On the Singular Control of Exchange Rates

Tiziano Vargiolu
vargiolu@math.unipd.it
(joint with Giorgio Ferrari, University of Bielefeld)

Department of Mathematics
University of Padova
I-35131 Padova, Italy

Frontiers in Mathematical Finance
Luminy (France), September 3 - 7, 2018

Outline

Outline

Outline

Outline

Outline

Outline

The problem

- Consider the problem of a central bank that wants to manage the exchange rate between its domestic currency and a foreign one.
- The central bank can purchase and sell the foreign currency, and each intervention on the exchange market leads to a proportional cost whose instantaneous marginal value depends on the current level of the exchange rate.
- The central bank aims at minimizing the total expected costs of interventions on the exchange market, plus a total expected holding cost.

How to model this?

The problem

- Consider the problem of a central bank that wants to manage the exchange rate between its domestic currency and a foreign one.
- The central bank can purchase and sell the foreign currency, and each intervention on the exchange market leads to a proportional cost whose instantaneous marginal value depends on the current level of the exchange rate.
- The central bank aims at minimizing the total expected costs of interventions on the exchange market, plus a total expected holding cost.

How to model this?

The problem

- Consider the problem of a central bank that wants to manage the exchange rate between its domestic currency and a foreign one.
- The central bank can purchase and sell the foreign currency, and each intervention on the exchange market leads to a proportional cost whose instantaneous marginal value depends on the current level of the exchange rate.
- The central bank aims at minimizing the total expected costs of interventions on the exchange market, plus a total expected holding cost.

How to model this?

The problem

- Consider the problem of a central bank that wants to manage the exchange rate between its domestic currency and a foreign one.
- The central bank can purchase and sell the foreign currency, and each intervention on the exchange market leads to a proportional cost whose instantaneous marginal value depends on the current level of the exchange rate.
- The central bank aims at minimizing the total expected costs of interventions on the exchange market, plus a total expected holding cost.

How to model this?

What a central bank does: central parity

A central bank controls the exchange rate by buying/selling foreign currency reserves.

As a result, in many cases one can observe that the exchange rate between two currencies is either kept below/above a given level ("pegging"), or it is maintained within announced margins on either side of a given value, ("central parity" or "central rate"). Similar regimes of the exchange rate are usually referred to as *target zones*.

Examples: Switzerland, Hong Kong, Denmark.

What a central bank does: central parity

A central bank controls the exchange rate by buying/selling foreign currency reserves.

As a result, in many cases one can observe that the exchange rate between two currencies is either kept below/above a given level ("pegging"), or it is maintained within announced margins on either side of a given value, ("central parity" or "central rate").

Similar regimes of the exchange rate are usually referred to as *target zones*.

Examples: Switzerland, Hong Kong, Denmark.

What a central bank does: central parity

A central bank controls the exchange rate by buying/selling foreign currency reserves.

As a result, in many cases one can observe that the exchange rate between two currencies is either kept below/above a given level ("pegging"), or it is maintained within announced margins on either side of a given value, ("central parity" or "central rate").

Similar regimes of the exchange rate are usually referred to as *target zones*.

Examples: Switzerland, Hong Kong, Denmark.

What a central bank does: central parity

A central bank controls the exchange rate by buying/selling foreign currency reserves.

As a result, in many cases one can observe that the exchange rate between two currencies is either kept below/above a given level ("pegging"), or it is maintained within announced margins on either side of a given value, ("central parity" or "central rate").

Similar regimes of the exchange rate are usually referred to as *target zones*.

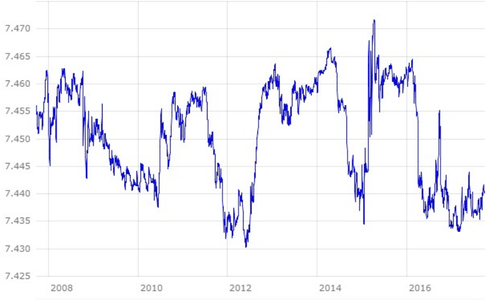
Examples: Switzerland, Hong Kong, Denmark.

The Danish case

January 12, 2017, marked the 30th anniversary of the Danish central parity (Mikkelsen 2017).

The decision to pursue a fixed exchange rate policy was made in the 1980s when the Danish economy was in a crisis: since then the Danish Krone (DKK) was anchored before to the German Mark and then, since 1999, to the Euro.

The central rate is 7.46038 Krone per Euro, and the Krone is allowed to increase or decrease by 2.25%.



The Danish case

January 12, 2017, marked the 30th anniversary of the Danish central parity (Mikkelsen 2017).

The decision to pursue a fixed exchange rate policy was made in the 1980s when the Danish economy was in a crisis: since then the Danish Krone (DKK) was anchored before to the German Mark and then, since 1999, to the Euro.

The central rate is 7.46038 Krone per Euro, and the Krone is allowed to increase or decrease by 2.25%.

The Danish case

January 12, 2017, marked the 30th anniversary of the Danish central parity (Mikkelsen 2017).
The decision to pursue a fixed exchange rate policy was made in the 1980s when the Danish economy was in a crisis: since then the Danish Krone (DKK) was anchored before to the German Mark and then, since 1999, to the Euro.
The central rate is 7.46038 Krone per Euro, and the Krone is allowed to increase or decrease by 2.25%.

The Swiss case

On the 6th of September 2011, the Swiss National Bank (SNB) stated in a press release (the New York Times):

[...] the current massive overvaluation of the Swiss Franc poses an acute threat to the Swiss economy and carries the risk of deflationary development. The Swiss National Bank is therefore aiming for a substantial and sustained weakening of the Swiss Franc. With immediate effect, it will no longer tolerate a EUR/CHF exchange rate below the minimum rate of CHF 1.20. The SNB will enforce this minimum rate with the utmost determination and is prepared to buy foreign currency in unlimited quantities [...]

The Swiss case

On the 6th of September 2011, the Swiss National Bank (SNB) stated in a press release (the New York Times):

[...] the current massive overvaluation of the Swiss Franc poses an acute threat to the Swiss economy and carries the risk of deflationary development. The Swiss National Bank is therefore aiming for a substantial and sustained weakening of the Swiss Franc. With immediate effect, it will no longer tolerate a EUR/CHF exchange rate below the minimum rate of CHF 1.20. The SNB will enforce this minimum rate with the utmost determination and is prepared to buy foreign currency in unlimited quantities [...]

Pegging EUR/CHF. . . and the end of it

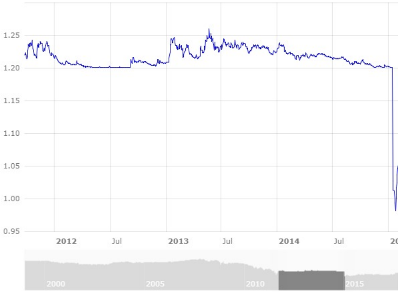


Figure: Plot EUR/CHF exchange rate from 2011 to 2015.

SNB adopted such an aggressive devaluation policy until the 15th of January 2015 (the Economist, Lloyd 2015), when SNB simply dropped its target zone policy with a very evident effect on the CHF/EUR exchange rate.

Pegging EUR/CHF. . . and the end of it

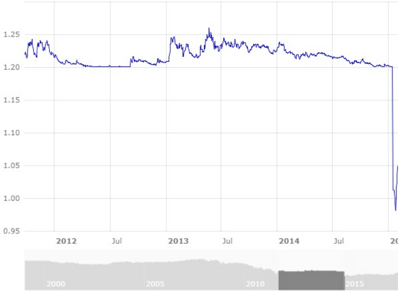


Figure: Plot EUR/CHF exchange rate from 2011 to 2015.

SNB adopted such an aggressive devaluation policy until the 15th of January 2015 (the Economist, Lloyd 2015), when SNB simply dropped its target zone policy with a very evident effect on the CHF/EUR exchange rate.

Two approaches in literature

It is not clear (nor of public knowledge) whether the width of the interval where the exchange rate is allowed to fluctuate is chosen according to some optimality criterion (e.g., maximization of social welfare or minimization of expected costs), or it is decided only on the basis of international and political agreements.

In the latest years the economic and mathematical literature experienced an intensive research on target zone models.

In particular, within the literature we can identify two main streams of research, which we could refer to "exogenous" or "endogenous" models.

Two approaches in literature

It is not clear (nor of public knowledge) whether the width of the interval where the exchange rate is allowed to fluctuate is chosen according to some optimality criterion (e.g., maximization of social welfare or minimization of expected costs), or it is decided only on the basis of international and political agreements.

In the latest years the economic and mathematical literature experienced an intensive research on target zone models.

In particular, within the literature we can identify two main streams of research, which we could refer to "exogenous" or "endogenous" models.

Two approaches in literature

It is not clear (nor of public knowledge) whether the width of the interval where the exchange rate is allowed to fluctuate is chosen according to some optimality criterion (e.g., maximization of social welfare or minimization of expected costs), or it is decided only on the basis of international and political agreements.
In the latest years the economic and mathematical literature experienced an intensive research on target zone models.
In particular, within the literature we can identify two main streams of research, which we could refer to "exogenous" or "endogenous" models.

"Exogenous" models: target zone

The pioneering paper is Krugman (1991), which assumes that the "fundamental" (NOT observed) exchange rate is a Brownian motion, instantaneously reflected at exogenously given upper and lower barriers.

This intrinsically defines a singular stochastic control problem (corresponding QVI also implicitly present in that paper), whose value function is the exchange rate really observed in the market.

Although many mathematical details are missing, the author finds that the observed exchange rate is mean-reverting inside the given target zone.

Following this, many refinements: Jørgensen-Mikkelsen (1996), De Jong *et al.* (2001), Larsen-Sørensen (2007), Bo *et al.* (2016), Yang *et al.* (2016), including ore general dynamics, reflecting or absorbing boundaries, regime switches, calibration to market data, etc.

"Exogenous" models: target zone

The pioneering paper is Krugman (1991), which assumes that the "fundamental" (NOT observed) exchange rate is a Brownian motion, instantaneously reflected at exogenously given upper and lower barriers.

This intrinsically defines a singular stochastic control problem (corresponding QVI also implicitly present in that paper), whose value function is the exchange rate really observed in the market.

Although many mathematical details are missing, the author finds that the observed exchange rate is mean-reverting inside the given target zone.

Following this, many refinements: Jørgensen-Mikkelsen (1996), De Jong *et al.* (2001), Larsen-Sørensen (2007), Bo *et al.* (2016), Yang *et al.* (2016), including ore general dynamics, reflecting or absorbing boundaries, regime switches, calibration to market data, etc.

"Exogenous" models: target zone

The pioneering paper is Krugman (1991), which assumes that the "fundamental" (NOT observed) exchange rate is a Brownian motion, instantaneously reflected at exogenously given upper and lower barriers.

This intrinsically defines a singular stochastic control problem (corresponding QVI also implicitly present in that paper), whose value function is the exchange rate really observed in the market.

Although many mathematical details are missing, the author finds that the observed exchange rate is mean-reverting inside the given target zone.

Following this, many refinements: Jørgensen-Mikkelsen (1996), De Jong *et al.* (2001), Larsen-Sørensen (2007), Bo *et al.* (2016), Yang *et al.* (2016), including ore general dynamics, reflecting or absorbing boundaries, regime switches, calibration to market data, etc.

"Exogenous" models: target zone

The pioneering paper is Krugman (1991), which assumes that the "fundamental" (NOT observed) exchange rate is a Brownian motion, instantaneously reflected at exogenously given upper and lower barriers.
This intrinsically defines a singular stochastic control problem (corresponding QVI also implicitly present in that paper), whose value function is the exchange rate really observed in the market.
Although many mathematical details are missing, the author finds that the observed exchange rate is mean-reverting inside the given target zone.
Following this, many refinements: Jørgensen-Mikkelsen (1996), De Jong *et al.* (2001), Larsen-Sørensen (2007), Bo *et al.* (2016), Yang *et al.* (2016), including ore general dynamics, reflecting or absorbing boundaries, regime switches, calibration to market data, etc.

"Endogenous" models: explicit optimal control

To endogenize the width of the target zone, several papers (Jeanblanc 1993, Mundaca-Øksendal 1998, Cadenillas-Zapatero 1999-2000, Bertola-Runngaldier-Yasuda 2016) formulate the exchange rates' optimal management problem as a stochastic optimal control problem.

In these papers, the central bank aims at adjusting the uncertain level of the exchange rate in order to minimize the spread between the instantaneous level of the exchange rate and a given central parity, by trading in the foreign currency, but at a cost.

In those papers such a cost has both a proportional and a fixed component, thus leading to a two-sided stochastic impulsive control problem. It is shown that the optimally controlled exchange rate is kept within endogenously determined levels and the interventions are of pure-jump type: at optimal times the exchange rate is pushed from a free boundary to another threshold level, also found endogenously.

"Endogenous" models: explicit optimal control

To endogenize the width of the target zone, several papers (Jeanblanc 1993, Mundaca-Øksendal 1998, Cadenillas-Zapatero 1999-2000, Bertola-Runngaldier-Yasuda 2016) formulate the exchange rates' optimal management problem as a stochastic optimal control problem.

In these papers, the central bank aims at adjusting the uncertain level of the exchange rate in order to minimize the spread between the instantaneous level of the exchange rate and a given central parity, by trading in the foreign currency, but at a cost.

In those papers such a cost has both a proportional and a fixed component, thus leading to a two-sided stochastic impulsive control problem. It is shown that the optimally controlled exchange rate is kept within endogenously determined levels and the interventions are of pure-jump type: at optimal times the exchange rate is pushed from a free boundary to another threshold level, also found endogenously.

"Endogenous" models: explicit optimal control

To endogenize the width of the target zone, several papers (Jeanblanc 1993, Mundaca-Øksendal 1998, Cadenillas-Zapatero 1999-2000, Bertola-Runngaldier-Yasuda 2016) formulate the exchange rates' optimal management problem as a stochastic optimal control problem.

In these papers, the central bank aims at adjusting the uncertain level of the exchange rate in order to minimize the spread between the instantaneous level of the exchange rate and a given central parity, by trading in the foreign currency, but at a cost.

In those papers such a cost has both a proportional and a fixed component, thus leading to a two-sided stochastic impulsive control problem. It is shown that the optimally controlled exchange rate is kept within endogenously determined levels and the interventions are of pure-jump type: at optimal times the exchange rate is pushed from a free boundary to another threshold level, also found endogenously.

"Endogenous" models: explicit optimal control

To endogenize the width of the target zone, several papers (Jeanblanc 1993, Mundaca-Øksendal 1998, Cadenillas-Zapatero 1999-2000, Bertola-Runngaldier-Yasuda 2016) formulate the exchange rates' optimal management problem as a stochastic optimal control problem.
In these papers, the central bank aims at adjusting the uncertain level of the exchange rate in order to minimize the spread between the instantaneous level of the exchange rate and a given central parity, by trading in the foreign currency, but at a cost.
In those papers such a cost has both a proportional and a fixed component, thus leading to a two-sided stochastic impulsive control problem. It is shown that the optimally controlled exchange rate is kept within endogenously determined levels and the interventions are of pure-jump type: at optimal times the exchange rate is pushed from a free boundary to another threshold level, also found endogenously.

Our approach

A closer look at the dynamics of the exchange rate EUR/CHF in the period 2011-2015 reveals NO jumps, but a *continuous reflection* of the exchange rate at the boundaries!

Such an observation suggests a *singular* stochastic control problem, rather than as an impulsive one.

