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On SEQUENTIAL TESTING of TWO STATISTICAL HYPOTHESES

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1. Suppose that we have two hypotheses

$$\begin{aligned} &\mathsf{H}_0: \quad dX_t = dB_t, && X_0 = 0, \\ &\mathsf{H}_1: \quad dX_t = \theta_t\,dt + dB_t, && X_0 = 0, \end{aligned}$$

where $(\theta_t)_{t\ge0}$ is interpreted as a signal and the Brownian motion $(B_t)_{t\ge0}$ is interpreted as noise.

We shall consider the sequential scheme $\Delta = \Delta(\tau,\delta)$, where $\tau = \tau(X)$ is a stopping time and $\delta = \delta(X)$ is an $\mathcal{F}_\tau$-measurable function taking two values 0 (if H_0 is true) and 1 (if H_1 is true).

In the book "Statistics of random processes" (R. Liptser and A. Shiryaev) there is the following theorem (Theorem 17.8).

In the class

$$\Delta = \Delta(\tau, \delta) \in \Delta_{\alpha,\beta},$$

where $\Delta_{\alpha,\beta}$ is a class of (τ, δ) such that the *errors of the first and second kind* satisfies

$$\mathsf{P}_1(\delta(X) = 0) \le \alpha, \quad \mathsf{P}_0(\delta(X) = 1) \le \beta$$

(α and β are given constants, $\alpha + \beta < 1$)

there exists a scheme $\tilde{\Delta} = \Delta(\tilde{\tau}, \tilde{\delta}) \in \Delta_{\alpha,\beta}$ which is optimal in the sense that for any other scheme $\Delta = \Delta(\tau, \delta) \in \Delta_{\alpha,\beta}$

$$\mathsf{E}_0 \int_0^{\tilde{\tau}(X)} m_t^2(X)\, ds \le \mathsf{E}_0 \int_0^{\tau(X)} m_t^2(X)\, dt,$$

$$\mathsf{E}_1 \int_0^{\tilde{\tau}(X)} m_t^2(X)\, ds \le \mathsf{E}_1 \int_0^{\tau(X)} m_t^2(X)\, dt,$$

$$m_t(X) := \mathsf{E}_1(\theta_t \,|\, \mathcal{F}_t^X) \quad (\mathsf{P}\text{-a.s.})$$

(we assume that $\theta = (\theta_t)_{t \ge 0}$ and $B = (B_t)_{t \ge 0}$ are independent and $\mathsf{E}_1|\theta_t| < \infty$, $t < \infty$).

In particular, if $\theta_t = \lambda t$, $\lambda \neq 0$, then we have a classical Wald result that there exists a test $(\tilde{\tau}, \tilde{\delta}) \in \Delta_{\alpha,\beta}$ such that

$$\mathsf{E}_1 \tilde{\tau} \leq \mathsf{E}_1 \tau, \quad \mathsf{E}_0 \tilde{\tau} \leq \mathsf{E}_0 \tau$$

for any test $(\tau, \delta) \in \Delta_{\alpha,\beta}$.

Optimal test $(\tilde{\tau}, \tilde{\delta})$ has a form

$$\tilde{\tau}(X) = \inf\{t \colon \lambda_t(X) \notin (A, B)\}, \qquad \tilde{\delta}(X) = \begin{cases} 1, & \lambda_{\tilde{\tau}(X)} \geq B, \\ 0, & \lambda_{\tilde{\tau}(X)} \leq A, \end{cases}$$

where $A = \ln \dfrac{\alpha}{1-\beta}$, $B = \ln \dfrac{1-\alpha}{\beta}$, and

$$\lambda_t(X) = \int_0^t m_s(X)\, dX_s - \frac{1}{2} \int_0^t m_s^2(X)\, ds.$$

In this case

$$\mathsf{E}_0 \int_0^{\tilde{\tau}(X)} m_t^2(X)\, ds = 2\omega(\beta, \alpha),$$
$$\mathsf{E}_1 \int_0^{\tilde{\tau}(X)} m_t^2(X)\, ds = 2\omega(\alpha, \beta),$$

where $\omega(x, y) = (1 - x) \ln \dfrac{1 - x}{y} + x \ln \dfrac{x}{1 - y}$.

