Pricing measures in the plural and their coherences issue

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Shiqi Song

Université d'Evry Val d'Essonne, France

An introduction

In this talk, there will be few conclusions. Rather, there will be examples and discussions.

The questions begin with the following fact: The market participants have their own beliefs about the financial market and they decide their own pricing strategies. These can be very different from each other without coordination. But in spite of the variety of individual pricing strategies, the market remains viable (generally speaking). The strategies of the different market participants should have followed some consistency principle.

From the viewpoint of modeling, we should have a general theory, which manage multiple trading systems (each composed of an arbitrage principle, an information flow, a numéraire, a pricing measure, and others) operating on the same market, and a corresponding general principle about the market stability. (Such a theory would be useful either for the participants or for the market regulators.)

Unfortunately, such a theory seems not exist. The classical FTAP does not serve for this purpose, because the FTAP supposes a very restrictive assumption: all market participants share a same price process and a same information flow (so that everyone takes the same trading strategy). We simply mention the market of O.T.C, which is much heavy than the organized markets, and where the pricing of O.T.C. contracts can be calculated with various different methods, not necessarily with a common price process, not necessarily based on the full information flow of the market.

Notice that, in particular situations, models of multiple trading systems exist and they yield useful results. But, it lacks a general notion to unify all the particular situations, and it lacks a general theory to examine the coherence of the multiple trading systems and the market stability. In this talk, we re-examine some of the known models, show the corresponding results and try to draw a profile about the expected theory.

Differentiated information flows

A first reason of different trading systems is the different level of information of the market participants. That is because pricing models depend on their filtrations of information.

Be careful: we not do speak about the insider trading. We speak of the information differentiation as a normal property of the market. A typical example is of the market makers (at least in old systems), who can profit extra information. (That said, market make seems not an eldorado, because of the obligations imposed to him.) The question is how a market can exist, despite the information differentiation of the participants.

Mathematically, such market exists. Consider the market model endowed with a brownian motion β and an independent random variable τ, whose law has the density function:

$$\frac{1}{\sqrt{2\pi t^3}} e^{-\frac{1}{2t}} 1\!\!1_{\{t>0\}}.$$

Consider the asset process $S = e^{X_t}, t \geq 0$, where X is the process determined by the following stochastic differential equation in the filtration $\mathbb{G}$ generated by β and by $\tau \in \mathcal{G}_0$:

$$(*) \begin{cases} dX_t &= d\beta_t - 1\!\!1_{\{t \leq \tau\}} \frac{1}{1-X_t} \left(1 - \frac{(1-X_t)^2}{\tau - t}\right) dt \\ X_0 &= 0 \end{cases}$$

We can check that this equation has a semimartingale solution X, which is related with τ by the relation $\tau = \inf\{s \geq 0 : X_s = 1\}$. People who operates in the filtration $\mathbb{G}$ on a horizon $[0, T]$ has an arbitrage strategy : buy S at time 0 if $\tau \leq T$ (recall that $\{\tau \leq T\} \in \mathcal{G}_0$) and sell it at the time τ.

However, for a market agent who observes only the asset S alone, he will find a nice Black-Scholes market. This is because X in its natural filtration is a brownian motion, according to [Jeulin Lemme(3.25)].

With different perceptions of the price process S, the two agents will have two different trading systems. The model seems well-defined and corresponding to what we wan. There is nevertheless something strange : the first market agent is granded an arbitrage opportunity, while the second agent is protected in perfect hedge in a Black-Scholes market. Who pays then the insider trading ?

Recall that the FTAP can not be applied to exclude this model from the consideration (under the assumption that arbitrage can exist in a normal market). One needs new results to qualify this model.

We present now a second study of market model with different information flows. (It is issued from a common work with Fontana and Jeanblanc.) On a stochastic basis, consider a filtration $\mathbb{F}$, a honest time in $\mathbb{F}$ and the progressive enlargement $\mathbb{G}$ of $\mathbb{F}$ with τ. Let Z be the Azema supermartingale of τ in $\mathbb{F}$ and let ν be the first zero of Z. Suppose that the filtration $\mathbb{F}$ is continuous, complete and satisfies NFLVR = NA+NA1. We have the following results.

i. NA1 holds in $\mathbb{G}$ on the time horizon $[0, \tau]$, but NA fails.

ii. For any $\mathbb{F}$-stopping time σ, NFLVR holds in $\mathbb{G}$ on the time horizon $[0, \tau \wedge \sigma]$, if and only if $\mathbb{Q}[\sigma \geq \nu] = 0$. (We had not at all expected such a result. In particular, when $\nu = \infty$ (which is not absurd), no insider trading will be possible.)

iii. For all $\epsilon > 0$, NA1 holds in $\mathbb{G}$ on the time horizon $[\tau + \epsilon, \infty]$, but NA fails.

iv NA1 fails to hold in $\mathbb{G}$ on the time horizon $[0, \infty)$.

