ for each $t \geq 0$. The process $V = (V_s)_{0 \leq s \leq t}$ is a Markov process w.r.t. to the natural filtration of the process $Y$, and with the infinitesimal generator $(\mathcal{A}_s^V)_{0 \leq s < t}$ given by

$$\mathcal{A}_s^V(f)(y) = (1 + y\, a_{t-s})\, f'(y) + \frac{1}{2} c_{t-s}\, f''(y)\, y^2 +$$

$$\int_{\mathbb{R}} \left[ f(ye^{-x}) - f(y) - f'(y) y (e^{-x} - 1) \right] K_{t-s}(dx)$$

where

$$a_{t-s} = -b_{t-s} + \frac{1}{2} c_{t-s} + \int_{\mathbb{R}} (e^{-x} - 1 + x) K_{t-s}(dx)$$

# Kolmogorov-type equation for the density of $I_t$

**THEOREM 2** For $0 \leq s \leq t$

$$\mathbf{E}(f(V_s)) = \int_0^s \mathbf{E}(\,\mathcal{A}_u^V(f)(V_u)\,)du$$

where $\mathcal{A}_t^V = \lim_{s \to t-} \mathcal{A}_s^V$.
If for $0 < s \leq t$ the density $p_s$ w.r.t. Lebesgue measure $\lambda$ of the law of $V_s$ exists and belongs to the class $\mathcal{C}^{1,2}(]0,t[\times \mathbb{R}^{+,*})$, then $\lambda$-a.s.

$$\frac{\partial}{\partial s} p_s(y) = \frac{1}{2} c_{t-s} \frac{\partial^2}{\partial y^2}(y^2\, p_s(y)) - \frac{\partial}{\partial y}((a_{t-s}\, y + 1)\, p_s(y)) +$$

$$\int_{\mathbb{R}} \left[ e^x p_s(ye^x) - p_s(y) + (e^{-x} - 1) \frac{\partial}{\partial y}(y p_s(y)) \right] K_{t-s}(dx)$$

In addition, the density of the law of $I_t$ is equal to the integral of the right-hand side of the previous expression w.r.t. $s$ on $[0, t[$.

# When $X$ is Levy process

let $X$ be Levy process with the parameters $(b_0, c_0, K_0)$ and

$$a_0 = -b_0 + \frac{1}{2} c_0 + \int_{\mathbb{R}} (e^{-x} - 1 + x) K_0(dx).$$

**PROPOSITION 2.1** Suppose that the density of the law of $I_t$ exists and belongs to the class $\mathcal{C}^{1,2}(]0,t[\times \mathbb{R}^{+,*})$. Then

$$\frac{\partial}{\partial t} p_t(y) = \frac{1}{2} c_0 \frac{\partial^2}{\partial y^2}(y^2\, p_t(y)) - \frac{\partial}{\partial y}((a_0 y + 1)\, p_t(y)) +$$

$$\int_{\mathbb{R}} \left[ e^x p_t(ye^x) - p_t(y) + (e^{-x} - 1) \frac{\partial}{\partial y}(y p_t(y)) \right] K_0(dx)$$

In particular case, when $I_\infty < \infty$ ($P$-a.s.) and the density $p_\infty$ of the law of $I_\infty$ exist and belongs to the class $\mathcal{C}^2(\mathbb{R}^{+,*})$ we have the same equation with l.h.s. replaced by 0.

**COROLLARY 2.1** Let us consider Brownian motion with drift:

$$dX_t = b_0 dt + \sqrt{c_0} dW_t$$

such that that $c_0 \neq 0$ and $b_0 > 0$. Then the law of exponential functional associated with $X$ has a density which verify :

$$\frac{\partial}{\partial t} p_t(y) = \frac{1}{2} c_0 \frac{\partial^2}{\partial y^2} (y^2\, p_t(y)) - \frac{\partial}{\partial y} ((a_0 y + 1)\, p_t(y))$$

with $a_0 = -b_0 + \frac{1}{2} c_0$. In particular, for $I_\infty$ (cf. Dufresne(2000), Borodin and Salminen(2002))

$$p_\infty(x) = \frac{1}{\Gamma(\frac{2\, b_0}{c_0})} \left(\frac{2}{c_0}\right)^{\frac{2b_0}{c_0}} x^{-(\frac{2b_0}{c_0}+1)} \exp\left(-\frac{2}{c_0 x}\right)$$

# Levy process with integrable jumps

Let us denote by $\nu^+$ and $\nu^-$ the Levy measure of positive and negative jumps respectively and put for $x > 0$

$$\nu^+(x) = \int_x^{+\infty} K_0(du), \;\; \nu^-(x) = \int_{-\infty}^{-x} K_0(du)$$

**COROLLARY 2.2** Suppose that $X$ is a Levy process with integrable jumps, Then the density $p_t$ of $I_t$, when it exists, verify :

$$\frac{\partial}{\partial t} p_t(y) = \frac{1}{2} c_0 \frac{\partial^2}{\partial y^2}(y^2\, p_t(y)) - \frac{\partial}{\partial y}((r_0 y + 1)\, p_t(y)) +$$

$$\int_y^{+\infty} p_t(z) \nu^+(\ln(\frac{z}{y}))\, dz - \int_0^y p_t(z) \nu^-(-\ln(\frac{z}{y}))\, dz$$

where $r_0 = -b_0 + \frac{1}{2}\, c_0 + \int_{\mathbb{R}} x\, K_0(dx)$

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