In this paper we introduce an infinite time, one-dimensional singular stochastic control problem to model the exchange rates' optimal management problem. In our model, the log-exchange rate is a one-dimensional Itô-diffusion satisfying a linearly controlled SDE. Such general dynamics allows us to cover classical models where the exchange rate evolves as a geometric Brownian motion, as well as more realistic mean-reverting models.

The cumulative amount of purchases and sales of the foreign currency (which are the control variables of the central bank) are monotone processes, adapted to the underlying filtration, and satisfying proper integrability conditions.

Our approach

A closer look at the dynamics of the exchange rate EUR/CHF in the period 2011-2015 reveals NO jumps, but a *continuous reflection* of the exchange rate at the boundaries!
Such an observation suggests a *singular* stochastic control problem, rather than as an impulsive one.
In this paper we introduce an infinite time, one-dimensional singular stochastic control problem to model the exchange rates' optimal management problem. In our model, the log-exchange rate is a one-dimensional Itô-diffusion satisfying a linearly controlled SDE. Such general dynamics allows us to cover classical models where the exchange rate evolves as a geometric Brownian motion, as well as more realistic mean-reverting models.
The cumulative amount of purchases and sales of the foreign currency (which are the control variables of the central bank) are monotone processes, adapted to the underlying filtration, and satisfying proper integrability conditions.

Our approach

A closer look at the dynamics of the exchange rate EUR/CHF in the period 2011-2015 reveals NO jumps, but a *continuous reflection* of the exchange rate at the boundaries!

Such an observation suggests a *singular* stochastic control problem, rather than as an impulsive one.

In this paper we introduce an infinite time, one-dimensional singular stochastic control problem to model the exchange rates' optimal management problem. In our model, the log-exchange rate is a one-dimensional Itô-diffusion satisfying a linearly controlled SDE. Such general dynamics allows us to cover classical models where the exchange rate evolves as a geometric Brownian motion, as well as more realistic mean-reverting models.

The cumulative amount of purchases and sales of the foreign currency (which are the control variables of the central bank) are monotone processes, adapted to the underlying filtration, and satisfying proper integrability conditions.

Our approach

A closer look at the dynamics of the exchange rate EUR/CHF in the period 2011-2015 reveals NO jumps, but a *continuous reflection* of the exchange rate at the boundaries!

Such an observation suggests a *singular* stochastic control problem, rather than as an impulsive one.

In this paper we introduce an infinite time, one-dimensional singular stochastic control problem to model the exchange rates' optimal management problem. In our model, the log-exchange rate is a one-dimensional Itô-diffusion satisfying a linearly controlled SDE. Such general dynamics allows us to cover classical models where the exchange rate evolves as a geometric Brownian motion, as well as more realistic mean-reverting models.

The cumulative amount of purchases and sales of the foreign currency (which are the control variables of the central bank) are monotone processes, adapted to the underlying filtration, and satisfying proper integrability conditions.

Our contribution

The contribution of this paper is twofold.

- We contribute to the literature from the modeling point of view. By modelling the exchange rates' optimal management problem with a singular stochastic control problem, we are able to mimic the continuous reflection of the exchange rate at the target zone's boundaries which seems to happen in reality, by only using variables which are observables (i.e. the real exchange rates), not recurring to "fundamental" and unobservable ones.
- From the mathematical point of view, we provide the explicit solution to a bounded variation singular stochastic control in a very general setting with general one-dimensional diffusion as state variable, and with state-dependent instantaneous marginal costs of control. To the best of our knowledge, the explicit solution to a similar problem was not available in the literature yet (somehow Matomaki 2012, but we go beyond).

Our contribution

The contribution of this paper is twofold.

- We contribute to the literature from the modeling point of view. By modelling the exchange rates' optimal management problem with a singular stochastic control problem, we are able to mimic the continuous reflection of the exchange rate at the target zone's boundaries which seems to happen in reality, by only using variables which are observables (i.e. the real exchange rates), not recurring to "fundamental" and unobservable ones.
- From the mathematical point of view, we provide the explicit solution to a bounded variation singular stochastic control in a very general setting with general one-dimensional diffusion as state variable, and with state-dependent instantaneous marginal costs of control. To the best of our knowledge, the explicit solution to a similar problem was not available in the literature yet (somehow Matomaki 2012, but we go beyond).

Our contribution

The contribution of this paper is twofold.

- We contribute to the literature from the modeling point of view. By modelling the exchange rates' optimal management problem with a singular stochastic control problem, we are able to mimic the continuous reflection of the exchange rate at the target zone's boundaries which seems to happen in reality, by only using variables which are observables (i.e. the real exchange rates), not recurring to "fundamental" and unobservable ones.
- From the mathematical point of view, we provide the explicit solution to a bounded variation singular stochastic control in a very general setting with general one-dimensional diffusion as state variable, and with state-dependent instantaneous marginal costs of control. To the best of our knowledge, the explicit solution to a similar problem was not available in the literature yet (somehow Matomaki 2012, but we go beyond).

The Probabilistic Setting

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, B a one-dimensional Brownian motion, and denote by $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$ a right-continuous filtration to which B is adapted.

We then consider on $(\Omega, \mathcal{F}, \mathbb{P})$ a process X defined by

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t + d\xi_t - d\eta_t, \quad X_0 = x \in \mathcal{I}. \tag{1}$$

Here $\mathcal{I} := (\underline{x}, \overline{x})$, with $-\infty \leq \underline{x} < \overline{x} \leq +\infty$, and μ and σ are suitable drift and diffusion coefficients. The process X represents the (log-)exchange rate between two currencies. The central bank can adjust the level of X through the processes ξ and η, which are an indication of the cumulative amount of the foreign currency which has been bought or sold up to time $t \geq 0$ in order to push the level of the exchange rate up or down, respectively.

We assume that $\xi - \eta$ is the minimal decomposition of a suitable process of bounded variation, as follows.

The Probabilistic Setting

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, B a one-dimensional Brownian motion, and denote by $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ a right-continuous filtration to which B is adapted.
We then consider on $(\Omega, \mathcal{F}, \mathbb{P})$ a process X defined by

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t + d\xi_t - d\eta_t, \quad X_0 = x \in \mathcal{I}. \tag{1}$$

Here $\mathcal{I} := (\underline{x}, \overline{x})$, with $-\infty \leq \underline{x} < \overline{x} \leq +\infty$, and μ and σ are suitable drift and diffusion coefficients. The process X represents the (log-)exchange rate between two currencies. The central bank can adjust the level of X through the processes ξ and η, which are an indication of the cumulative amount of the foreign currency which has been bought or sold up to time $t \geq 0$ in order to push the level of the exchange rate up or down, respectively.
We assume that $\xi - \eta$ is the minimal decomposition of a suitable process of bounded variation, as follows.

The Probabilistic Setting

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, B a one-dimensional Brownian motion, and denote by $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$ a right-continuous filtration to which B is adapted.
We then consider on $(\Omega, \mathcal{F}, \mathbb{P})$ a process X defined by

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t + d\xi_t - d\eta_t, \quad X_0 = x \in \mathcal{I}. \tag{1}$$

Here $\mathcal{I} := (\underline{x}, \overline{x})$, with $-\infty \leq \underline{x} < \overline{x} \leq +\infty$, and μ and σ are suitable drift and diffusion coefficients. The process X represents the (log-)exchange rate between two currencies. The central bank can adjust the level of X through the processes ξ and η, which are an indication of the cumulative amount of the foreign currency which has been bought or sold up to time $t \geq 0$ in order to push the level of the exchange rate up or down, respectively.
We assume that $\xi - \eta$ is the minimal decomposition of a suitable process of bounded variation, as follows.

The Probabilistic Setting

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, B a one-dimensional Brownian motion, and denote by $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ a right-continuous filtration to which B is adapted.
We then consider on $(\Omega, \mathcal{F}, \mathbb{P})$ a process X defined by

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t + d\xi_t - d\eta_t, \quad X_0 = x \in \mathcal{I}. \tag{1}$$

Here $\mathcal{I} := (\underline{x}, \overline{x})$, with $-\infty \leq \underline{x} < \overline{x} \leq +\infty$, and μ and σ are suitable drift and diffusion coefficients. The process X represents the (log-)exchange rate between two currencies. The central bank can adjust the level of X through the processes ξ and η, which are an indication of the cumulative amount of the foreign currency which has been bought or sold up to time $t \geq 0$ in order to push the level of the exchange rate up or down, respectively.
We assume that $\xi - \eta$ is the minimal decomposition of a suitable process of bounded variation, as follows.

The Probabilistic Setting

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, B a one-dimensional Brownian motion, and denote by $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$ a right-continuous filtration to which B is adapted.
We then consider on $(\Omega, \mathcal{F}, \mathbb{P})$ a process X defined by

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t + d\xi_t - d\eta_t, \quad X_0 = x \in \mathcal{I}. \tag{1}$$

Here $\mathcal{I} := (\underline{x}, \overline{x})$, with $-\infty \leq \underline{x} < \overline{x} \leq +\infty$, and μ and σ are suitable drift and diffusion coefficients. The process X represents the (log-)exchange rate between two currencies. The central bank can adjust the level of X through the processes ξ and η, which are an indication of the cumulative amount of the foreign currency which has been bought or sold up to time $t \geq 0$ in order to push the level of the exchange rate up or down, respectively.
We assume that $\xi - \eta$ is the minimal decomposition of a suitable process of bounded variation, as follows.

Processes of bounded variation

We introduce the nonempty sets

$$\mathcal{S} := \{\nu : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+,\ \mathcal{F}\text{-adapted and s.t. } t \mapsto \nu_t \text{ is a.s.}$$
$$\text{(locally) of bounded variation, left-continuous and s.t. } \nu_0 = 0\},$$
$$\mathcal{U} := \{\vartheta : \vartheta \in \mathcal{S} \ \text{ and } t \mapsto \vartheta_t \text{ is nondecreasing}\}.$$

Then, for any $\nu \in \mathcal{S}$, we denote by $\xi, \eta \in \mathcal{U}$ the two processes providing the minimal decomposition of ν; that is, such that

$$\nu_t = \xi_t - \eta_t, \quad t \geq 0,$$

and the increments $\Delta\xi_t = \xi_{t+} - \xi_t$ and $\Delta\eta_t := \eta_{t+} - \eta_t$ are supported on disjoint subsets of $\mathbb{R}_+$.

For frequent future use, notice that any $\nu \in \mathcal{S}$ satisfies

$$\nu_t = \nu^c_t + \nu^j_t, \qquad t \geq 0.$$

Here ν^c is the continuous part of ν, and the jump part ν^j is such that $\nu^j_t := \sum_{0 \leq s < t} \Delta\nu_s$, where $\Delta\nu_t := \nu_{t+} - \nu_t$, $t \geq 0$.

Processes of bounded variation

We introduce the nonempty sets

$$\mathcal{S} := \{\nu : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+,\ \mathcal{F}\text{-adapted and s.t. } t \mapsto \nu_t \text{ is a.s. (locally) of bounded variation, left-continuous and s.t. } \nu_0 = 0\},$$
$$\mathcal{U} := \{\vartheta : \vartheta \in \mathcal{S} \text{ and } t \mapsto \vartheta_t \text{ is nondecreasing}\}.$$

Then, for any $\nu \in \mathcal{S}$, we denote by $\xi, \eta \in \mathcal{U}$ the two processes providing the minimal decomposition of ν; that is, such that

$$\nu_t = \xi_t - \eta_t, \quad t \geq 0,$$

and the increments $\Delta\xi_t = \xi_{t+} - \xi_t$ and $\Delta\eta_t := \eta_{t+} - \eta_t$ are supported on disjoint subsets of $\mathbb{R}_+$.

For frequent future use, notice that any $\nu \in \mathcal{S}$ satisfies

$$\nu_t = \nu^c_t + \nu^j_t, \qquad t \geq 0.$$

Here ν^c is the continuous part of ν, and the jump part ν^j is such that $\nu^j_t := \sum_{0 \leq s < t} \Delta\nu_s$, where $\Delta\nu_t := \nu_{t+} - \nu_t$, $t \geq 0$.

Processes of bounded variation

We introduce the nonempty sets

$$\mathcal{S} := \{\nu : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+,\ \mathcal{F}\text{-adapted and s.t. } t \mapsto \nu_t \text{ is a.s. (locally) of bounded variation, left-continuous and s.t. } \nu_0 = 0\},$$
$$\mathcal{U} := \{\vartheta : \vartheta \in \mathcal{S} \text{ and } t \mapsto \vartheta_t \text{ is nondecreasing}\}.$$

Then, for any $\nu \in \mathcal{S}$, we denote by $\xi, \eta \in \mathcal{U}$ the two processes providing the minimal decomposition of ν; that is, such that

$$\nu_t = \xi_t - \eta_t, \quad t \geq 0,$$

and the increments $\Delta\xi_t = \xi_{t+} - \xi_t$ and $\Delta\eta_t := \eta_{t+} - \eta_t$ are supported on disjoint subsets of $\mathbb{R}_+$.
For frequent future use, notice that any $\nu \in \mathcal{S}$ satisfies

$$\nu_t = \nu_t^c + \nu_t^j, \qquad t \geq 0.$$

Here ν^c is the continuous part of ν, and the jump part ν^j is such that $\nu_t^j := \sum_{0 \leq s < t} \Delta\nu_s$, where $\Delta\nu_t := \nu_{t+} - \nu_t$, $t \geq 0$.

Existence and uniqueness (up to exit)

The following assumption ensures that, for any $\nu \in \mathcal{S}$, there exists a unique strong solution to (1) (see Protter(1990), Theorem V.7).

Assumption 1. The coefficients $\mu : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to (0, \infty)$ belong to $C^1(\mathbb{R})$. Moreover, there exists $L > 0$ such that for all $x, y \in \mathcal{I}$,

$$|\mu(x) - \mu(y)| + |\sigma(x) - \sigma(y)| \leq L|x - y|.$$

From now on, in order to stress its dependence on the initial value $x \in \mathcal{I}$ and on the two processes ξ and η, we refer to the (left-continuous) solution to (1) as $X^{x;\xi,\eta}$, where appropriate. We also denote by

$$\sigma_{\mathcal{I}} := \inf\{t \geq 0 \mid X_t^{x;\xi,\eta} \notin \mathcal{I}\}$$

the first time when the controlled process $X_t^{x;\xi,\eta}$ leaves $\mathcal{I}$. Also, in the rest of the paper we use the notation $\mathbb{E}_x[f(X_t^{\xi,\eta})] = \mathbb{E}[f(X_t^{x,\xi,\eta})]$, where $\mathbb{E}_x$ is the expectation under the measure $\mathbb{P}_x(\,\cdot\,) := \mathbb{P}(\,\cdot\,|X_0^{\xi,\eta} = x)$.

Existence and uniqueness (up to exit)

The following assumption ensures that, for any $\nu \in \mathcal{S}$, there exists a unique strong solution to (1) (see Protter(1990), Theorem V.7).

Assumption 1. The coefficients $\mu : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to (0, \infty)$ belong to $C^1(\mathbb{R})$. Moreover, there exists $L > 0$ such that for all $x, y \in \mathcal{I}$,

$$|\mu(x) - \mu(y)| + |\sigma(x) - \sigma(y)| \leq L|x - y|.$$

From now on, in order to stress its dependence on the initial value $x \in \mathcal{I}$ and on the two processes ξ and η, we refer to the (left-continuous) solution to (1) as $X^{x;\xi,\eta}$, where appropriate. We also denote by

$$\sigma_{\mathcal{I}} := \inf\{t \geq 0 \mid X_t^{x;\xi,\eta} \notin \mathcal{I}\}$$

the first time when the controlled process $X_t^{x;\xi,\eta}$ leaves $\mathcal{I}$. Also, in the rest of the paper we use the notation $\mathbb{E}_x[f(X_t^{\xi,\eta})] = \mathbb{E}[f(X_t^{x,\xi,\eta})]$, where $\mathbb{E}_x$ is the expectation under the measure $\mathbb{P}_x(\,\cdot\,) := \mathbb{P}(\,\cdot\,|X_0^{\xi,\eta} = x)$.