Roughly speaking, "NA fails" means some arbitrage gains possible, while "NA1 fails" means a unbounded arbitrage profit.

This is one of the successful models which modelize the normal arbitrages in the market. This model shows that, in general, the market can be composed with agants working in different information flow (together with their potential arbitrage opportunities), and remains viable. The arbitrage profit of the insider remains generally limited, except he successes to operate at exactly the instance τ (fairly impossible).

Notice that, if this model is taken for the market regulation, one will conclude that, to prevent the market instability from a default at τ, it is enough to stop the trading for a short while after the default. (That seems already a custom, and we have now a theory to approve it.)

Consider again the question of market maker, for whom the arbitrage opportunities yield only limited profits. In the light of the above example, the NA1 would be a sufficient framework for a general theory on the system of market makers.

Pricing with changes of probability measure

It is not surprising to have different trading systems, because of different information flows. But, in default pricing, multiple trading systems exist for quite a different reason.

Let us consider a market model $(\Omega, \mathbb{G}, \mathbb{Q})$ defaultable at τ a totally inaccessible $\mathbb{G}$ stopping time with an intensity process h. In applying the standardized pricing formula ($\mathbb{P}$ being a martingale measure), the price process of a defaultable security at the maturity $T > 0$ (under suitable integrability condition) is given by:

$$\begin{aligned} S_t &= \mathbb{E}\left[e^{-\int_t^\tau r_v dv} R_\tau 1\!\!1_{\{t<\tau\le T\}} + e^{-\int_t^T r_v dv} \xi 1\!\!1_{\{T<\tau\}} \ |\mathcal{G}_t \right] \\ &= \mathbb{E}\left[\int_t^T e^{-\int_t^u r_v dv} R_u h_u 1\!\!1_{\{u\le\tau\}} du + e^{-\int_t^T r_v dv} \xi 1\!\!1_{\{T<\tau\}} \ |\mathcal{G}_t \right], \ t < T, \end{aligned} \tag{1}$$

where r denotes the interest rate, ξ denotes the promised payoff at the maturity, and R denotes the recovery rule at default.

However, it is a very poor formula, as indicated by Duffie-Schroder-Skiadas, because the formula depends explicitly on the default time τ (which can not be calibrated from the market data before the default), while a desirable formula should be a functional of the only market fundamental quantities (the dividend process, the interest rate process, the default intensity process, etc.) that a market participant can obtain from the market data before the default event.

Duffie-Schroder-Skiadas propose a new formula: Let

$$V_t = \mathbb{E}\left[\int_t^T e^{-\int_t^u r_v + h_v dv} R_u h_u du + e^{-\int_t^T r_v + h_v dv} \xi \ |\mathcal{G}_t\right], \ t < T. \tag{2}$$

Suppose $\Delta_\tau V = 0$ on $\{\tau < T\}$. Then, the price process S coincides with V on $[0, \tau \wedge T)$ (V being then a pre-default value process of S).

The formula (2) is very good. It has the required form to be a suitable formula in default pricing and it is a first proof that default pricing is simply the normal pricing, with a new divident Rh and with an increase of the interest rate by h points.

However, the condition $\Delta_\tau V = 0$ has no economical meaning and is difficult to verify. Collin.Dufresne-Goldstein-Hugonnier propose then a different computation. The process

$$M_t = 1\!\!1_{\{t<\tau\}} e^{\int_0^t h_v dv},\ t \geq 0.$$

is in general a local martingale under $\mathbb{Q}$. Suppose it a true martingale and define a new probability measure $\overline{\mathbb{Q}} = M_T.\mathbb{Q}$ on $\mathcal{G}_T$. Then,

$$\begin{array}{rcl} S_t & = & \mathbb{E}\left[\int_t^T e^{-\int_t^u r_v dv} R_u h_u 1\!\!1_{\{u\leq\tau\}} du + e^{-\int_t^T r_v dv} \xi 1\!\!1_{\{T<\tau\}}\ |\mathcal{G}_t\right] \\ & = & 1\!\!1_{\{t<\tau\}} \frac{1}{M_t} \mathbb{E}\left[\left(\int_t^T e^{-\int_t^u r_v + h_v dv} R_u h_u du + e^{-\int_t^T r_v + h_v dv} \xi\right) M_T\ |\mathcal{G}_t\right] \\ & = & 1\!\!1_{\{t<\tau\}} \overline{\mathbb{E}}\left[\int_t^T e^{-\int_t^u r_v + h_v dv} R_u h_u du + e^{-\int_t^T r_v + h_v dv} \xi\ |\mathcal{G}_t\right],\ t < T. \end{array} \quad (3)$$

We have here all the virtue of the first formula without the condition $\Delta_\tau V = 0$.