For Bayesian test of two hypotheses ($\theta_t = \lambda t$) with $\pi = 1/2$ the answer will be the same.

It is interesting that the Wald test has a form

$$\tilde{\tau} = \inf\{t \geq 0 \colon \pi_t \notin (A, B)\}.$$

Now we will consider a problem of testing two Wald's hypotheses for FINITE time of observation $[0, T]$. In this case optimal test (in Bayesian setting) will have the form

$$\tilde{\tau}_T = \inf\{t \leq T \colon \pi_t \notin (g_0(t), g_1(t))\}.$$

Boundaries are complicated and in principle can be found as solutions to a Stefan problem. Now we remark only that behavior of the boundaries near TERMINAL point T is the following:

$$\lim_{t \uparrow T} \frac{dg_0(t)}{dt} = +\infty, \qquad \lim_{t \uparrow T} \frac{dg_1(t)}{dt} = -\infty$$

(S. Gorban, *Russian Math. Surveys*, **70**:4 (2016)).

2. Now we consider the following two Markov–Gaussian processes:

$$\mathsf{H}_0\colon \quad dX_t = -\gamma X_t\,dt + dB_t, \quad X_0 \sim \mathcal{N}\Big(0, \frac{1}{2\gamma}\Big),$$

$$\mathsf{H}_1\colon \quad dX_t = -\theta X_t\,dt + dB_t, \quad X_0 \sim \mathcal{N}\Big(0, \frac{1}{2\theta}\Big),$$

where $\gamma > 0$, $\theta > 0$, and $\gamma > \theta$. It means that we consider two stationary Ornstein–Uhlenbeck processes and we want to test two hypotheses H_0 and H_1.

As above, by $\Delta_{\alpha,\beta}$ we will denote a class $\Delta = \Delta(\tau,\delta)$ such that $\mathsf{P}_1(\delta(X)=0) \le \alpha$, $\mathsf{P}_0(\delta(X)=1) \le \beta$, $0<\alpha+\beta<1$, and

$$\mathsf{E}_0 \int_0^{\tau(X)} X_s^2\,ds < \infty, \quad \mathsf{E}_1 \int_0^{\tau(X)} X_s^2\,ds < \infty.$$

The central results are the following two theorems.

THEOREM 1. In the class $\Delta = \Delta_{\alpha,\beta}$ there exists the asymptotically optimal test $(\tilde\tau, \tilde\delta)$ such that for any test $(\tau, \delta) \in \Delta_{\alpha,\beta}$

$$\mathsf{E}_0 \int_0^{\tilde\tau(X)} X_s^2\,ds \le \mathsf{E}_0 \int_0^{\tau(X)} X_s^2\,ds + O(\beta \ln(\alpha\beta)),$$
$$\mathsf{E}_1 \int_0^{\tilde\tau(X)} X_s^2\,ds \le \mathsf{E}_1 \int_0^{\tau(X)} X_s^2\,ds + O\left(\alpha \ln(\alpha\beta) + \beta^{\theta/(\gamma-\theta)}\sqrt{\ln\frac{1}{\beta}}\right)$$

as $\alpha \to 0$, $\beta \to 0$. The plan $(\tilde\tau, \tilde\delta)$ is defined by

$$\tilde\tau(X) = \inf\{t \ge 0\colon \lambda_t(X) \notin (A, B)\}, \qquad \tilde\delta(X) = \begin{cases} 1, & \lambda_{\tilde\tau(X)} > B, \\ 0, & \lambda_{\tilde\tau(X)} \le A, \end{cases} \quad (*)$$

where $A = \ln\alpha$, $B = -\ln\beta$, and

$$\lambda_t(X) = \ln\sqrt{\frac{\theta}{\gamma}} + \frac{\gamma-\theta}{2}(X_0^2 + X_t^2 - t) + \frac{\gamma^2-\theta^2}{2}\int_0^t X_s^2\,ds.$$