Existence and uniqueness (up to exit)

The following assumption ensures that, for any $\nu \in \mathcal{S}$, there exists a unique strong solution to (1) (see Protter(1990), Theorem V.7).

Assumption 1. The coefficients $\mu : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to (0, \infty)$ belong to $C^1(\mathbb{R})$. Moreover, there exists $L > 0$ such that for all $x, y \in \mathcal{I}$,

$$|\mu(x) - \mu(y)| + |\sigma(x) - \sigma(y)| \leq L|x - y|.$$

From now on, in order to stress its dependence on the initial value $x \in \mathcal{I}$ and on the two processes ξ and η, we refer to the (left-continuous) solution to (1) as $X^{x;\xi,\eta}$, where appropriate.

We also denote by

$$\sigma_{\mathcal{I}} := \inf\{t \geq 0 \mid X_t^{x;\xi,\eta} \notin \mathcal{I}\}$$

the first time when the controlled process $X_t^{x;\xi,\eta}$ leaves $\mathcal{I}$.

Also, in the rest of the paper we use the notation $\mathbb{E}_x[f(X_t^{\xi,\eta})] = \mathbb{E}[f(X_t^{x,\xi,\eta})]$, where $\mathbb{E}_x$ is the expectation under the measure $\mathbb{P}_x(\,\cdot\,) := \mathbb{P}(\,\cdot\,|X_0^{\xi,\eta} = x)$.

Existence and uniqueness (up to exit)

The following assumption ensures that, for any $\nu \in \mathcal{S}$, there exists a unique strong solution to (1) (see Protter(1990), Theorem V.7).

Assumption 1. The coefficients $\mu : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to (0, \infty)$ belong to $C^1(\mathbb{R})$. Moreover, there exists $L > 0$ such that for all $x, y \in \mathcal{I}$,

$$|\mu(x) - \mu(y)| + |\sigma(x) - \sigma(y)| \leq L|x - y|.$$

From now on, in order to stress its dependence on the initial value $x \in \mathcal{I}$ and on the two processes ξ and η, we refer to the (left-continuous) solution to (1) as $X^{x;\xi,\eta}$, where appropriate.

We also denote by

$$\sigma_{\mathcal{I}} := \inf\{t \geq 0 \mid X_t^{x;\xi,\eta} \notin \mathcal{I}\}$$

the first time when the controlled process $X_t^{x;\xi,\eta}$ leaves $\mathcal{I}$.

Also, in the rest of the paper we use the notation $\mathbb{E}_x[f(X_t^{\xi,\eta})] = \mathbb{E}[f(X_t^{x,\xi,\eta})]$, where $\mathbb{E}_x$ is the expectation under the measure $\mathbb{P}_x(\cdot) := \mathbb{P}(\cdot \,|X_0^{\xi,\eta} = x)$.

Existence and uniqueness (up to exit)

The following assumption ensures that, for any $\nu \in \mathcal{S}$, there exists a unique strong solution to (1) (see Protter(1990), Theorem V.7).

Assumption 1. The coefficients $\mu : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to (0, \infty)$ belong to $C^1(\mathbb{R})$. Moreover, there exists $L > 0$ such that for all $x, y \in \mathcal{I}$,

$$|\mu(x) - \mu(y)| + |\sigma(x) - \sigma(y)| \leq L|x - y|.$$

From now on, in order to stress its dependence on the initial value $x \in \mathcal{I}$ and on the two processes ξ and η, we refer to the (left-continuous) solution to (1) as $X^{x;\xi,\eta}$, where appropriate.

We also denote by

$$\sigma_{\mathcal{I}} := \inf\{t \geq 0 \mid X_t^{x;\xi,\eta} \notin \mathcal{I}\}$$

the first time when the controlled process $X_t^{x;\xi,\eta}$ leaves $\mathcal{I}$.

Also, in the rest of the paper we use the notation $\mathbb{E}_x[f(X_t^{\xi,\eta})] = \mathbb{E}[f(X_t^{x,\xi,\eta})]$, where $\mathbb{E}_x$ is the expectation under the measure $\mathbb{P}_x(\,\cdot\,) := \mathbb{P}(\,\cdot\,|X_0^{\xi,\eta} = x)$.

The Optimal Control Problem

In this section we introduce the optimization problem faced by the central bank. The central bank can adjust the level of the exchange rate by purchasing or selling one of the two currencies (i.e. by properly exerting ξ and η), and we suppose that a policy of currency's (d)evaluation results into proportional costs, c_1 and c_2, that possibly depend on the current level of the exchange rate. Also, we assume that, being X_t the level of the (log-)exchange rate at time $t \geq 0$, the central bank faces an holding cost $h(X_t)$. The total expected cost associated to a central bank's policy $\nu \in \mathcal{S}$ is therefore

$$\begin{aligned}\mathcal{J}_x(\nu) \;:=\; \mathbb{E}_x\Big[\int_0^{\sigma_{\mathcal{I}}} e^{-rs} h(X_s^{\xi,\eta})\, ds + \\ + \int_0^{\sigma_{\mathcal{I}}} e^{-rs}\Big(c_1(X_s^{\xi,\eta}) \oplus d\xi_s + c_2(X_s^{\xi,\eta}) \ominus d\eta_s\Big)\Big].\end{aligned}$$

In the second line, we have pathwise integrals, but with left-continuous integrals and integrators.

The Optimal Control Problem

In this section we introduce the optimization problem faced by the central bank. The central bank can adjust the level of the exchange rate by purchasing or selling one of the two currencies (i.e. by properly exerting ξ and η), and we suppose that a policy of currency's (d)evaluation results into proportional costs, c_1 and c_2, that possibly depend on the current level of the exchange rate.

Also, we assume that, being X_t the level of the (log-)exchange rate at time $t \geq 0$, the central bank faces an holding cost $h(X_t)$.

The total expected cost associated to a central bank's policy $\nu \in \mathcal{S}$ is therefore

$$\begin{aligned} \mathcal{J}_x(\nu) \;:=\; \mathbb{E}_x\Big[& \int_0^{\sigma_{\mathcal{I}}} e^{-rs} h(X_s^{\xi,\eta})\, ds + \\ & + \int_0^{\sigma_{\mathcal{I}}} e^{-rs} \Big(c_1(X_s^{\xi,\eta}) \oplus d\xi_s + c_2(X_s^{\xi,\eta}) \ominus d\eta_s \Big) \Big]. \end{aligned}$$

In the second line, we have pathwise integrals, but with left-continuous integrals and integrators.

The Optimal Control Problem

In this section we introduce the optimization problem faced by the central bank. The central bank can adjust the level of the exchange rate by purchasing or selling one of the two currencies (i.e. by properly exerting ξ and η), and we suppose that a policy of currency's (d)evaluation results into proportional costs, c_1 and c_2, that possibly depend on the current level of the exchange rate. Also, we assume that, being X_t the level of the (log-)exchange rate at time $t \geq 0$, the central bank faces an holding cost $h(X_t)$.

The total expected cost associated to a central bank's policy $\nu \in \mathcal{S}$ is therefore

$$\begin{aligned} \mathcal{J}_x(\nu) := \mathbb{E}_x\Bigg[& \int_0^{\sigma_{\mathcal{I}}} e^{-rs} h(X_s^{\xi,\eta})\, ds + \\ & + \int_0^{\sigma_{\mathcal{I}}} e^{-rs}\Big(c_1(X_s^{\xi,\eta}) \oplus d\xi_s + c_2(X_s^{\xi,\eta}) \ominus d\eta_s\Big)\Bigg]. \end{aligned}$$

In the second line, we have pathwise integrals, but with left-continuous integrals and integrators.

The Optimal Control Problem

In this section we introduce the optimization problem faced by the central bank. The central bank can adjust the level of the exchange rate by purchasing or selling one of the two currencies (i.e. by properly exerting ξ and η), and we suppose that a policy of currency's (d)evaluation results into proportional costs, c_1 and c_2, that possibly depend on the current level of the exchange rate. Also, we assume that, being X_t the level of the (log-)exchange rate at time $t \geq 0$, the central bank faces an holding cost $h(X_t)$. The total expected cost associated to a central bank's policy $\nu \in \mathcal{S}$ is therefore

$$\begin{aligned}\mathcal{J}_x(\nu) \;&:=\; \mathbb{E}_x\bigg[\int_0^{\sigma_{\mathcal{I}}} e^{-rs} h(X_s^{\xi,\eta})\, ds + \\ &\quad + \int_0^{\sigma_{\mathcal{I}}} e^{-rs}\Big(c_1(X_s^{\xi,\eta}) \oplus d\xi_s + c_2(X_s^{\xi,\eta}) \ominus d\eta_s\Big)\bigg].\end{aligned}$$

In the second line, we have pathwise integrals, but with left-continuous integrals and integrators.

The Optimal Control Problem

In this section we introduce the optimization problem faced by the central bank. The central bank can adjust the level of the exchange rate by purchasing or selling one of the two currencies (i.e. by properly exerting ξ and η), and we suppose that a policy of currency's (d)evaluation results into proportional costs, c_1 and c_2, that possibly depend on the current level of the exchange rate. Also, we assume that, being X_t the level of the (log-)exchange rate at time $t \geq 0$, the central bank faces an holding cost $h(X_t)$.
The total expected cost associated to a central bank's policy $\nu \in \mathcal{S}$ is therefore

$$\begin{aligned} \mathcal{J}_x(\nu) \quad := \quad & \mathbb{E}_x\bigg[\int_0^{\sigma_\mathcal{I}} e^{-rs} h(X_s^{\xi,\eta})\ ds + \\ & + \int_0^{\sigma_\mathcal{I}} e^{-rs}\Big(c_1(X_s^{\xi,\eta}) \oplus d\xi_s + c_2(X_s^{\xi,\eta}) \ominus d\eta_s\Big)\bigg]. \end{aligned}$$

In the second line, we have pathwise integrals, but with left-continuous integrals and integrators.

Pathwise integrals (left-continuous integrals & integrators)

The definition of the costs of control with $\oplus$ and $\ominus$ has been introduced in Zhu (1992), and it is now common in the singular stochastic control literature:

$$\int_0^{\sigma_{\mathcal{I}}} e^{-rs} c_1(X_s^{x,\xi,\eta}) \oplus d\xi_s := \int_0^{\sigma_{\mathcal{I}}} e^{-rs} c_1(X_s^{x,\xi,\eta})\, d\xi_s^c + \sum_{s<\sigma_{\mathcal{I}}} e^{-rs} \int_0^{\Delta\xi_s} c_1(X_s^{\xi,\eta} + z)\, dz,$$

$$\int_0^{\sigma_{\mathcal{I}}} e^{-rs} c_2(X_s^{x,\xi,\eta}) \ominus d\eta_s := \int_0^{\sigma_{\mathcal{I}}} e^{-rs} c_2(X_s^{x,\xi,\eta})\, d\eta_s^c + \sum_{s<\sigma_{\mathcal{I}}} e^{-rs} \int_0^{\Delta\eta_s} c_2(X_s^{\xi,\eta} - z)\, dz,$$

where ξ^c and η^c denote the continuous parts of ξ and η, respectively.

Admissible controls

The following definition characterizes the class of admissible controls.

Definition 2. For any $x \in \mathcal{I}$ we say that $\nu \in \mathcal{S}$ is an admissible control, and we write $\nu \in \mathcal{A}(x)$, if $X_t^{x,\xi,\eta} \in \mathcal{I}$ for all $t > 0$ (i.e., $\sigma_{\mathcal{I}} = +\infty$ $\mathbb{P}_x$-a.s.) and the following hold true:

(a) $\mathbb{E}_x\left[\int_0^\infty e^{-rs}\left(|c_1(X_s^{\xi,\eta})| \oplus d\xi_s + |c_2(X_s^{\xi,\eta})| \ominus d\eta_s\right)\right] < +\infty;$

(b) $\mathbb{E}_x\left[\int_0^\infty e^{-rs} h(X_s^{\xi,\eta})\, ds\right] < +\infty;$

(c) $\mathbb{E}_x\left[\sup_{t\geq 0} e^{-\frac{r}{2}t}|X_t^{\xi,\eta}|^{1+\gamma}\right] < +\infty$ for γ such that

$$|c_i(x)| \leq K_i(1 + |x|^\gamma), \quad x \in \mathcal{I}.$$

for some K_i, $i = 1, 2$.

Admissible controls

The following definition characterizes the class of admissible controls.

Definition 2. For any $x \in \mathcal{I}$ we say that $\nu \in \mathcal{S}$ is an admissible control, and we write $\nu \in \mathcal{A}(x)$, if $X_t^{x,\xi,\eta} \in \mathcal{I}$ for all $t > 0$ (i.e., $\sigma_{\mathcal{I}} = +\infty$ $\mathbb{P}_x$-a.s.) and the following hold true:

(a) $\mathbb{E}_x\left[\int_0^{\infty} e^{-rs}\left(|c_1(X_s^{\xi,\eta})| \oplus d\xi_s + |c_2(X_s^{\xi,\eta})| \ominus d\eta_s\right)\right] < +\infty;$

(b) $\mathbb{E}_x\left[\int_0^{\infty} e^{-rs} h(X_s^{\xi,\eta})\, ds\right] < +\infty;$

(c) $\mathbb{E}_x\left[\sup_{t\geq 0} e^{-\frac{r}{2}t}|X_t^{\xi,\eta}|^{1+\gamma}\right] < +\infty$ for γ such that

$$|c_i(x)| \leq K_i(1+|x|^{\gamma}), \quad x \in \mathcal{I}.$$

for some K_i, $i = 1, 2$.

Admissible controls

The following definition characterizes the class of admissible controls.

Definition 2. For any $x \in \mathcal{I}$ we say that $\nu \in \mathcal{S}$ is an admissible control, and we write $\nu \in \mathcal{A}(x)$, if $X_t^{x,\xi,\eta} \in \mathcal{I}$ for all $t > 0$ (i.e., $\sigma_{\mathcal{I}} = +\infty$ $\mathbb{P}_x$-a.s.) and the following hold true:

(a) $$\mathbb{E}_x\left[\int_0^\infty e^{-rs}\left(|c_1(X_s^{\xi,\eta})| \oplus d\xi_s + |c_2(X_s^{\xi,\eta})| \ominus d\eta_s\right)\right] < +\infty;$$

(b) $$\mathbb{E}_x\left[\int_0^\infty e^{-rs} h(X_s^{\xi,\eta})\, ds\right] < +\infty;$$

(c) $$\mathbb{E}_x\left[\sup_{t\geq 0} e^{-\frac{r}{2}t}|X_t^{\xi,\eta}|^{1+\gamma}\right] < +\infty$$ for γ such that

$$|c_i(x)| \leq K_i(1+|x|^\gamma), \quad x \in \mathcal{I}.$$

for some K_i, $i = 1, 2$.

Admissible controls

The following definition characterizes the class of admissible controls.