Collin.Dufresne-Goldstein-Hugonnier's computation has inspired from Schonbucher's works with survival measure. Schonbucher has studied if one can use defaultable security as numéraire and if in this case the duality between numéraire and pricing measure remains again valid. His conclude is affirmative. For example, to price defaultable zero-coupon bond, one can take the price process $\overline{B}$ of the unit defaultable zero-coupon bond as the numéraire and write the other defaultable zero-coupon bond prices X in $\overline{B}$-units as follows:

$$X_t = \mathbb{E}[e^{-\int_0^T r_v dv} \xi 1\!\!1_{\{T<\tau\}} \mid \mathcal{G}_t] = e^{-\int_0^t r_v dv} B_t \overline{\mathbb{E}}[e^{-\int_0^t r_v dv} \xi \mid \mathcal{G}_t], \tag{4}$$

where $\overline{\mathbb{E}}$ denotes the expectation under the probability measure $\overline{\mathbb{P}}$ defined by the density function $\frac{1}{\overline{B}_0} e^{-\int_0^T r_v dv} 1\!\!1_{\{T<\tau\}}$ with respect to the pricing measure $\mathbb{Q}$, called the survival measure.

Four points: *It is an example with multiple trading systems. *It is an example where the different systems yield the same price. *The trading systems are established with changes of probability measure. *The best pricing formula is made in term of singular numéraires and of non equivalent probability changes. It is a good exemple to explain the necessity to accept the "exotic" trading system.

There seems be some fuzziness in the literature around whether the use of singular numéraires and of non equivalent probability change should be considered as exotic or as standard. The recent development of default pricing seems be in favor of the second option. There exist attempts such as Fisher-Pulido-Ruf (with Martingale valuation operator) to officialize the use of singular numéraires, with some advantages and drawbacks (not adapted to our idea of a theory on multiple trading systems).

Another definition of pricing measure

We prefer an approach based on the notion of pricing measure (as Schonbucher has suggested). We need to settle down the singularity. We recall a result of Kunita about pair of probability measures. For any probability measure $\mathbb{P}$, there exists a stopping time $T = \nu^{\mathbb{Q}/\mathbb{P}}$ such that, for any $t \geq 0$, on $\mathcal{G}_t \cap \{t < T\}$, the real world probability measure $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$, while $\mathbb{P}[t \geq T] = 0$ so that $\mathbb{Q}$ is completely singular to $\mathbb{P}$ on $\mathcal{G}_t \cap \{t \geq T\}$. According to this result, any process defined under $\mathbb{P}$ will be well-defined under $\mathbb{Q}$ over the time horizon $[0, T[$.

Definition 1 *Let X be a given gain-and-loss process (the arbitrage process). Let $N \geq 0$ be a given numéraire. Let R be a stopping time. A probability measure $\mathbb{P}$ is call a pricing measure for X in N-numéraire on the time horizon $[0, R[$, if $\nu^{\mathbb{Q}/\mathbb{P}} \geq R$ and if the value in N-numéraire of X is a local martingale under $\mathbb{P}$ (so that it is a martingale measure for $\frac{X}{N}$).*

This notion unifies the previous examples. For instance, the survival measure $\overline{\mathbb{P}}$ of Schonbucher is a pricing measure with

. the arbitrage process is $X_t = \mathbb{E}\left[e^{-\int_t^T r_v dv} \xi \mathbb{1}_{\{T<\tau\}} \, | \mathcal{G}_t\right]$;

. the numraire is $N_t = \overline{B}_t e^{\int_0^t r_v dv}$;

. the explosion time $\nu^{\mathbb{G},\mathbb{Q}/\overline{\mathbb{P}}} = \inf\{s \geq 0 : \overline{B}_s = 0\}$;

. $R = \tau \leq \nu^{\mathbb{G},\mathbb{Q}/\overline{\mathbb{P}}}$ (because $\overline{B}_t > 0$ on $\{t < \tau\}$).

As for the measure $\overline{\mathbb{Q}}$ of Collin.Dufresne-Goldstein-Hugonnier, it is a pricing measure with

. the arbitrage process is $X_t = S_t + e^{\int_0^t r_v + h_v dv} \int_0^t e^{-\int_0^u r_v + h_v dv} R_u h_u du$;

. the numraire is $N_t = e^{\int_0^t r_v + h_v dv}$;

. the explosion time $\nu^{\mathbb{G},\mathbb{Q}/\overline{\mathbb{Q}}} = \tau$;

. $R = \tau = \nu^{\mathbb{G},\mathbb{Q}/\overline{\mathbb{Q}}}$.