THEOREM 2. In the class $\Delta = \Delta_{\alpha,\beta}$ there exists the asymptotically optimal test $(\tilde{\tau}, \tilde{\delta})$ such that if

$$\gamma \to \infty, \quad \theta \to \infty, \quad \text{but} \quad \gamma - \theta = c > 0,$$

then for any test $(\tau, \delta) \in \Delta_{\alpha,\beta}$

$$\mathsf{E}_0 \int_0^{\tilde{\tau}(X)} X_s^2\, ds \le \mathsf{E}_0 \int_0^{\tau(X)} X_s^2\, ds + O\left(\gamma^{-5/2}\left(\frac{\beta}{1-\alpha}\right)^{\gamma/c}\right),$$

$$\mathsf{E}_1 \int_0^{\tilde{\tau}(X)} X_s^2\, ds \le \mathsf{E}_1 \int_0^{\tau(X)} X_s^2\, ds + O\left(\gamma^{-5/2}\left(\frac{\beta}{1-\alpha}\right)^{\gamma/c}\right),$$

where $0 < \alpha + \beta < 1$, and the plan $(\tilde{\tau}, \tilde{\delta})$ is defined by $(*)$ with

$$A = \ln \frac{\alpha}{1-\beta}, \quad B = \ln \frac{1-\alpha}{\beta}.$$

3. The proof of these theorems is based on Lemmas 1–5.

LEMMA 1. The following inequalities hold:

$$\mathsf{E}_0 \ln \frac{d\mathsf{P}_0}{d\mathsf{P}_1}(\tau, X) \ge \omega(\beta, \alpha),$$
$$\mathsf{E}_1 \ln \frac{d\mathsf{P}_1}{d\mathsf{P}_0}(\tau, X) \ge \omega(\alpha, \beta),$$

where

$$\omega(x, y) = (1 - x) \ln \frac{1 - x}{y} + x \ln \frac{x}{1 - y}.$$

LEMMA 2. For the plan $(\tilde{\tau}, \tilde{\delta})$ defined in $(*)$, where A and B are arbitrary but such that $A \le \ln\sqrt{\theta/\gamma} \le B$, we have

$$\mathsf{E}_0 \int_0^{\tilde{\tau}(X)} X_s^2\, ds = G_0(\theta, \gamma, A, B), \qquad \mathsf{E}_1 \int_0^{\tilde{\tau}(X)} X_s^2\, ds = G_1(\theta, \gamma, A, B),$$

where

$$\begin{aligned} G_1(\theta, \gamma, A, B) = \frac{2}{(\gamma-\theta)^2}\Bigg[&-\frac{A-B}{e^A - e^B} f(\gamma, B) \\ &- \left(\ln\sqrt{\frac{\theta}{\gamma}} + \frac{\gamma-\theta}{2\gamma} + \frac{Be^B - Ae^A}{e^A - e^B} \right) f(\theta, B) \\ &- \sqrt{\frac{(\gamma-\theta)(B - \ln\sqrt{\theta/\gamma})}{\pi\theta}} \exp\left\{ -\frac{\theta(B - \ln\sqrt{\theta/\gamma})}{\gamma-\theta} \right\} \Bigg] \end{aligned}$$

$$f(x,y) = \mathrm{Erf}\left(\sqrt{\frac{x(y - \ln\sqrt{\theta/\gamma})}{\gamma-\theta}} \right), \qquad \mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-v^2/2}\, dv$$

and the expression for $G_0(\theta, \gamma, A, B)$ is similar.

PROOF of Lemma 2 is based on Itô's formula for X_t^2. For example, with respect to P_1

$$X_t^2 = X_0^2 + t - 2\theta \int_0^t X_s^2\, ds + \int_0^t X_s\, dB_s,$$

whence for

$$\lambda_t(X) = \ln \frac{d\mathsf{P}_1}{d\mathsf{P}_0}(t, X) = \ln \sqrt{\frac{\theta}{\gamma}} + \frac{\gamma - \theta}{2}(X_0^2 + X_t^2 - t) + \frac{\gamma^2 - \theta^2}{2} \int_0^t X_s^2\, ds$$

we obtain that (P_1-a.s.)

$$\lambda_t(X) = \ln \sqrt{\frac{\theta}{\gamma}} + (\gamma - \theta)\left(X_0^2 + \int_0^t X_s^2\, dB_s\right) + \frac{(\gamma - \theta)^2}{2} \int_0^t X_s^2\, ds.$$