Definition 2. For any $x \in \mathcal{I}$ we say that $\nu \in \mathcal{S}$ is an admissible control, and we write $\nu \in \mathcal{A}(x)$, if $X_t^{x,\xi,\eta} \in \mathcal{I}$ for all $t > 0$ (i.e., $\sigma_{\mathcal{I}} = +\infty$ $\mathbb{P}_x$-a.s.) and the following hold true:

(a) $\mathbb{E}_x \Big[\int_0^\infty e^{-rs} \Big(|c_1(X_s^{\xi,\eta})| \oplus d\xi_s + |c_2(X_s^{\xi,\eta})| \ominus d\eta_s \Big) \Big] < +\infty;$

(b) $\mathbb{E}_x \Big[\int_0^\infty e^{-rs} h(X_s^{\xi,\eta})\, ds \Big] < +\infty;$

(c) $\mathbb{E}_x \Big[\sup_{t\geq 0} e^{-\frac{r}{2}t} |X_t^{\xi,\eta}|^{1+\gamma} \Big] < +\infty$ for γ such that

$$|c_i(x)| \leq K_i(1+|x|^\gamma), \quad x \in \mathcal{I}.$$

for some K_i, $i = 1, 2$.

Admissible controls

The following definition characterizes the class of admissible controls.

Definition 2. For any $x \in \mathcal{I}$ we say that $\nu \in \mathcal{S}$ is an admissible control, and we write $\nu \in \mathcal{A}(x)$, if $X_t^{x,\xi,\eta} \in \mathcal{I}$ for all $t > 0$ (i.e., $\sigma_{\mathcal{I}} = +\infty$ $\mathbb{P}_x$-a.s.) and the following hold true:

(a) $\mathbb{E}_x\left[\int_0^\infty e^{-rs}\left(|c_1(X_s^{\xi,\eta})| \oplus d\xi_s + |c_2(X_s^{\xi,\eta})| \ominus d\eta_s\right)\right] < +\infty;$

(b) $\mathbb{E}_x\left[\int_0^\infty e^{-rs} h(X_s^{\xi,\eta})\, ds\right] < +\infty;$

(c) $\mathbb{E}_x\left[\sup_{t\geq 0} e^{-\frac{r}{2}t}|X_t^{\xi,\eta}|^{1+\gamma}\right] < +\infty$ for γ such that

$$|c_i(x)| \leq K_i(1 + |x|^\gamma), \quad x \in \mathcal{I}.$$

for some K_i, $i = 1, 2$.

The optimal control problem

The central bank aims at picking an admissible $\nu^\star$ such that the total expected cost functional

$$\mathcal{J}_x(\nu) := \mathbb{E}_x\Big[\int_0^{\sigma_\mathcal{I}} e^{-rs} h(X_s^{\xi,\eta})\, ds + \int_0^{\sigma_\mathcal{I}} e^{-rs}\Big(c_1(X_s^{\xi,\eta}) \oplus d\xi_s + c_2(X_s^{\xi,\eta}) \ominus d\eta_s\Big)\Big].$$

is minimized; that is, it aims at solving

$$v(x) := \inf_{\nu \in \mathcal{A}(x)} \mathcal{J}_x(\nu), \qquad x \in \mathcal{I}. \tag{2}$$

Problem (2) takes the form of a singular stochastic control problem (see, e.g., Shreve (1988) for an introduction); that is, a problem where the (random) measure on $\mathbb{R}_+$ induced by a control process might be singular with respect to the Lebesgue measure.

The optimal control problem

The central bank aims at picking an admissible $\nu^{\star}$ such that the total expected cost functional

$$\mathcal{J}_x(\nu) := \mathbb{E}_x\bigg[\int_0^{\sigma_{\mathcal{I}}} e^{-rs} h(X_s^{\xi,\eta})\, ds + \\ + \int_0^{\sigma_{\mathcal{I}}} e^{-rs}\Big(c_1(X_s^{\xi,\eta}) \oplus d\xi_s + c_2(X_s^{\xi,\eta}) \ominus d\eta_s\Big)\bigg].$$

is minimized; that is, it aims at solving

$$v(x) := \inf_{\nu\in\mathcal{A}(x)} \mathcal{J}_x(\nu), \qquad x \in \mathcal{I}. \tag{2}$$

Problem (2) takes the form of a singular stochastic control problem (see, e.g., Shreve (1988) for an introduction); that is, a problem where the (random) measure on $\mathbb{R}_+$ induced by a control process might be singular with respect to the Lebesgue measure.

The optimal control problem

The central bank aims at picking an admissible $\nu^{\star}$ such that the total expected cost functional

$$\mathcal{J}_x(\nu) := \mathbb{E}_x\bigg[\int_0^{\sigma_{\mathcal{I}}} e^{-rs} h(X_s^{\xi,\eta})\, ds + \\ + \int_0^{\sigma_{\mathcal{I}}} e^{-rs}\Big(c_1(X_s^{\xi,\eta}) \oplus d\xi_s + c_2(X_s^{\xi,\eta}) \ominus d\eta_s\Big)\bigg].$$

is minimized; that is, it aims at solving

$$v(x) := \inf_{\nu \in \mathcal{A}(x)} \mathcal{J}_x(\nu), \qquad x \in \mathcal{I}. \tag{2}$$

Problem (2) takes the form of a singular stochastic control problem (see, e.g., Shreve (1988) for an introduction); that is, a problem where the (random) measure on $\mathbb{R}_+$ induced by a control process might be singular with respect to the Lebesgue measure.

The verification theorem

i) Suppose that Assumption 1 holds true and assume that the Hamilton-Jacobi-Bellman equation

$$\min\Big\{(\mathcal{L}_X - r)u(x) + h(x), c_2(x) - u'(x), u'(x) + c_1(x)\Big\} = 0, \quad x \in \mathcal{I}.$$

with

$$(\mathcal{L}_X f)(x) := \frac{1}{2}\sigma^2(x)f''(x) + \mu(x)f'(x), \quad f \in C^2(\overline{\mathcal{I}}), x \in \mathcal{I},$$

admits a C^2 solution $u : \mathcal{I} \to \mathbb{R}$ such that

$$|u(x)| \leq K(1 + |x|^{1+\gamma}), \quad x \in \mathcal{I},$$

for some $K > 0$. Then one has that $u \leq v$ on $\mathcal{I}$.

The verification theorem - ii

ii) Suppose also that there exists $\widehat{\nu} = \widehat{\xi} - \widehat{\eta} \in \mathcal{A}(x)$ such that

$$X_t^{x,\widehat{\xi},\widehat{\eta}} \in \Big\{ x \in \mathcal{I} : \ (\mathcal{L}_X - r) u(x) + h(x) = 0 \Big\}, \qquad (3)$$

Lebesgue-a.e. $\mathbb{P}$-a.s., the process

$$\left(\int_0^t e^{-rs} \sigma(X_s^{x;\widehat{\xi},\widehat{\eta}}) u'(X_s^{x;\widehat{\xi},\widehat{\eta}}) \ dB_s \right)_{t \geq 0} \quad \text{is an } \mathbb{F}\text{-martingale}, \qquad (4)$$

and

$$\begin{cases} \displaystyle\int_0^T \left(u'(X_t^{x,\widehat{\xi},\widehat{\eta}}) + c_1(X_t^{x,\widehat{\xi},\widehat{\eta}}) \right) \oplus d\widehat{\xi}_t = 0, \\ \displaystyle\int_0^T \left(c_2(X_t^{x,\widehat{\xi},\widehat{\eta}}) - u'(X_t^{x,\widehat{\xi},\widehat{\eta}}) \right) \ominus d\widehat{\eta}_t = 0, \end{cases} \qquad (5)$$

for all $T \geq 0$ $\mathbb{P}$-a.s. Then $u = v$ on $\mathcal{I}$ and $\widehat{\nu}$ is optimal for (2).

Proof (sketch - i)

Step 1. Let $x \in \mathcal{I}$ and $\nu \in \mathcal{A}(x)$. Since $u \in C^2(\mathcal{I})$ we can apply Itô-Meyer's formula for semimartingales to $(e^{-rt}u(X_t^{x,\xi,\eta}))_{t\geq 0}$ on an arbitrary time interval $[0, T]$, $T > 0$. Then

$$
\begin{aligned}
u(x) = {} & e^{-rT}u(X_T^{x;\xi,\eta}) - \int_0^T e^{-rs}(\mathcal{L}_X - r)u(X_s^{x;\xi,\eta})\, ds - M_T^{x;\xi,\eta} \\
& - \int_0^T e^{-rs}u'(X_s^{x;\xi,\eta})\, d\xi_s^c + \int_0^T e^{-rs}u'(X_s^{x;\xi,\eta})\, d\eta_s^c \qquad (6) \\
& - \sum_{0\leq s<T} e^{-rs}\Big(u(X_{s+}^{x;\xi,\eta}) - u(X_s^{x;\xi,\eta})\Big),
\end{aligned}
$$

where

$$
M_T^{x;\xi,\eta} := \int_0^T e^{-rs}\sigma(X_s^{x;\xi,\eta})u'(X_s^{x;\xi,\eta})\, dB_s.
$$

is a martingale.

Proof (sketch - i)

Since the processes ξ and η jump on disjoint subsets of $\mathbb{R}_+$ we can write

$$\sum_{0\leq s<T} e^{-rs}(u(X_{s+}) - u(X_s)) =$$

$$= \sum_{0\leq s<T} e^{-rs}\left[\int_0^{\Delta\xi_s} u'(X_s^{x;\xi,\eta} + z)\, dz - \int_0^{\Delta\eta_s} u'(X_s^{x;\xi,\eta} - z)\, dz\right],$$

and because $(\mathcal{L}_X - r)u \geq -h$ and $-c_1 \leq u' \leq c_2$ on $\mathcal{I}$ by the HJB equation, we end up with

$$u(x) \leq e^{-rT}u(X_T^{x;\xi,\eta}) + \int_0^T e^{-rs}h(X_s^{x;\xi,\eta})\, ds - M_T^{x;\xi,\eta}$$
$$+ \int_0^T e^{-rs}c_1(X_s^{x;\xi,\eta}) \oplus d\xi_s + \int_0^T e^{-rs}c_2(X_s^{x;\xi,\eta}) \ominus d\eta_s,$$

Proof (sketch - i)

By assumption, for all $x \in \mathcal{I}$ one has $|u(x)| \leq K(1+|x|^{\gamma+1})$, and therefore we can write for some $K>0$ that

$$u(x) \leq e^{-\frac{r}{2}T} K\Big(1+\sup_{t\geq 0} e^{-\frac{r}{2}t}|X_t^{x;\xi,\eta}|^{\gamma+1}\Big) + \int_0^T e^{-rs} h(X_s^{x;\xi,\eta})\, ds - M_T^{x;\xi,\eta}$$
$$+ \int_0^T e^{-rs} c_1(X_s^{x;\xi,\eta}) \oplus d\xi_s + \int_0^T e^{-rs} c_2(X_s^{x;\xi,\eta}) \ominus d\eta_s.$$

From the previous equation we have that, for all $T>0$,

$$M_T^{x;\xi,\eta} \leq -u(x) + K\Big(1+\sup_{t\geq 0} e^{-\frac{r}{2}t}|X_t^{x;\xi,\eta}|^{\gamma+1}\Big) +$$
$$+ \int_0^\infty e^{-rs}|c_1(X_s^{x;\xi,\eta})| \oplus d\xi_s + \int_0^\infty e^{-rs}|c_2(X_s^{x;\xi,\eta})| \ominus d\eta_s$$

so that $M_T^{x;\xi,\eta} \in L^1(\mathbb{P})$ by admissibility of ν; hence, $(M_T^{x;\xi,\eta})_{T\geq 0}$ is a submartingale.

Proof (sketch - i)

Then, taking expectations we have

$$u(x) \leq e^{-\frac{r}{2}T}\mathbb{E}_x\Big[K\Big(1 + \sup_{t\geq 0} e^{-\frac{r}{2}t}|X_t^{x;\xi,\eta}|^{\gamma+1}\Big)\Big]$$
$$+\mathbb{E}_x\Big[\int_0^T e^{-rs}h(X_s^{x;\xi,\eta})\,ds + \int_0^T e^{-rs}\Big(c_1(X_s^{x;\xi,\eta}) \oplus d\xi_s + c_2(X_s^{x;\xi,\eta})$$

Taking limits as $T \uparrow +\infty$, and using the fact that ν is admissible, by the dominated convergence theorem we get $u(x) \leq \mathcal{J}_x(\nu)$. Since the latter holds for any $x \in \mathcal{I}$ and $\nu \in \mathcal{A}(x)$ we conclude that $u \leq v$ on $\mathcal{I}$.

Proof (sketch - ii)

Step 2. Let again $x \in \mathcal{I}$ be given and fixed, and take the admissible $\widehat{\nu}$ satisfying (3), (4) and (5). Then all the inequalities leading to (**??**) become equalities, and taking expectations we obtain

$$u(x) = \mathbb{E}_x\bigg[e^{-rT}u(X_T^{x;\widehat{\xi},\widehat{\eta}}) + \int_0^T e^{-rs}h(X_s^{x;\widehat{\xi},\widehat{\eta}})\ ds + \\ + \int_0^T e^{-rs}c_1(X_s^{x;\widehat{\xi},\widehat{\eta}}) \oplus d\widehat{\xi}_s + \int_0^T e^{-rs}c_2(X_s^{x;\widehat{\xi},\widehat{\eta}}) \ominus d\widehat{\eta}_s\bigg].$$

By assumption we have that $u(x) \geq -K(1+|x|^{1+\gamma})$, so that

$$u(x) \geq -\ e^{-\frac{r}{2}T}\mathbb{E}_x\Big[K\Big(1 + \sup_{t\geq 0} e^{-\frac{r}{2}t}|X_t^{x;\widehat{\xi},\widehat{\eta}}|^{\gamma+1}\Big)\Big] + \mathbb{E}_x\bigg[\int_0^T e^{-rs}h(X_s^{x;\widehat{\xi},\widehat{\eta}}) \\ + \int_0^T e^{-rs}c_1(X_s^{x;\widehat{\xi},\widehat{\eta}}) \oplus d\widehat{\xi}_s + \int_0^T e^{-rs}c_2(X_s^{x;\widehat{\xi},\widehat{\eta}}) \ominus d\widehat{\eta}_s\bigg].$$

Proof (sketch - ii)

By admissibility of $\widehat{\nu}$ we can take limits as $T \uparrow \infty$, invoke the dominated convergence theorem for the second expectation in the right hand-side of (**??**), and finally find that

$$\begin{aligned} u(x) \;\; &\geq \;\; \mathbb{E}_x \Big[\int_0^{\infty} e^{-rs} h(X_s^{x;\widehat{\xi},\widehat{\eta}}) \, ds + \\ &+ \int_0^{\infty} e^{-rs} c_1(X_s^{x;\widehat{\xi},\widehat{\eta}}) \oplus d\widehat{\xi}_s + \int_0^{\infty} e^{-rs} c_2(X_s^{x;\widehat{\xi},\widehat{\eta}}) \ominus d\widehat{\eta}_s \Big]. \end{aligned}$$

Hence $u(x) \geq \mathcal{J}_x(\widehat{\nu}) \geq v(x)$. Combining this inequality with the fact that $u \leq v$ on $\mathcal{I}$ by *Step 1*, we conclude that $u = v$ on $\mathcal{I}$ and that $\widehat{\nu}$ is optimal.

Construction of the solution

Since we have a one-dimensional state variable, we can "explicitly" build a solution to the HJB quasi-variational inequality with the general theory of ODEs and one-dimensional diffusions.

The usual interpretation of the HJB (quasi)-variational inequality in this kind of problems (i.e., singular, but also impulsive and stopping) is that the max operator divides the domain into two regions:

- the **continuation region** $\mathcal{C}$ (open), where the controls are not exercised:

$$\mathcal{C} \subset \left\{ x \in \mathcal{I} : \left(\mathcal{L}_X - r\right) u(x) + h(x) = 0 \right\}$$

- the **exercising region** $\mathcal{C}^c$ (closed), where active controls are used:

$$\mathcal{C}^c \supset \left\{ x \in \mathcal{I} : \left(\mathcal{L}_X - r\right) u(x) + h(x) < 0 \right\}$$

What have we got in our case?

