Utility maximization with proportional transaction cost under model uncertainty

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Outline

1. Introduction
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Financial market : trading, arbitrage, and the FTAP

• Discrete time market framework : $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a filtered probability space, S is the adapted price process of the underlying assets, H is a predictable process as trading strategy process. Assume $r = 0$. For a self-financing strategy H, its P&L is given by

$$(H \circ S)_T := \sum_{t=1}^{T} H_t \Delta S_t.$$

• No arbitrage condition (NA) :

$$(H \circ S)_T \geq 0, \text{ a.s. } \Rightarrow (H \circ S)_T = 0, \text{ a.s.}$$

• Fundamental Theorem of Asset Pricing (FTAP) : (i) (NA) is equivalent to the existence of equivalent martingale measure, i.e.

$$\mathcal{M} := \{\mathbb{Q} \sim \mathbb{P} : S \text{ is a } \mathbb{Q} \text{-martingale}\} \neq \emptyset.$$

(ii) The market is complete iff $\mathcal{M}$ is a singleton.

Option pricing

- Complete market : pricing by replication for a derivative option $\xi : \Omega \to \mathbb{R}$:

$$\pi(\xi) = y \quad \text{where there is } H \text{ s.t. } y + (H \circ S)_T = \xi, \text{ a.s.}$$

- Incomplete market : super-replication problem :

$$\pi(\xi) \quad := \quad \inf\left\{ y \in \mathbb{R} \ : y + (H \circ S)_T \geq \xi, \text{ a.s.} \right\}$$

- The superhedging duality :

$$\pi(\xi) \ = \ \sup_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}^{\mathbb{Q}}[\xi].$$

Portfolio choice, utility maximization

- A utility maximization problem : given a utility function $U : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$, one solves

$$V_0 \ := \ \sup_H \mathbb{E}^{\mathbb{P}} \big[U\big(x + (H \circ S)_T\big)\big].$$

- A convex duality result :

$$V_0 \ = \ \inf_{y>0} \Big(\mathbb{E}^{\mathbb{P}} \Big[U^*\Big(y \frac{d\mathbb{Q}}{d\mathbb{P}} \Big) \Big] + xy \Big),$$

where

$$U^*(y) := \sup_{x \in \mathrm{dom}(U)} \big(U(x) - xy \big).$$

- Existence of optimal trading strategy, optimal dual optimizer ?

Utility indifference price

- Utility maximization problem with initial wealth 0 and random endowment $\xi : \Omega \to \mathbb{R}$:

$$V(0,\xi) \ := \ \sup_H \mathbb{E}^{\mathbb{P}}\big[U\big(0 + (H \circ S)_T + \xi\big)\big].$$

- Utility indifference price : find a constant p such that

$$\begin{aligned} V(p,0) \ &:= \ \sup_H \mathbb{E}^{\mathbb{P}}\big[U\big(p + (H \circ S)_T + 0\big)\big] \\ &= \ V(0,\xi) \end{aligned}$$

- Denote the utility indifference price

$$\pi_u(\xi) := p.$$

Trading with proportional transaction cost

- Trading with proportional transaction cost $(d = 1)$:

$$(H \circ S)_T \quad \to \quad (H \circ S)_T - c \sum_t |H_t - H_{t-1}| S_t.$$

- More general proportional transaction cost $(d \geq 2)$ modelling by solvency cone.

- Consistent Price System : $(X, \mathbb{Q})$ such that X is a $\mathbb{Q}$-martingale, $\mathbb{Q} \sim \mathbb{P}$, $X_t \in [(1-c)S_t, (1+c)S_t]$.

- FTAP, Duality, existence of the optimal strategy, shadow process, etc. Kabanov and Safarian (2009), etc.

Robust finance

- For a measurable space $(\Omega, \mathcal{F})$, in place of a fixed probability $\mathbb{P}$, one considers a family $\mathcal{P}$ of (singular) probability measures.

- The "robust" FTAP and "robust" super-hedging problem :

$$\inf \left\{ y \in \mathbb{R} \ : y + (H \circ S)_T \geq \xi, \ \mathcal{P}\text{-q.s.} \right\}.$$

- The "robust" utility maximization problem :

$$\sup_H \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}} \left[U\left(x + (H \circ S)_T \right) \right].$$

- Some references in discrete time framework :
  - Nutz (F&S, 2016), existence of the optimal strategy.
  - Blanchard and Carassus (AAP, 2018), existence under more general conditions.
  - Blanchard and Carassus (2017), indifference pricing.
  - Neufeld and Sikic (SICON, 2018), with friction.
  - Bartl (2016), exponential utility function.