We believe that the approach with pricing measure would be better suitable to the study of multiple trading systems. There is a big advantage of the approach with pricing measure: one can then make "the computation in law". A typical example of "the computation in law" is the resolution of a stochastic differential equation by a PDE. If we want to apply formula (4) to price default, we need not to go back under the original probability measure $\mathbb{Q}$. It is enough to know the law of ξ under the survival measure $\overline{\mathbb{P}}$, which may very be possible. (Consider the situation where the model has a Markovian factor process determined by a stochastic differential equation with respect to a Brownian motion.)

The XVA computation

In the previous example, there is one only filtration. We present now an default pricing with two filtrations (issued from works with Crepey). The two different filtrations will yield two trading systems. But we will show that both the trading systems yield the same price process, just like in the previous example.

On a stochastic basis we consider two filtrations $\mathbb{F}$ and $\mathbb{G}$. Let τ be a $\mathbb{G}$ stopping time. We suppose the two following conditions.

Definition 2 condition (B). *There exists a sub-filtration $\mathbb{F} \subset \mathbb{G}$ such that τ is not an $\mathbb{F}$ stopping time and*

$$\mathcal{P}(\mathbb{F}) \cap (0, \tau] = \mathcal{P}(\mathbb{G}) \cap (0, \tau].$$

Definition 3 Condition (A). *Under the condition (B), there exists a probability measure $\mathbb{P}$, equivalent to $\mathbb{Q}$ on $\mathcal{F}_T$, such that, for any $(\mathbb{F}, \mathbb{P})$ local martingale M, $M^{\tau-}$ is a $(\mathbb{G}, \mathbb{Q})$ local martingale on $[0, T]$.*

Lemma 4 *The condition (B) implies $\mathcal{O}(\mathbb{F}) \cap [0, \tau) = \mathcal{O}(\mathbb{G}) \cap [0, \tau)$.*

The relationships $\mathcal{P}(\mathbb{G}) \to \mathcal{P}(\mathbb{F})$ and $\mathcal{O}(\mathbb{G}) \to \mathcal{O}(\mathbb{F})$ will be called "reduction".

We consider a XVA computation in $\mathbb{G}$. (Recall Crepey's talk.) Let $\mathbb{P}$ be the probability in the condition (A) (called an invariance probability measure)

Theorem 5 *(A gentle version) Under the condition (B) and the condition (A), the pre-default value process of the solution of the XVA equation in $\mathbb{G}$ under $\mathbb{Q}$ coincides with the solution of the same (modulo reduction) XVA equation in $\mathbb{F}$ under $\mathbb{P}$ without default.*

It is an important result. This result reassures people (approving some practical custums) in their computation of default pricing, who have the feeling to not have all information about defaults.

Consider another point. One knows that pricing measure may not be unique. One know how to change the numéraire and the pricing measure. But, when one speaks about this non uniqueness, one does not think that the non uniqueness can also come from the change of the filtration. The invariance probability measure $\mathbb{P}$ of condition (A) is a pricing measure in the sense of Definition 1. One also remembers the survival measure as pricing measure (of Schonbucher and of Collin.Dufresne-Goldstein-Hugonnier). We have the following unexpected result. Let Z denote the Azéma supermartingale in $\mathbb{F}$ of τ.

Theorem 6 *Suppose that $Z_T > 0$. Suppose that the survival measure exists. Then, the restriction of the survival measure on $\mathcal{F}_T$ is the invariance probability measure $\mathbb{P}$.*

This theorem shows that Definition 1 is a good definition, because it exhibits a big coherence in default models.

From local martingale deflator to pricing measure

The notion of pricing measure has shown its utility in the previous examples. On the other hand, a lot of market models are founded on the NA1 principle (linked with deflators). We ask naturally if we can also consider a deflator as pricing measure (so that we can make "the computation in law"). The answer to that question is not straightforward. We have to reconsider the very basic notion of market model.

"**Definition**" *A (mathematical) market is an isomorphical equivalence class of stochastic basis with arbitrage processes and pricing formulas. Each member of the equivalence class will be called a market model.*

With this in mind, we would be able to prove the following result.

"**Theorem**" *For any market with a local martingale deflator, it is imbedded into a market model which has a pricing measure.*

A conclusion

To have a general theory dealing with multiple trading systems, we need a tractable representation of the trading systems. The new definition of pricing measure envelop what are necessary for pricing. The previous examples seem show that the pricing measure constitutes a good candidat to represent the various trading systems. A future theory would be, therefore, a theory of pricing measures.

We have already something to do: achieve the theorem in quotes, or define a notion of system coherence in the model of Theorem 6.

Thank you very much