Construction of the solution

Since we have a one-dimensional state variable, we can "explicitly" build a solution to the HJB quasi-variational inequality with the general theory of ODEs and one-dimensional diffusions.
The usual interpretation of the HJB (quasi)-variational inequality in this kind of problems (i.e., singular, but also impulsive and stopping) is that the max operator divides the domain into two regions:

- the **continuation region** $\mathcal{C}$ (open), where the controls are not exercised:

$$\mathcal{C} \subset \left\{ x \in \mathcal{I} : \ (\mathcal{L}_X - r)u(x) + h(x) = 0 \right\}$$

- the **exercising region** $\mathcal{C}^c$ (closed), where active controls are used:

$$\mathcal{C}^c \supset \left\{ x \in \mathcal{I} : \ (\mathcal{L}_X - r)u(x) + h(x) < 0 \right\}$$

What have we got in our case?

Construction of the solution

Since we have a one-dimensional state variable, we can "explicitly" build a solution to the HJB quasi-variational inequality with the general theory of ODEs and one-dimensional diffusions.
The usual interpretation of the HJB (quasi)-variational inequality in this kind of problems (i.e., singular, but also impulsive and stopping) is that the max operator divides the domain into two regions:

- the **continuation region** $\mathcal{C}$ (open), where the controls are not exercised:

$$\mathcal{C} \subset \Big\{x \in \mathcal{I} : \; (\mathcal{L}_X - r)u(x) + h(x) = 0\Big\}$$

- the **exercising region** $\mathcal{C}^c$ (closed), where active controls are used:

$$\mathcal{C}^c \supset \Big\{x \in \mathcal{I} : \; (\mathcal{L}_X - r)u(x) + h(x) < 0\Big\}$$

What have we got in our case?

Construction of the solution

Since we have a one-dimensional state variable, we can "explicitly" build a solution to the HJB quasi-variational inequality with the general theory of ODEs and one-dimensional diffusions.
The usual interpretation of the HJB (quasi)-variational inequality in this kind of problems (i.e., singular, but also impulsive and stopping) is that the max operator divides the domain into two regions:

- the **continuation region** $\mathcal{C}$ (open), where the controls are not exercised:

$$\mathcal{C} \subset \Big\{ x \in \mathcal{I} : \; \big(\mathcal{L}_X - r\big) u(x) + h(x) = 0 \Big\}$$

- the **exercising region** $\mathcal{C}^c$ (closed), where active controls are used:

$$\mathcal{C}^c \supset \Big\{ x \in \mathcal{I} : \; \big(\mathcal{L}_X - r\big) u(x) + h(x) < 0 \Big\}$$

What have we got in our case?

Construction of the solution

Since we have a one-dimensional state variable, we can "explicitly" build a solution to the HJB quasi-variational inequality with the general theory of ODEs and one-dimensional diffusions.
The usual interpretation of the HJB (quasi)-variational inequality in this kind of problems (i.e., singular, but also impulsive and stopping) is that the max operator divides the domain into two regions:

- the **continuation region** $\mathcal{C}$ (open), where the controls are not exercised:

$$\mathcal{C} \subset \Big\{x \in \mathcal{I} : \; \big(\mathcal{L}_X - r\big)u(x) + h(x) = 0\Big\}$$

- the **exercising region** $\mathcal{C}^c$ (closed), where active controls are used:

$$\mathcal{C}^c \supset \Big\{x \in \mathcal{I} : \; \big(\mathcal{L}_X - r\big)u(x) + h(x) < 0\Big\}$$

What have we got in our case?

Construction of the solution - more concrete

Assume that in our case the continuation region is an interval (a, b) such that

$$\left\{x \in \mathcal{I} : \left(\mathcal{L}_X - r\right) u(x) + h(x) = 0\right\} = [a, b]$$

and that the exercising region divides in the two half-lines

$$\{x \in \mathcal{I} : u'(x) = -c_1(x)\} = (\underline{x}, a],$$

$$\{x \in \mathcal{I} : u'(x) = c_2(x)\} = [b, \overline{x}).$$

Thus, u is the solution of a second-order ODE in the continuation region, of very simple first-order ODEs in the exercising region, and is C^2 everywhere, including a and b $\rightsquigarrow$ algorithm!

Construction of the solution - more concrete

Assume that in our case the continuation region is an interval (a, b) such that

$$\Big\{x \in \mathcal{I} : \big(\mathcal{L}_X - r\big)u(x) + h(x) = 0\Big\} = [a, b]$$

and that the exercising region divides in the two half-lines

$$\{x \in \mathcal{I} : u'(x) = -c_1(x)\} = (\underline{x}, a],$$

$$\{x \in \mathcal{I} : u'(x) = c_2(x)\} = [b, \overline{x}).$$

Thus, u is the solution of a second-order ODE in the continuation region, of very simple first-order ODEs in the exercising region, and is C^2 everywhere, including a and b $\rightsquigarrow$ algorithm!

Construction of the solution - more concrete

Assume that in our case the continuation region is an interval (a, b) such that

$$\Big\{x \in \mathcal{I} : \big(\mathcal{L}_X - r\big)u(x) + h(x) = 0\Big\} = [a, b]$$

and that the exercising region divides in the two half-lines

$$\big\{x \in \mathcal{I} : u'(x) = -c_1(x)\big\} = (\underline{x}, a],$$

$$\big\{x \in \mathcal{I} : u'(x) = c_2(x)\big\} = [b, \overline{x}).$$

Thus, u is the solution of a second-order ODE in the continuation region, of very simple first-order ODEs in the exercising region, and is C^2 everywhere, including a and b $\rightsquigarrow$ algorithm!

Construction of the solution - more concrete

Assume that in our case the continuation region is an interval (a, b) such that

$$\Big\{x \in \mathcal{I} : \big(\mathcal{L}_X - r\big)u(x) + h(x) = 0\Big\} = [a, b]$$

and that the exercising region divides in the two half-lines

$$\{x \in \mathcal{I} : u'(x) = -c_1(x)\} = (\underline{x}, a],$$

$$\{x \in \mathcal{I} : u'(x) = c_2(x)\} = [b, \overline{x}).$$

Thus, u is the solution of a second-order ODE in the continuation region, of very simple first-order ODEs in the exercising region, and is C^2 everywhere, including a and b $\rightsquigarrow$ algorithm!

Solution in the continuation region

From the general theory of ODEs, the non-homogeneous equation

$$\big(\mathcal{L}_X - r\big)u(x) + h(x) = 0$$

admits a two-dimensional affine space of solutions.

More in detail, one can describe this space by the use of the so-called **fundamental solutions** ψ and φ of the homogeneous ODE

$$\big(\mathcal{L}_X - r\big)u(x) = 0$$

such that ψ is strictly increasing and φ is strictly decreasing: Then we can represent the general solution of the non-homogeneous ODE in (a, b) as

$$u(x) = A\psi(x) + B\varphi(x) + (Rh)(x), \quad x \in (a, b),$$

where $(Rh)(x)$ is "any particular" solution of the non-homogeneous ODE.

Solution in the continuation region

From the general theory of ODEs, the non-homogeneous equation

$$\left(\mathcal{L}_X - r\right)u(x) + h(x) = 0$$

admits a two-dimensional affine space of solutions.
More in detail, one can describe this space by the use of the so-called **fundamental solutions** ψ and φ of the homogeneous ODE

$$\left(\mathcal{L}_X - r\right)u(x) = 0$$

such that ψ is strictly increasing and φ is strictly decreasing:
Then we can represent the general solution of the non-homogeneous ODE in (a, b) as

$$u(x) = A\psi(x) + B\varphi(x) + (Rh)(x), \quad x \in (a, b),$$

where $(Rh)(x)$ is "any particular" solution of the non-homogeneous ODE.

Solution in the continuation region

From the general theory of ODEs, the non-homogeneous equation

$$\big(\mathcal{L}_X - r\big)u(x) + h(x) = 0$$

admits a two-dimensional affine space of solutions.
More in detail, one can describe this space by the use of the so-called **fundamental solutions** ψ and φ of the homogeneous ODE

$$\big(\mathcal{L}_X - r\big)u(x) = 0$$

such that ψ is strictly increasing and φ is strictly decreasing:
Then we can represent the general solution of the non-homogeneous ODE in (a, b) as

$$u(x) = A\psi(x) + B\varphi(x) + (Rh)(x), \quad x \in (a, b),$$

where $(Rh)(x)$ is "any particular" solution of the non-homogeneous ODE.

Solution in the continuation region - more explicit

As the particular solution of the non-homogeneous ODE, we choose the resolvent operator R computed on h, defined as

$$(Rf)(x) := \mathbb{E}_x\left[\int_0^\infty e^{-rs} f(X_s^{0,0})\, ds\right], \quad x \in \mathcal{I},$$

Then the solution of the HJB equation on (a, b) is determined by

$$u(x) = A\psi(x) + B\varphi(x) + (Rh)(x), \quad x \in (a, b),$$

where everything is known up to the four real constants $a < b$, A and B.

To find them, use the C^2 conditions in the points a and b!

Solution in the continuation region - more explicit

As the particular solution of the non-homogeneous ODE, we choose the resolvent operator R computed on h, defined as

$$(Rf)(x) := \mathbb{E}_x \left[\int_0^\infty e^{-rs} f(X_s^{0,0}) \, ds \right], \quad x \in \mathcal{I},$$

Then the solution of the HJB equation on (a, b) is determined by

$$u(x) = A\psi(x) + B\varphi(x) + (Rh)(x), \quad x \in (a, b),$$

where everything is known up to the four real constants $a < b$, A and B.

To find them, use the C^2 conditions in the points a and b!

Solution in the continuation region - more explicit

As the particular solution of the non-homogeneous ODE, we choose the resolvent operator R computed on h, defined as

$$(Rf)(x) := \mathbb{E}_x \left[\int_0^\infty e^{-rs} f(X_s^{0,0}) \, ds \right], \quad x \in \mathcal{I},$$

Then the solution of the HJB equation on (a, b) is determined by

$$u(x) = A\psi(x) + B\varphi(x) + (Rh)(x), \quad x \in (a, b),$$

where everything is known up to the four real constants $a < b$, A and B.
To find them, use the C^2 conditions in the points a and b!

Solution in the exercising region

Luckily, constructing a solution in the exercising region is easier: just take

$$u(x) = u(a) + \int_x^a c_1(y)dy = A\psi(a) + B\varphi(a) + (Rh)(a) + \int_x^a c_1(y)dy$$

for $x \in (\underline{x}, a]$, and

$$u(x) = u(b) + \int_b^x c_2(y)dy = A\psi(b) + B\varphi(b) + (Rh)(b) + \int_b^x c_2(y)dy$$

for $x \in [b, \overline{x})$. Notice that in this way the function u is automatically continuous at a and b.

If we now impose C^2 in a and b, we have four conditions over the four free parameters $a < b$, A and B.

Solution in the exercising region

Luckily, constructing a solution in the exercising region is easier: just take

$$u(x) = u(a) + \int_x^a c_1(y)dy = A\psi(a) + B\varphi(a) + (Rh)(a) + \int_x^a c_1(y)dy$$

for $x \in (\underline{x}, a]$, and

$$u(x) = u(b) + \int_b^x c_2(y)dy = A\psi(b) + B\varphi(b) + (Rh)(b) + \int_b^x c_2(y)dy$$

for $x \in [b, \overline{x})$. Notice that in this way the function u is automatically continuous at a and b.

If we now impose C^2 in a and b, we have four conditions over the four free parameters $a < b$, A and B.

Solution in the exercising region

Luckily, constructing a solution in the exercising region is easier: just take

$$u(x) = u(a) + \int_x^a c_1(y)dy = A\psi(a) + B\varphi(a) + (Rh)(a) + \int_x^a c_1(y)dy$$

for $x \in (\underline{x}, a]$, and

$$u(x) = u(b) + \int_b^x c_2(y)dy = A\psi(b) + B\varphi(b) + (Rh)(b) + \int_b^x c_2(y)dy$$

for $x \in [b, \overline{x})$. Notice that in this way the function u is automatically continuous at a and b.
If we now impose C^2 in a and b, we have four conditions over the four free parameters $a < b$, A and B.

The system

Imposing $u \in C^2$ in a and b, we have

$$A\psi'(a) + B\varphi'(a) + (Rh)'(a) = (\widehat{R}\,\widehat{c}_1)(a), \tag{7}$$

$$A\psi''(a) + B\varphi''(a) + (Rh)''(a) = (\widehat{R}\,\widehat{c}_1)'(a), \tag{8}$$

$$A\psi'(b) + B\varphi'(b) + (Rh)'(b) = -(\widehat{R}\,\widehat{c}_2)(b), \tag{9}$$

$$A\psi''(b) + B\varphi''(b) + (Rh)''(b) = -(\widehat{R}\,\widehat{c}_2)'(b). \tag{10}$$

which is linear in A and B, but heavily non-linear in a and b (which, by the way, are essential elements of the optimal strategy!) However, we can **prove** that a unique solution (a, b, A, B) exists, with $a < b$!

The system

Imposing $u \in C^2$ in a and b, we have

$$A\psi'(a) + B\varphi'(a) + (Rh)'(a) = (\widehat{R}\,\widehat{c}_1)(a), \tag{7}$$

$$A\psi''(a) + B\varphi''(a) + (Rh)''(a) = (\widehat{R}\,\widehat{c}_1)'(a), \tag{8}$$

$$A\psi'(b) + B\varphi'(b) + (Rh)'(b) = -(\widehat{R}\,\widehat{c}_2)(b), \tag{9}$$

$$A\psi''(b) + B\varphi''(b) + (Rh)''(b) = -(\widehat{R}\,\widehat{c}_2)'(b). \tag{10}$$

which is linear in A and B, but heavily non-linear in a and b (which, by the way, are essential elements of the optimal strategy!)
However, we can **prove** that a unique solution (a, b, A, B) exists, with $a < b$!

The system

Imposing $u \in C^2$ in a and b, we have

$$A\psi'(a) + B\varphi'(a) + (Rh)'(a) = (\widehat{R}\,\widehat{c}_1)(a), \tag{7}$$

$$A\psi''(a) + B\varphi''(a) + (Rh)''(a) = (\widehat{R}\,\widehat{c}_1)'(a), \tag{8}$$

$$A\psi'(b) + B\varphi'(b) + (Rh)'(b) = -(\widehat{R}\,\widehat{c}_2)(b), \tag{9}$$

$$A\psi''(b) + B\varphi''(b) + (Rh)''(b) = -(\widehat{R}\,\widehat{c}_2)'(b). \tag{10}$$

which is linear in A and B, but heavily non-linear in a and b (which, by the way, are essential elements of the optimal strategy!) However, we can **prove** that a unique solution (a, b, A, B) exists, with $a < b$!

Solving the nonlinear system

The first step is to solve the first two equation in A and B, and to do the same for the last two. We thus obtain two expressions for both A and B, i.e.

$$A = I_1(a) = I_2(b), \qquad B = J_1(a) = J_2(b)$$

where I_i and J_i are suitable functions (of scale and speed functions of the uncontrolled process $X^{0,0}$, and of another suitable optimal stopping game).

Thus we are lead to solve

$$I_1(a) - I_2(b) = 0, \qquad J_1(a) - J_2(b) = 0$$

which is "easier" (i.e., two nonlinear equations in two unknowns).