Outline

One-period with transaction cost at time 1

- Let us consider a one-period market ($t = 0, 1$) with model uncertainty, $U : \Omega \times \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ be a utility function, one consider the utility maximization problem :

$$V := \sup_{H \in \mathbb{R}} \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}} \Big[U\Big(H(S_1 - S_0) - |H| c S_1 \Big) \Big].$$

- Notice that

$$\begin{aligned} H(S_1 - S_0) - |H| c S_1 &= \min_{\theta \in [-1,1]} H(S_1 - S_0) - H c \theta S_1 \\ &= \min_{\theta \in [-1,1]} H\Big((1 - c\theta) S_1 - S_0 \Big). \end{aligned}$$

- Let $X_0 := S_0$ and $X_1(\omega, \theta) := (1 - c\theta) S_1(\omega)$, then

$$V = \sup_{H \in \mathbb{R}} \inf_{\mathbb{P} \in \mathcal{P}} \inf_{\theta \in [-1,1]} \mathbb{E}^{\mathbb{P}} \Big[U\Big(H(X_1 - X_0) \Big) \Big].$$

One-period with transaction cost at time 1

- Let $\overline{\Omega} := \Omega \times [-1, 1]$, $X_0 = S_0$, $X_1(\omega, \theta) := S_1(\omega)(1 + c\theta)$, $\overline{\mathcal{P}} := \{\overline{\mathbb{P}} : \overline{\mathbb{P}}|_\Omega \in \mathcal{P}\}$, then

$$\begin{aligned} V &= \sup_{H \in \mathbb{R}} \inf_{\mathbb{P} \in \mathcal{P}} \inf_{\theta \in [-1,1]} \mathbb{E}^{\mathbb{P}}\big[U\big(H(X_1 - X_0)\big)\big] \\ &= \sup_{H \in \mathbb{R}} \inf_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}} \mathbb{E}^{\overline{\mathbb{P}}}\big[U\big(H(X_1 - X_0)\big)\big]. \end{aligned}$$

- Market with underlying X : let

$$\overline{\mathcal{Q}} := \{\overline{\mathbb{Q}} \lll \overline{\mathcal{P}} \ : X \text{ is a } \overline{\mathbb{Q}}\text{-martingale}\},$$

then $(\overline{\mathbb{Q}}, X)$ is a CPS since $X_t \in [(1-c)S_t, (1+c)S_t]$ by construction.

One-period with transaction cost at both 0 and 1

• Let us consider the one-period market ($t = 0, 1$) with proportional transaction cost at both time 0 and 1 :

$$V := \sup_{H\in\mathbb{R}} \inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}\Big[U\big(H(S_1 - S_0) - |H|cS_1 - |H|cS_0\big)\Big].$$

• By the same argument, one introduce and enlarged space $\overline{\Omega} := \Omega \times [-1, 1] \times [-1, 1]$, $X_0 := (1 + c\theta_0)S_0$ and $X_1 := (1 + c\theta_1)S_1$, then

$$\begin{aligned} V &= \sup_{H\in\mathbb{R}} \inf_{\mathbb{P}\in\mathcal{P}} \inf_{\theta_0\in[-1,1]} \inf_{\theta_1\in[-1,1]} \mathbb{E}^{\mathbb{P}}\Big[U\big(H(X_1 - X_0)\big)\Big] \\ &= \sup_{H\in\mathbb{R}} \inf_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}} \mathbb{E}^{\overline{\mathbb{P}}}\Big[U\big(H(X_1 - X_0)\big)\Big]. \end{aligned}$$

• Problem : H is a deterministic constant rather than $\mathbb{F}^X-$adapted.

Super-replication under proportional transaction cost

• For the exponential utility function $U(x) := -\exp(-\gamma x)$, one uses MiniMax Theorem :

$$\begin{aligned} V &= \sup_{H\in\mathbb{R}} \inf_{\theta_0\in[-1,1]} \Big(\inf_{\mathbb{P}\in\mathcal{P}} \inf_{\theta_1\in[-1,1]} \mathbb{E}^{\mathbb{P}}\big[U\big(H(X_1 - X_0)\big)\big]\Big) \\ &= \inf_{\theta_0\in[-1,1]} \sup_{H\in\mathbb{R}} \Big(\inf_{\mathbb{P}\in\mathcal{P}} \inf_{\theta_1\in[-1,1]} \mathbb{E}^{\mathbb{P}}\big[U\big(H(X_1 - X_0)\big)\big]\Big), \end{aligned}$$