Solving the nonlinear system

The first step is to solve the first two equation in A and B, and to do the same for the last two. We thus obtain two expressions for both A and B, i.e.

$$A = I_1(a) = I_2(b), \qquad B = J_1(a) = J_2(b)$$

where I_i and J_i are suitable functions (of scale and speed functions of the uncontrolled process $X^{0,0}$, and of another suitable optimal stopping game).
Thus we are lead to solve

$$I_1(a) - I_2(b) = 0, \qquad J_1(a) - J_2(b) = 0$$

which is "easier" (i.e., two nonlinear equations in two unknowns).

Solving the reduced system

Proposition. Assume that $c_i \in C^2$ with polynomial growth and $c_1 + c_2 > 0$, and that there exist $\widetilde{x}_1$ and $\widetilde{x}_2$ such that $\underline{x} < \widetilde{x}_1 < \widetilde{x}_2 < \overline{x}$ and

$$-(\mathcal{L}_X - (r - \mu'))c_1(x) + h'(x) \begin{cases} < 0, & x < \widetilde{x}_1, \\ = 0, & x = \widetilde{x}_1, \\ > 0, & x > \widetilde{x}_1, \end{cases}$$

$$(\mathcal{L}_X - (r - \mu'))c_2(x) + h'(x) \begin{cases} < 0, & x < \widetilde{x}_2, \\ = 0, & x = \widetilde{x}_2, \\ > 0, & x > \widetilde{x}_2, \end{cases}$$

Then there exists a unique couple $(a^*, b^*) \in \mathcal{I} \times \mathcal{I}$, such that $a^* < \widetilde{x}_1 < \widetilde{x}_2 < b^*$, that solves

$$I_1(a) - I_2(b) = 0, \qquad J_1(a) - J_2(b) = 0$$

Solving the reduced system

Proposition. Assume that $c_i \in C^2$ with polynomial growth and $c_1 + c_2 > 0$, and that there exist $\widetilde{x}_1$ and $\widetilde{x}_2$ such that $\underline{x} < \widetilde{x}_1 < \widetilde{x}_2 < \overline{x}$ and

$$-(\mathcal{L}_X - (r-\mu'))c_1(x) + h'(x) \begin{cases} < 0, & x < \widetilde{x}_1, \\ = 0, & x = \widetilde{x}_1, \\ > 0, & x > \widetilde{x}_1, \end{cases}$$

$$(\mathcal{L}_X - (r-\mu'))c_2(x) + h'(x) \begin{cases} < 0, & x < \widetilde{x}_2, \\ = 0, & x = \widetilde{x}_2, \\ > 0, & x > \widetilde{x}_2, \end{cases}$$

Then there exists a unique couple $(a^*, b^*) \in \mathcal{I} \times \mathcal{I}$, such that $a^* < \widetilde{x}_1 < \widetilde{x}_2 < b^*$, that solves

$$I_1(a) - I_2(b) = 0, \qquad J_1(a) - J_2(b) = 0$$

Closing the circle

Theorem. The function u, defined as

$$u(x) := \begin{cases} A\psi(a^*) + B\varphi(a^*) + (Rh)(a^*) + \int_x^{a^*} c_1(y)\, dy, & x \in (\underline{x}, a^*], \\ A\psi(x) + B\varphi(x) + (Rh)(x), & x \in (a^*, b^*), \\ A\psi(b^*) + B\varphi(b^*) + (Rh)(b^*) + \int_{b^*}^{x} c_2(y)\, dy, & x \in [b^*, \overline{x}). \end{cases}$$

is a classical solution to the HJB equation.
Moreover, there exists $K > 0$ such that $|u(x)| \leq K(1 + |x|^{\gamma+1})$,
where $\gamma \geq 1$ is the growth coefficient of c_i, $i = 1, 2$.
Thus, $u \leq v$ after the verification theorem - i).

Closing the circle

Theorem. The function u, defined as

$$u(x) := \begin{cases} A\psi(a^*) + B\varphi(a^*) + (Rh)(a^*) + \int_x^{a^*} c_1(y)\ dy, & x \in (\underline{x}, a^*], \\ A\psi(x) + B\varphi(x) + (Rh)(x), & x \in (a^*, b^*), \\ A\psi(b^*) + B\varphi(b^*) + (Rh)(b^*) + \int_{b^*}^{x} c_2(y)\ dy, & x \in [b^*, \overline{x}). \end{cases}$$

is a classical solution to the HJB equation.
Moreover, there exists $K > 0$ such that $|u(x)| \leq K\left(1 + |x|^{\gamma+1}\right)$,
where $\gamma \geq 1$ is the growth coefficient of c_i, $i = 1, 2$.
Thus, $u \leq v$ after the verification theorem - i).

Closing the circle

Theorem. The function u, defined as

$$u(x) := \begin{cases} A\psi(a^*) + B\varphi(a^*) + (Rh)(a^*) + \int_x^{a^*} c_1(y)\ dy, & x \in (\underline{x}, a^*], \\ A\psi(x) + B\varphi(x) + (Rh)(x), & x \in (a^*, b^*), \\ A\psi(b^*) + B\varphi(b^*) + (Rh)(b^*) + \int_{b^*}^{x} c_2(y)\ dy, & x \in [b^*, \overline{x}). \end{cases}$$

is a classical solution to the HJB equation.
Moreover, there exists $K > 0$ such that $|u(x)| \leq K\left(1 + |x|^{\gamma+1}\right)$,
where $\gamma \geq 1$ is the growth coefficient of c_i, $i = 1, 2$.
Thus, $u \leq v$ after the verification theorem - i).

The optimal strategies

Let $(\xi^\star, \eta^\star)$ be the couple of nondecreasing processes that solves the following double Skorokhod reflection problem:
find $(\xi, \eta) \in \mathcal{U} \times \mathcal{U}$ s.t.

$$\begin{cases} X_t^{x,\xi,\eta} \in [a^*, b^*], \mathbb{P}\text{-a.s. for } t > 0, \\ \int_0^T \mathbf{1}_{\{X_t^{x,\xi,\eta} > a^*\}} d\xi_t = 0, \mathbb{P}\text{-a.s. for any } T > 0, \\ \int_0^T \mathbf{1}_{\{X_t^{x,\xi,\eta} < b^*\}} d\eta_t = 0, \mathbb{P}\text{-a.s. for any } T > 0. \end{cases} \tag{11}$$

Under our assumptions, this problem admits a unique pathwise solution $(\xi_t^\star, \eta_t^\star)$, continuous apart possible jumps at time zero of amplitude $(a^* - x)^+$ and $(x - b^*)^+$, respectively.
Proposition. The process $\nu^\star := \xi^\star - \eta^\star$ is an admissible control. Thus, $u = v$ by the verification theorem - ii).

The optimal strategies

Let $(\xi^\star, \eta^\star)$ be the couple of nondecreasing processes that solves the following double Skorokhod reflection problem:
find $(\xi, \eta) \in \mathcal{U} \times \mathcal{U}$ s.t.

$$\begin{cases} X_t^{x,\xi,\eta} \in [a^*, b^*], \mathbb{P}\text{-a.s. for } t > 0, \\ \int_0^T \mathbf{1}_{\{X_t^{x,\xi,\eta} > a^*\}} d\xi_t = 0, \mathbb{P}\text{-a.s. for any } T > 0, \\ \int_0^T \mathbf{1}_{\{X_t^{x,\xi,\eta} < b^*\}} d\eta_t = 0, \mathbb{P}\text{-a.s. for any } T > 0. \end{cases} \tag{11}$$

Under our assumptions, this problem admits a unique pathwise solution $(\xi_t^\star, \eta_t^\star)$, continuous apart possible jumps at time zero of amplitude $(a^* - x)^+$ and $(x - b^*)^+$, respectively.
Proposition. The process $\nu^\star := \xi^\star - \eta^\star$ is an admissible control.
Thus, $u = v$ by the verification theorem - ii).

The optimal strategies

Let $(\xi^\star, \eta^\star)$ be the couple of nondecreasing processes that solves the following double Skorokhod reflection problem:
find $(\xi, \eta) \in \mathcal{U} \times \mathcal{U}$ s.t.

$$\begin{cases} X_t^{x,\xi,\eta} \in [a^*, b^*], \mathbb{P}\text{-a.s. for } t > 0, \\ \int_0^T \mathbf{1}_{\{X_t^{x,\xi,\eta} > a^*\}} d\xi_t = 0, \mathbb{P}\text{-a.s. for any } T > 0, \\ \int_0^T \mathbf{1}_{\{X_t^{x,\xi,\eta} < b^*\}} d\eta_t = 0, \mathbb{P}\text{-a.s. for any } T > 0. \end{cases} \tag{11}$$

Under our assumptions, this problem admits a unique pathwise solution $(\xi_t^\star, \eta_t^\star)$, continuous apart possible jumps at time zero of amplitude $(a^* - x)^+$ and $(x - b^*)^+$, respectively.

Proposition. The process $\nu^\star := \xi^\star - \eta^\star$ is an admissible control. Thus, $u = v$ by the verification theorem - ii).

The optimal strategies

Let $(\xi^\star, \eta^\star)$ be the couple of nondecreasing processes that solves the following double Skorokhod reflection problem:
find $(\xi, \eta) \in \mathcal{U} \times \mathcal{U}$ s.t.

$$\begin{cases} X_t^{x,\xi,\eta} \in [a^*, b^*], \mathbb{P}\text{-a.s. for } t > 0, \\ \int_0^T \mathbf{1}_{\{X_t^{x,\xi,\eta} > a^*\}} d\xi_t = 0, \mathbb{P}\text{-a.s. for any } T > 0, \\ \int_0^T \mathbf{1}_{\{X_t^{x,\xi,\eta} < b^*\}} d\eta_t = 0, \mathbb{P}\text{-a.s. for any } T > 0. \end{cases} \tag{11}$$

Under our assumptions, this problem admits a unique pathwise solution $(\xi_t^\star, \eta_t^\star)$, continuous apart possible jumps at time zero of amplitude $(a^* - x)^+$ and $(x - b^*)^+$, respectively.
Proposition. The process $\nu^\star := \xi^\star - \eta^\star$ is an admissible control.
Thus, $u = v$ by the verification theorem - ii).

The optimal strategies

Let $(\xi^\star, \eta^\star)$ be the couple of nondecreasing processes that solves the following double Skorokhod reflection problem:
find $(\xi, \eta) \in \mathcal{U} \times \mathcal{U}$ s.t.

$$\begin{cases} X_t^{x,\xi,\eta} \in [a^*, b^*], \mathbb{P}\text{-a.s. for } t > 0, \\ \int_0^T \mathbf{1}_{\{X_t^{x,\xi,\eta} > a^*\}} d\xi_t = 0, \mathbb{P}\text{-a.s. for any } T > 0, \\ \int_0^T \mathbf{1}_{\{X_t^{x,\xi,\eta} < b^*\}} d\eta_t = 0, \mathbb{P}\text{-a.s. for any } T > 0. \end{cases} \tag{11}$$

Under our assumptions, this problem admits a unique pathwise solution $(\xi_t^\star, \eta_t^\star)$, continuous apart possible jumps at time zero of amplitude $(a^* - x)^+$ and $(x - b^*)^+$, respectively.
Proposition. The process $\nu^\star := \xi^\star - \eta^\star$ is an admissible control. Thus, $u = v$ by the verification theorem - ii).

Mean-reverting (log-)exchange rate

Let us assume now that the log-exchange rate follows a (controlled) Ornstein-Uhlenbeck process

$$dX_t = \rho(m - X_t)\ dt + \sigma\ dB_t + d\xi_t - d\eta_t, \qquad X_0 = x \in \mathbb{R}.$$

In absence of interventions (i.e. $\nu \equiv 0$), this specification is the simplest dynamics which keeps X in a given (suitable) region with a high probability.

Empirical studies have concluded that it well describes several exchange rates among the main world countries.

Mean-reverting (log-)exchange rate

Let us assume now that the log-exchange rate follows a (controlled) Ornstein-Uhlenbeck process

$$dX_t = \rho(m - X_t)\ dt + \sigma\ dB_t + d\xi_t - d\eta_t, \qquad X_0 = x \in \mathbb{R}.$$

In absence of interventions (i.e. $\nu \equiv 0$), this specification is the simplest dynamics which keeps X in a given (suitable) region with a high probability.
Empirical studies have concluded that it well describes several exchange rates among the main world countries.

Costs of the central bank

Let us also assume that the central bank has instantaneous costs $c_i(x) \equiv c_i$ for all $x \in \mathbb{R}$, and a running cost function of the form

$$h(x;\theta) = \frac{1}{2}(x-\theta)^2.$$

The parameter $\theta > 0$ represents a so-called *reference target*, and it can be also viewed as the logarithm of the *central parity* (introduced in Krugman 1991).

The function h penalizes any displacement of the (log-)exchange rate from such a value.

Costs of the central bank

Let us also assume that the central bank has instantaneous costs $c_i(x) \equiv c_i$ for all $x \in \mathbb{R}$, and a running cost function of the form

$$h(x;\theta) = \frac{1}{2}(x-\theta)^2.$$

The parameter $\theta > 0$ represents a so-called *reference target*, and it can be also viewed as the logarithm of the *central parity* (introduced in Krugman 1991).

The function h penalizes any displacement of the (log-)exchange rate from such a value.

Costs of the central bank

Let us also assume that the central bank has instantaneous costs $c_i(x) \equiv c_i$ for all $x \in \mathbb{R}$, and a running cost function of the form

$$h(x;\theta) = \frac{1}{2}(x-\theta)^2.$$

The parameter $\theta > 0$ represents a so-called *reference target*, and it can be also viewed as the logarithm of the *central parity* (introduced in Krugman 1991).
The function h penalizes any displacement of the (log-)exchange rate from such a value.

Fundamental solutions

It turns out that the homogeneous equation

$$\frac{1}{2}\sigma^2 f'' + \rho(m - x)f' - ru = 0$$

has "explicit" fundamental solutions

$$\varphi(x) := e^{\frac{\rho(x-m)^2}{2\sigma^2}} D_{-\frac{r}{\rho}}\left(\frac{(x-m)}{\sigma}\sqrt{2\rho}\right),$$
$$\psi(x) := e^{\frac{\rho(x-m)^2}{2\sigma^2}} D_{-\frac{r}{\rho}}\left(-\frac{(x-m)}{\sigma}\sqrt{2\rho}\right),$$

where D_α is the cylinder function of order α (Bateman 1981) given by

$$D_\alpha(x) := \frac{e^{-\frac{x^2}{4}}}{\Gamma(-\alpha)}\int_0^\infty t^{-\alpha-1}e^{-\frac{t^2}{2}-xt}dt, \quad \text{Re}(\alpha) < 0,$$

and $\Gamma(\cdot)$ is the Euler's Gamma function.

Fundamental solutions

It turns out that the homogeneous equation

$$\frac{1}{2}\sigma^2 f'' + \rho(m-x)f' - ru = 0$$

has "explicit" fundamental solutions

$$\begin{aligned} \varphi(x) &:= e^{\frac{\rho(x-m)^2}{2\sigma^2}} D_{-\frac{r}{\rho}}\left(\frac{(x-m)}{\sigma}\sqrt{2\rho}\right), \\ \psi(x) &:= e^{\frac{\rho(x-m)^2}{2\sigma^2}} D_{-\frac{r}{\rho}}\left(-\frac{(x-m)}{\sigma}\sqrt{2\rho}\right), \end{aligned}$$

where D_α is the cylinder function of order α (Bateman 1981) given by

$$D_\alpha(x) := \frac{e^{-\frac{x^2}{4}}}{\Gamma(-\alpha)} \int_0^\infty t^{-\alpha-1} e^{-\frac{t^2}{2}-xt} dt, \quad \text{Re}(\alpha) < 0,$$

and $\Gamma(\cdot)$ is the Euler's Gamma function.

Comparative statics for a^* and b^*: central bank's costs

Proposition. The optimal intervention boundaries a^* and b^* are such that $c_1 \mapsto a^*(c_1)$ is decreasing, and $c_1 \mapsto b^*(c_1)$ is increasing. Also, $c_2 \mapsto a^*(c_2)$ is decreasing and $c_2 \mapsto b^*(c_2)$ is increasing.

Interpretation: the more the intervention costs, the less the bank intervenes.

Proposition. The optimal intervention boundaries a^* and b^* are such that $\theta \mapsto a^*(\theta)$ and $\theta \mapsto b^*(\theta)$ are increasing.

Interpretation: the target zone moves together with the (log-)central parity decided by the central bank.

Comparative statics for a^* and b^*: central bank's costs

Proposition. The optimal intervention boundaries a^* and b^* are such that $c_1 \mapsto a^*(c_1)$ is decreasing, and $c_1 \mapsto b^*(c_1)$ is increasing. Also, $c_2 \mapsto a^*(c_2)$ is decreasing and $c_2 \mapsto b^*(c_2)$ is increasing.
Interpretation: the more the intervention costs, the less the bank intervenes.

Proposition. The optimal intervention boundaries a^* and b^* are such that $\theta \mapsto a^*(\theta)$ and $\theta \mapsto b^*(\theta)$ are increasing.
Interpretation: the target zone moves together with the (log-)central parity decided by the central bank.

Comparative statics for a^* and b^*: central bank's costs

Proposition. The optimal intervention boundaries a^* and b^* are such that $c_1 \mapsto a^*(c_1)$ is decreasing, and $c_1 \mapsto b^*(c_1)$ is increasing. Also, $c_2 \mapsto a^*(c_2)$ is decreasing and $c_2 \mapsto b^*(c_2)$ is increasing.

Interpretation: the more the intervention costs, the less the bank intervenes.

Proposition. The optimal intervention boundaries a^* and b^* are such that $\theta \mapsto a^*(\theta)$ and $\theta \mapsto b^*(\theta)$ are increasing.

Interpretation: the target zone moves together with the (log-)central parity decided by the central bank.

Comparative statics for a^* and b^*: central bank's costs

Proposition. The optimal intervention boundaries a^* and b^* are such that $c_1 \mapsto a^*(c_1)$ is decreasing, and $c_1 \mapsto b^*(c_1)$ is increasing. Also, $c_2 \mapsto a^*(c_2)$ is decreasing and $c_2 \mapsto b^*(c_2)$ is increasing.
Interpretation: the more the intervention costs, the less the bank intervenes.

Proposition. The optimal intervention boundaries a^* and b^* are such that $\theta \mapsto a^*(\theta)$ and $\theta \mapsto b^*(\theta)$ are increasing.
Interpretation: the target zone moves together with the (log-)central parity decided by the central bank.

Comparative statics for a^* and b^*: market parameters

Proposition. The optimal intervention boundaries a^* and b^* are such that $\sigma \mapsto a^*(\sigma)$ is decreasing, and $\sigma \mapsto b^*(\sigma)$ is increasing.

Interpretation: the more the exchange market is volatile, the more the central bank is reluctant to intervene.

Proposition. The optimal intervention boundaries a^* and b^* are decreasing in the long-run equilibrium level m; that is, $m \mapsto a^*(m)$ and $m \mapsto b^*(m)$ are decreasing.

Interpretation: this instead is quite conterintuitive. In fact, the target zone moves *against* market movements, in particular against the foreign exchange's natural trend!

Comparative statics for a^* and b^*: market parameters

Proposition. The optimal intervention boundaries a^* and b^* are such that $\sigma \mapsto a^*(\sigma)$ is decreasing, and $\sigma \mapsto b^*(\sigma)$ is increasing.
Interpretation: the more the exchange market is volatile, the more the central bank is reluctant to intervene.

Proposition. The optimal intervention boundaries a^* and b^* are decreasing in the long-run equilibrium level m; that is, $m \mapsto a^*(m)$ and $m \mapsto b^*(m)$ are decreasing.
Interpretation: this instead is quite conterintuitive. In fact, the target zone moves *against* market movements, in particular against the foreign exchange's natural trend!

Comparative statics for a^* and b^*: market parameters

Proposition. The optimal intervention boundaries a^* and b^* are such that $\sigma \mapsto a^*(\sigma)$ is decreasing, and $\sigma \mapsto b^*(\sigma)$ is increasing.
Interpretation: the more the exchange market is volatile, the more the central bank is reluctant to intervene.

Proposition. The optimal intervention boundaries a^* and b^* are decreasing in the long-run equilibrium level m; that is, $m \mapsto a^*(m)$ and $m \mapsto b^*(m)$ are decreasing.
Interpretation: this instead is quite conterintuitive. In fact, the target zone moves *against* market movements, in particular against the foreign exchange's natural trend!

Comparative statics for a^* and b^*: market parameters

Proposition. The optimal intervention boundaries a^* and b^* are such that $\sigma \mapsto a^*(\sigma)$ is decreasing, and $\sigma \mapsto b^*(\sigma)$ is increasing.
Interpretation: the more the exchange market is volatile, the more the central bank is reluctant to intervene.
Proposition. The optimal intervention boundaries a^* and b^* are decreasing in the long-run equilibrium level m; that is, $m \mapsto a^*(m)$ and $m \mapsto b^*(m)$ are decreasing.
Interpretation: this instead is quite conterintuitive. In fact, the target zone moves *against* market movements, in particular against the foreign exchange's natural trend!

Exit time from the target zone

Define the exit time of $X^\star_t$ from (a^*, b^*), which is a.s. finite, as

$$\tau_{(a^*,b^*)} := \inf\{t > 0 \ : \ X^\star_t \notin (a^*, b^*)\},$$

We can compute the probabilities that $X^\star$ touches a^* or b^* for the first time:

$$\mathbb{P}_x\{X_{\tau_{(a^*,b^*)}} = a^*\} = \frac{\displaystyle\int_x^{b^*} \exp\left(\rho\frac{(y-m)^2}{\sigma^2}\right)\, dy}{\displaystyle\int_{a^*}^{b^*} \exp\left(\rho\frac{(y-m)^2}{\sigma^2}\right)\, dy}$$

$$\mathbb{P}_x\{X_{\tau_{(a^*,b^*)}} = b^*\} = 1 - \mathbb{P}_x\{X_{\tau_{(a^*,b^*)}} = a^*\}$$

Exit time from the target zone

Define the exit time of $X_t^\star$ from (a^*, b^*), which is a.s. finite, as

$$\tau_{(a^*,b^*)} := \inf\{t > 0 \ : \ X_t^\star \notin (a^*, b^*)\},$$

We can compute the probabilities that $X^\star$ touches a^* or b^* for the first time:

$$\mathbb{P}_x\{X_{\tau_{(a^*,b^*)}} = a^*\} = \frac{\displaystyle\int_x^{b^*} \exp\left(\rho \frac{(y-m)^2}{\sigma^2}\right) dy}{\displaystyle\int_{a^*}^{b^*} \exp\left(\rho \frac{(y-m)^2}{\sigma^2}\right) dy}$$

$$\mathbb{P}_x\{X_{\tau_{(a^*,b^*)}} = b^*\} = 1 - \mathbb{P}_x\{X_{\tau_{(a^*,b^*)}} = a^*\}$$

Expected exit time from the target zone

Furthermore, we know that the function $q(x) := \mathbb{E}_x[\tau_{(a^*,b^*)}]$, $x \in (a^*, b^*)$, satisfies the boundary value differential problem

$$\mathcal{L}_X q + 1 = 0, \qquad q(a) = q(b) = 0,$$

whose solution is

$$\begin{aligned} q(x) \;&=\; A_1 + B_1 \int_{\frac{\sqrt{2\rho}}{\sigma}(a^*-m)}^{\frac{\sqrt{2\rho}}{\sigma}(x-m)} e^{\frac{1}{2}w^2}\, dw \, - \\ &\quad -\frac{1}{\rho}\int_{\frac{\sqrt{2\rho}}{\sigma}(x-m)}^{\frac{\sqrt{2\rho}}{\sigma}(b^*-m)} e^{\frac{1}{2}w^2} \int_{w}^{\frac{\sqrt{2\rho}}{\sigma}(b^*-m)} e^{-\frac{1}{2}u^2}\, du\, dw, \end{aligned}$$

with given constants A_1 and B_1:

$$A_1 = \frac{1}{\rho}\int_{\frac{\sqrt{2\rho}}{\sigma}(a^*-m)}^{\frac{\sqrt{2\rho}}{\sigma}(b^*-m)} e^{\frac{1}{2}w^2} \int_{w}^{\frac{\sqrt{2\rho}}{\sigma}(b^*-m)} e^{-\frac{1}{2}u^2}\, du\, dw, \quad B_1 = \frac{-A_1}{\int_{\frac{\sqrt{2\rho}}{\sigma}(a^*-m)}^{\frac{\sqrt{2\rho}}{\sigma}(b^*-m)} e^{\frac{1}{2} \cdots}}$$

Expected exit time from the target zone

Furthermore, we know that the function $q(x) := \mathbb{E}_x[\tau_{(a^*,b^*)}]$, $x \in (a^*, b^*)$, satisfies the boundary value differential problem

$$\mathcal{L}_X q + 1 = 0, \qquad q(a) = q(b) = 0,$$

whose solution is

$$\begin{aligned} q(x) &= A_1 + B_1 \int_{\frac{\sqrt{2\rho}}{\sigma}(a^*-m)}^{\frac{\sqrt{2\rho}}{\sigma}(x-m)} e^{\frac{1}{2}w^2}\, dw - \\ &\quad -\frac{1}{\rho} \int_{\frac{\sqrt{2\rho}}{\sigma}(x-m)}^{\frac{\sqrt{2\rho}}{\sigma}(b^*-m)} e^{\frac{1}{2}w^2} \int_{w}^{\frac{\sqrt{2\rho}}{\sigma}(b^*-m)} e^{-\frac{1}{2}u^2}\, du\, dw, \end{aligned}$$

with given constants A_1 and B_1:

$$A_1 = \frac{1}{\rho} \int_{\frac{\sqrt{2\rho}}{\sigma}(a^*-m)}^{\frac{\sqrt{2\rho}}{\sigma}(b^*-m)} e^{\frac{1}{2}w^2} \int_{w}^{\frac{\sqrt{2\rho}}{\sigma}(b^*-m)} e^{-\frac{1}{2}u^2}\, du\, dw, \; B_1 = \frac{-A_1}{\int_{\frac{\sqrt{2\rho}}{\sigma}(a^*-m)}^{\frac{\sqrt{2\rho}}{\sigma}(b^*-m)} e^{\frac{1}{2}\dots}}$$

Case study: DKK/EUR

Since it seems that in 30 years there was no need to intervene from the Danish Central Bank, we can safely assume that the long-run mean corresponds to the logarithm of the central parity fixed to 7.46038 DKK/EUR.

Remembering that the Ornstein-Uhlenbeck process represents the log-exchange rate, we thus let $m = \theta = \log 7.46038 \simeq 2.01$.

From time series of market data, other plausible parameters for the Ornstein-Uhlenbeck dynamics could be $\rho = 0.001$ and $\sigma = 0.015$.

Given the interest rates in the current economy, a plausible value for r could be $r = 0.5\% = 0.005$.

The values above are characteristic of the Danish and European economies, and still do not reflect the Danish Central Bank's policy, which is instead implemented in the three parameters θ, c_1 and c_2.

Case study: DKK/EUR

Since it seems that in 30 years there was no need to intervene from the Danish Central Bank, we can safely assume that the long-run mean corresponds to the logarithm of the central parity fixed to 7.46038 DKK/EUR.
Remembering that the Ornstein-Uhlenbeck process represents the log-exchange rate, we thus let $m = \theta = \log 7.46038 \simeq 2.01$.
From time series of market data, other plausible parameters for the Ornstein-Uhlenbeck dynamics could be $\rho = 0.001$ and $\sigma = 0.015$.
Given the interest rates in the current economy, a plausible value for r could be $r = 0.5\% = 0.005$.
The values above are characteristic of the Danish and European economies, and still do not reflect the Danish Central Bank's policy, which is instead implemented in the three parameters θ, c_1 and c_2.

Case study: DKK/EUR

Since it seems that in 30 years there was no need to intervene from the Danish Central Bank, we can safely assume that the long-run mean corresponds to the logarithm of the central parity fixed to 7.46038 DKK/EUR.

Remembering that the Ornstein-Uhlenbeck process represents the log-exchange rate, we thus let $m = \theta = \log 7.46038 \simeq 2.01$.

From time series of market data, other plausible parameters for the Ornstein-Uhlenbeck dynamics could be $\rho = 0.001$ and $\sigma = 0.015$.

Given the interest rates in the current economy, a plausible value for r could be $r = 0.5\% = 0.005$.

The values above are characteristic of the Danish and European economies, and still do not reflect the Danish Central Bank's policy, which is instead implemented in the three parameters θ, c_1 and c_2.

Case study: DKK/EUR

Since it seems that in 30 years there was no need to intervene from the Danish Central Bank, we can safely assume that the long-run mean corresponds to the logarithm of the central parity fixed to 7.46038 DKK/EUR.

Remembering that the Ornstein-Uhlenbeck process represents the log-exchange rate, we thus let $m = \theta = \log 7.46038 \simeq 2.01$.

From time series of market data, other plausible parameters for the Ornstein-Uhlenbeck dynamics could be $\rho = 0.001$ and $\sigma = 0.015$.

Given the interest rates in the current economy, a plausible value for r could be $r = 0.5\% = 0.005$.

The values above are characteristic of the Danish and European economies, and still do not reflect the Danish Central Bank's policy, which is instead implemented in the three parameters θ, c_1 and c_2.

Case study: DKK/EUR

Since it seems that in 30 years there was no need to intervene from the Danish Central Bank, we can safely assume that the long-run mean corresponds to the logarithm of the central parity fixed to 7.46038 DKK/EUR.
Remembering that the Ornstein-Uhlenbeck process represents the log-exchange rate, we thus let $m = \theta = \log 7.46038 \simeq 2.01$.
From time series of market data, other plausible parameters for the Ornstein-Uhlenbeck dynamics could be $\rho = 0.001$ and $\sigma = 0.015$.
Given the interest rates in the current economy, a plausible value for r could be $r = 0.5\% = 0.005$.
The values above are characteristic of the Danish and European economies, and still do not reflect the Danish Central Bank's policy, which is instead implemented in the three parameters θ, c_1 and c_2.

Reverse engineering the Danish central bank's parameters

What we know about the parameters θ, c_1 and c_2 is that they imply a target zone centered of $\pm 2.25\%$ around a (log-)central parity of 2.01.

Thus, we are led to solve the following inverse problem: find c_1, c_2 such that, with the parameters above, the optimal a^* and b^* are

$$\begin{aligned} a^* &= \log 7.46038(1 - 0.0225) = 1.98685, \\ b^* &= \log 7.46038(1 + 0.0225) = 2.03186 \end{aligned}$$

Given the (approximate) symmetry of our problem, since

$$\begin{aligned} \log(1 + 0.0225) &= 0.02225 \simeq 0.0225, \\ \log(1 - 0.0225) &= -0.02276 \simeq -0.0225 \end{aligned}$$

we search for c_1 and c_2 such that $c_1 = c_2 =: c$.