since

$$H \mapsto \log\left(\mathbb{E}^{\mathbb{P}}\big[\exp(-\gamma(H(X_1 - X_0)))\big]\right) \text{ is concave,}$$

and

$$\begin{aligned} & \log\left(\mathbb{E}^{\mathbb{P}}\big[\exp(-\gamma(H(X_1 - (1+c\theta_0)S_0)))\big]\right) \\ = \; & \log\left(\mathbb{E}^{\mathbb{P}}\big[\exp(-\gamma H X_1)\big] + \gamma(1+c\theta_0)S_0\right. \end{aligned}$$

is linear in θ_0.

Outline

A general framework of Bouchard & Nutz (2014)

• Let $\Omega_0 := \{\omega_0\}$, Ω_1 be a Polish space, $\Omega_t := \Omega_0 \times \Omega_1^t$, and $\Omega := \Omega_T$, $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ be defined by $\mathcal{F}_t := \mathcal{B}(\Omega_t)$ for all $t \geq 0$. Given $t < T$ and $\omega \in \Omega_t$, the set $\mathcal{P}_t(\omega)$ defines the set of all probability measure on Ω_{t+1} known the path previous path ω, let

$$\mathcal{P} := \left\{\mathbb{P}_0 \otimes \mathbb{P}_1 \otimes \cdots \otimes \mathbb{P}_{T-1} \;\; : \mathbb{P}_t(\cdot) \in \mathcal{P}_t(\cdot)\right\}.$$

• Transaction cost modelling with solvency cone $(K_t)_{t\geq 0}$, where $K_t : \Omega \to 2^{\mathbb{R}^d}$ is a $\mathcal{F}_t$-measurable random set. K_t represents the collection of positions, labelled in units of different d financial assets, that can be turned into non-negative ones (component by component) by performing immediately exchanges between the assets.

• An admissible trading strategy is an $\mathbb{F}$-adapted process $(\eta_t)_{0\leq t\leq T}$ such that $-\eta_t \in K_t$. No-arbitrage condition is equivalent to the existence of consistent price system.

Main result : superhedging duality

• Let $U(x) = -\exp(-\gamma x)$, $\xi : \Omega \to \mathbb{R}^d$ be some derivative option, for $i = 1, \cdots, e$, $\zeta_i : \Omega \to \mathbb{R}^d$ be some liquid options whose price is 0, we consider the (numéraire-based) utility maximization problem :

$$V := \sup_{(\ell,\eta)} \inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}\Big[U\Big(\Big(\xi + \sum_{i=1}^{e} (\ell_i\zeta_i - |\ell_i|c_i 1_d) + \sum_{t=0}^{T} \eta_t\Big)^d\Big)\Big].$$

• Let

$$\mathcal{S}_e := \Big\{(\mathbb{Q}, Z) : \mathbb{Q} \lll \mathcal{P},\ Z_t \in K_t^*,\ Z \text{ is a } \mathbb{Q}\text{-martingale s.t. } \mathbb{E}^{\mathbb{Q}}\big[\zeta_i \cdot Z_T\big] \in [-c_i, c_i],\ i = 1, \cdots, e\Big\},$$

where $K_t^*(\omega) := \big\{y \in \mathbb{R}^d : x \cdot y \geq 0 \text{ for all } x \in K_t(\omega)\big\}$.

Main result : superhedging duality

• Main result 1 : under some no-arbitrage condition, one has the existence of the optimal trading strategy and the duality result

$$V = -\exp\Big(- \inf_{(\mathbb{Q},Z)\in\mathcal{S}_e} \big\{ \mathbb{E}^{\mathbb{Q}}[\gamma\xi \cdot Z_T] + \mathcal{E}(\mathbb{Q},\mathcal{P}) \big\} \Big),$$

where $\mathcal{E}(\mathbb{Q},\mathbb{P}) := \mathbb{E}^{\mathbb{P}}\big[\frac{d\mathbb{P}}{d\mathbb{Q}} \log(\frac{d\mathbb{P}}{d\mathbb{Q}})\big]$, $\mathcal{E}(\mathbb{Q},\mathcal{P}) := \inf_{\mathbb{P}\in\mathcal{P}} \mathcal{E}(\mathbb{Q},\mathbb{P})$ are entropie functions.

- Extensions in the framework of Bartl (2016).

• Main result 2 : Some consequences of the duality result : convergence and stability of the utility indifference price.