Reverse engineering the Danish central bank's parameters

What we know about the parameters θ, c_1 and c_2 is that they imply a target zone centered of $\pm 2.25\%$ around a (log-)central parity of 2.01.

Thus, we are led to solve the following inverse problem: find c_1, c_2 such that, with the parameters above, the optimal a^* and b^* are

$$\begin{aligned} a^* &= \log 7.46038(1 - 0.0225) = 1.98685, \\ b^* &= \log 7.46038(1 + 0.0225) = 2.03186 \end{aligned}$$

Given the (approximate) symmetry of our problem, since

$$\begin{aligned} \log(1 + 0.0225) &= 0.02225 \simeq 0.0225, \\ \log(1 - 0.0225) &= -0.02276 \simeq -0.0225 \end{aligned}$$

we search for c_1 and c_2 such that $c_1 = c_2 =: c$.

Reverse engineering the Danish central bank's parameters

What we know about the parameters θ, c_1 and c_2 is that they imply a target zone centered of $\pm 2.25\%$ around a (log-)central parity of 2.01.
Thus, we are led to solve the following inverse problem: find c_1, c_2 such that, with the parameters above, the optimal a^* and b^* are

$$\begin{aligned} a^* &= \log 7.46038(1 - 0.0225) = 1.98685, \\ b^* &= \log 7.46038(1 + 0.0225) = 2.03186 \end{aligned}$$

Given the (approximate) symmetry of our problem, since

$$\begin{aligned} \log(1 + 0.0225) &= 0.02225 \simeq 0.0225, \\ \log(1 - 0.0225) &= -0.02276 \simeq -0.0225 \end{aligned}$$

we search for c_1 and c_2 such that $c_1 = c_2 =: c$.

Finding the Danish central bank's parameters

From the previous monotonicity results, we know that, by increasing (decreasing) the common proportional cost c, the continuation region (a^*, b^*) will enlarge (shrink): this is a positive sign that our inverse problem can have a unique solution.

With this in mind, by any numerical method one arrives at this:

c	a^*	b^*	$a^* - m$	$b^* - m$
1	1.93729	2.08193	−0.07232	0.07232
0.5	1.95302	2.0662	−0.0565905	0.0565905
0.1	1.97703	2.04218	−0.0325786	0.0325786
0.05	1.98383	2.03539	−0.0257803	0.0257803
0.04	1.98569	2.03352	−0.0239155	0.0239155
0.035	1.98674	2.03247	−0.0228658	0.0228658
0.034	1.98696	2.03225	−0.0226442	0.0226442
0.0335	**1.98707**	**2.03214**	**−0.0225317**	**0.0225317**
0.033	1.98719	2.03202	−0.0224182	0.0224182
0.03	1.98789	2.03132	−0.021712	0.021712

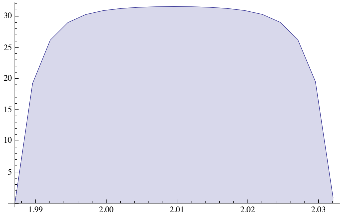
Finding the Danish central bank's parameters

From the previous monotonicity results, we know that, by increasing (decreasing) the common proportional cost c, the continuation region (a^*, b^*) will enlarge (shrink): this is a positive sign that our inverse problem can have a unique solution.
With this in mind, by any numerical method one arrives at this:

c	a^*	b^*	$a^* - m$	$b^* - m$
1	1.93729	2.08193	−0.07232	0.07232
0.5	1.95302	2.0662	−0.0565905	0.0565905
0.1	1.97703	2.04218	−0.0325786	0.0325786
0.05	1.98383	2.03539	−0.0257803	0.0257803
0.04	1.98569	2.03352	−0.0239155	0.0239155
0.035	1.98674	2.03247	−0.0228658	0.0228658
0.034	1.98696	2.03225	−0.0226442	0.0226442
0.0335	**1.98707**	**2.03214**	**−0.0225317**	**0.0225317**
0.033	1.98719	2.03202	−0.0224182	0.0224182
0.03	1.98789	2.03132	−0.021712	0.021712

The model predicts DKK's observed stability

We can compute the expected exit time of the exchange rate from the target zone: We can plot the average exit time from the target zone (in years) as a function of initial (log-)exchange rate x:

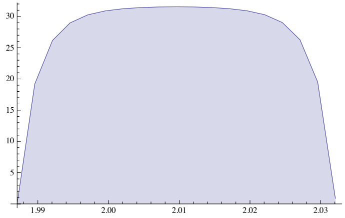


We can see that the maximal expected time is obtained (as expected) when the deviation from central parity is null, i.e., for $x = \log 7.46038 \simeq 2.01$, and decreases as the exchange rate nears the target zone's boundaries.

This maximum expected time is around 31.11 years: in line with the last 30 years of DKK/EUR exchange rate!

The model predicts DKK's observed stability

We can compute the expected exit time of the exchange rate from the target zone: We can plot the average exit time from the target zone (in years) as a function of initial (log-)exchange rate x:

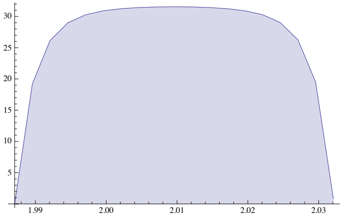


We can see that the maximal expected time is obtained (as expected) when the deviation from central parity is null, i.e., for $x = \log 7.46038 \simeq 2.01$, and decreases as the exchange rate nears the target zone's boundaries.

This maximum expected time is around 31.11 years: in line with the last 30 years of DKK/EUR exchange rate!

The model predicts DKK's observed stability

We can compute the expected exit time of the exchange rate from the target zone: We can plot the average exit time from the target zone (in years) as a function of initial (log-)exchange rate x:



We can see that the maximal expected time is obtained (as expected) when the deviation from central parity is null, i.e., for $x = \log 7.46038 \simeq 2.01$, and decreases as the exchange rate nears the target zone's boundaries.

This maximum expected time is around 31.11 years: in line with the last 30 years of DKK/EUR exchange rate!

Conclusions

- In this paper we have studied the optimal management problem of exchange rates faced by a central bank.
- We have formulated it as an infinite time-horizon singular stochastic control problem for a one-dimensional diffusion that is linearly controlled through a process of bounded variation.
- We have provided the explicit expression of the value function, as well as the complete characterization of the optimal control.
- At each instant of time, the optimally controlled exchange rate is kept within an optimal band (continuation region), whose boundaries (the so-called free boundaries) are endogenously determined as part of the solution to the problem.

Conclusions

- In this paper we have studied the optimal management problem of exchange rates faced by a central bank.
- We have formulated it as an infinite time-horizon singular stochastic control problem for a one-dimensional diffusion that is linearly controlled through a process of bounded variation.
- We have provided the explicit expression of the value function, as well as the complete characterization of the optimal control.
- At each instant of time, the optimally controlled exchange rate is kept within an optimal band (continuation region), whose boundaries (the so-called free boundaries) are endogenously determined as part of the solution to the problem.

Conclusions

- In this paper we have studied the optimal management problem of exchange rates faced by a central bank.
- We have formulated it as an infinite time-horizon singular stochastic control problem for a one-dimensional diffusion that is linearly controlled through a process of bounded variation.
- We have provided the explicit expression of the value function, as well as the complete characterization of the optimal control.
- At each instant of time, the optimally controlled exchange rate is kept within an optimal band (continuation region), whose boundaries (the so-called free boundaries) are endogenously determined as part of the solution to the problem.

Conclusions

- In this paper we have studied the optimal management problem of exchange rates faced by a central bank.
- We have formulated it as an infinite time-horizon singular stochastic control problem for a one-dimensional diffusion that is linearly controlled through a process of bounded variation.
- We have provided the explicit expression of the value function, as well as the complete characterization of the optimal control.
- At each instant of time, the optimally controlled exchange rate is kept within an optimal band (continuation region), whose boundaries (the so-called free boundaries) are endogenously determined as part of the solution to the problem.

Conclusions - ii

- A detailed comparative statics analysis of the free boundaries is provided when the (log-)exchange rate evolves as an Ornstein-Uhlenbeck process.
- If the minimum of h is very near to the long-term mean of the exchange dynamics, then the exchange rate stays naturally with a high probability in the continuation region (Krugman's "target zone", seen in the Danish case).
- Instead, if the rate's long-term mean is far from the minimum of h, then it is very probable that the exchange rate hits one boundary of the continuation region much more often than the other one ("pegging", CHF/EUR 2011–2015).

Conclusions - ii

- A detailed comparative statics analysis of the free boundaries is provided when the (log-)exchange rate evolves as an Ornstein-Uhlenbeck process.
- If the minimum of h is very near to the long-term mean of the exchange dynamics, then the exchange rate stays naturally with a high probability in the continuation region (Krugman's "target zone", seen in the Danish case).
- Instead, if the rate's long-term mean is far from the minimum of h, then it is very probable that the exchange rate hits one boundary of the continuation region much more often than the other one ("pegging", CHF/EUR 2011–2015).

Conclusions - ii

- A detailed comparative statics analysis of the free boundaries is provided when the (log-)exchange rate evolves as an Ornstein-Uhlenbeck process.
- If the minimum of h is very near to the long-term mean of the exchange dynamics, then the exchange rate stays naturally with a high probability in the continuation region (Krugman's "target zone", seen in the Danish case).
- Instead, if the rate's long-term mean is far from the minimum of h, then it is very probable that the exchange rate hits one boundary of the continuation region much more often than the other one ("pegging", CHF/EUR 2011–2015).

. . . but not the end!

- 1995 Pisa
- 1997 "Paris" (maybe Marne-La-Vallée)
- 1997 Trento

⋮

- 2000 Besancon

⋮

- 2018 Metabief
- 2018 Luminy

⋮

Thanks Yuri!

. . . but not the end!

- 1995 Pisa
- 1997 "Paris" (maybe Marne-La-Vallée)
- 1997 Trento

⋮

- 2000 Besancon

⋮

- 2018 Metabief
- 2018 Luminy

⋮

Thanks Yuri!

. . . but not the end!

- 1995 Pisa
- 1997 "Paris" (maybe Marne-La-Vallée)
- 1997 Trento

⋮

- 2000 Besancon

⋮

- 2018 Metabief
- 2018 Luminy

⋮

Thanks Yuri!

... but not the end!

- 1995 Pisa
- 1997 "Paris" (maybe Marne-La-Vallée)
- 1997 Trento

⋮

- 2000 Besancon

⋮

- 2018 Metabief
- 2018 Luminy

⋮

Thanks Yuri!

. . . but not the end!

- 1995 Pisa
- 1997 "Paris" (maybe Marne-La-Vallée)
- 1997 Trento

⋮

- 2000 Besancon

⋮

- 2018 Metabief
- 2018 Luminy

⋮

Thanks Yuri!

. . . but not the end!

- 1995 Pisa
- 1997 "Paris" (maybe Marne-La-Vallée)
- 1997 Trento

⋮

- 2000 Besancon

⋮

- 2018 Metabief
- 2018 Luminy

⋮

Thanks Yuri!

. . . but not the end!

- 1995 Pisa
- 1997 "Paris" (maybe Marne-La-Vallée)
- 1997 Trento

⋮

- 2000 Besancon

⋮

- 2018 Metabief
- 2018 Luminy

⋮

Thanks Yuri!

. . . but not the end!

- 1995 Pisa
- 1997 "Paris" (maybe Marne-La-Vallée)
- 1997 Trento

⋮

- 2000 Besancon

⋮

- 2018 Metabief
- 2018 Luminy

⋮

Thanks Yuri!

. . . but not the end!

- 1995 Pisa
- 1997 "Paris" (maybe Marne-La-Vallée)
- 1997 Trento

⋮

- 2000 Besancon

⋮

- 2018 Metabief
- 2018 Luminy

⋮

Thanks Yuri!

. . . but not the end!

- 1995 Pisa
- 1997 "Paris" (maybe Marne-La-Vallée)
- 1997 Trento

⋮

- 2000 Besancon

⋮

- 2018 Metabief
- 2018 Luminy

⋮

Thanks Yuri!

Bibliography

- Bateman, H. (1981). *Higher Trascendental Functions*, Volume II. McGraw-Hill Book Company.
- Bertola, G., Runggaldier, W. J., Yasuda, K. (2016). *On classical and restricted impulse stochastic control for the exchange rate*. Appl. Math. Optim. **74(2)**, pp. 423–454.
- Bo, L., Li, D., Ren, G. Wang, Y., Yang, X. (2016). *Modeling the exchange rates in a target zone by reflected Ornstein-Uhlenbeck process*. Preprint. Available at SSRN: https://ssrn.com/abstract=2107686 or http://dx.doi.org/10.2139/ssrn.2107686
- Cadenillas, A., Zapatero, F. (1999). *Optimal central bank intervention in the foreign exchange market*. J. Econ. Theory **87**, pp. 218–242.

- Cadenillas, A., Zapatero, F. (2000). *Classical and impulse stochastic control of the exchange rate using interest rates and reserves*. Math. Finance **10**, pp. 141–156.
- De Jong, F., Drost, F.C., Werker, B.J.M. (2001). *A jump-diffusion model for exchange rates in a target zone*. Statist. Neerlandica **55(3)**, pp. 270–300.
- Jeanblanc-Picqué, M. (1993). *Impulse control method and exchange rate*. Math. Finance **3**, pp. 161–177.
- Jørgensen, B., Mikkelsen, H. O. (1996). *An arbitrage free trilateral target zone model*. J. Int. Money Finance **15(1)**, pp. 117–134.
- Krugman, P.R. (1991). *Target zones and exchange rate dynamics*. Quart. J. Econ. **106(3)**, pp. 669–682.
- Larsen, K.S., Sørensen, M. (2007). *Diffusion models for exchange rates in a target zone*. Math. Finance **17(2)**, pp. 285–306.

C. Lloyd (2015). *On the end of the EUR CHF peg*. SNBCHF.com, February 6, 2015
https://snbchf.com/chf/colin-lloyd-end-eur-chf-peg/

Matomäki, P. (2012). *On solvability of a two-sided singular control problem*. Math. Meth. Oper. Res **76**, pp. 239–271.

Mikkelsen, O. (2017). *Denmark's fixed exchange rate policy: 30th anniversary of unchanged central rate*. News — Danmarks Nationalbank, Jan. 2017 n. 1.
http://www.nationalbanken.dk/en/publications/Pages/2017/01/Denmark's-fixed-exchange-rate-policy-30th-anniversary-of-unchanged-central-rate.aspx

Mundaca, G., Øksendal, B. (1998). *Optimal stochastic intervention control with application to the exchange rate*. J. Math. Econ. **29**, pp. 223–241.

Protter, P. (1990). *Stochastic Integration and Differential Equations*. Springer, Berlin.

- SHREVE S.E. (1988). *An Introduction to Singular Stochastic Control*, in Stochastic Differential Systems, Stochastic Control Theory and Applications, IMA Vol. 10, W. Fleming and P.-L. Lions, ed. Springer-Verlag, New York.
- YANG, X., REN, G., WANG, Y., BO, L., LI, D. (2016). *Modeling the exchange rates in a target zone by reflected Ornstein-Uhlenbeck process*. Preprint. Available at SSRN, https://ssrn.com/abstract=2107686 or http://dx.doi.org/10.2139/ssrn.2107686.
- *Swiss Central Bank Acts to Put a Cap on Franc's Rise*. The New York Times, Sept. 6, 2011. http://www.nytimes.com/2011/09/07/business/global/swiss-franc.html
- *The Economist explains: Why the Swiss unpegged the franc*. The Economist, Jan. 18, 2015. http://www.economist.com/blogs/economist-explains/2015/01/economist-explains-13