Denote by $g = g(x)$ the solution to the differential equation

$$g''(x) - g'(x) = 1, \qquad g(A) = 0, \quad g(B) = 0.$$

Then $g(x) = (e^A - e^B)^{-1}\Big[Be^A - Ae^B + (A-B)e^x - (e^A - e^B)x\Big]$, and applying Itô's formulas, we find

$$\mathsf{E}_1 g(\lambda_{\tilde{\tau}(X)}(X)) = \mathsf{E}_1 g(\lambda_0(X)) + \frac{(\gamma-\theta)^2}{2}\mathsf{E}_1 \int_0^{\tilde{\tau}(X)} X_s^2\, ds.$$

But

$$\lambda_{\tilde{\tau}(X)}(X) = \begin{cases} A \text{ or } B, & \lambda_0 \in [A,B], \\ \lambda_0(X) \equiv \ln\sqrt{\theta/\gamma} + (\gamma-\theta)X_0^2, & \lambda_0 \notin [A,B]. \end{cases}$$

So, for case $A \le \ln\sqrt{\theta/\gamma} \le B$ we have

$$\frac{(\gamma-\theta)^2}{2}\mathsf{E}_1 \int_0^{\tilde{\tau}(X)} X_s^2\, ds = \mathsf{E}_1\Big(-g(\lambda_0(X)) I\Big\{\lambda_0(X) \in \Big[\ln\sqrt{\frac{\theta}{\gamma}}, B\Big]\Big\}\Big).$$

Integration of the right-hand side leads to the given formulas.

The following Lemma 3 gives, for the plan $(\tilde{\tau}, \tilde{\delta})$, the structure of the errors $\mathsf{P}_1(\tilde{\delta}(X) = 0)$ and $\mathsf{P}_0(\tilde{\delta}(X) = 1)$.

LEMMA 3. **(a)** If $B \geq A \geq \ln\sqrt{\theta/\gamma}$, then

$$\mathsf{P}_1(\tilde{\delta}(X) = 0) = f(\theta, A) + \frac{e^A}{e^A - e^B}\Big[e^B(f(\gamma, A) - f(\gamma, B)) - (f(\theta, A) - f(\theta, B))\Big],$$

$$\mathsf{P}_0(\tilde{\delta}(X) = 1) = 1 - f(\gamma, B) + \frac{1}{e^A - e^B}\Big[e^A(f(\gamma, B) - f(\gamma, A)) - (f(\theta, A) - f(\theta, B))\Big].$$

(b) If $A \le \ln\sqrt{\theta/\gamma} \le B$, then

$$\mathsf{P}_1(\tilde{\delta}(X) = 0) = \frac{e^A}{e^A - e^B}\Big[f(\theta, B) - e^B f(\gamma, B)\Big],$$

$$\mathsf{P}_0(\tilde{\delta}(X) = 1) = 1 - f(\gamma, B) + \frac{1}{e^A - e^B}\Big[e^A f(\gamma, B) - f(\theta, B)\Big].$$

Again, the proof is based on Itô's formula.

We observe that

$$f(x,t) = 1 - c(x)\, y^{-1/2}\,\mathrm{exp}\left\{-\frac{xy}{\gamma-\theta}\right\}(1+O(y^{-1})), \qquad x>0,\ y\to\infty,$$

$$\text{with} \quad c(x) = \sqrt{\frac{\gamma-\theta}{\pi x}}\left(-\frac{\theta}{\gamma}\right)^{\theta/(2(\gamma-\theta))};$$

$$f(\theta,y) = 1 - \frac{\sqrt{c}\, e^{y-1/2}}{\sqrt{\pi y}}\,\gamma^{-1/2}\,\mathrm{exp}\left\{-\frac{\gamma y}{c}\right\}(1+O(\gamma^{-1})),$$

$$\theta\to\infty, \qquad \gamma-\theta = c;$$

$$f(\gamma,y) = 1 - \frac{\sqrt{c}\, e^{-1/2}}{\sqrt{\pi y}}\,\gamma^{-1/2}\,\mathrm{exp}\left\{-\frac{\gamma y}{c}\right\}(1+O(\gamma^{-1})),$$

$$\gamma\to\infty, \qquad \gamma-\theta = c.$$

LEMMA 4. For the plan $\tilde{\Delta} = (\tilde{\tau}, \tilde{\delta})$ we have as $A \to -\infty$, $B \to \infty$:

$$\frac{(\gamma-\theta)^2}{2} \mathsf{E}_0 \int_0^{\tilde{\tau}(X)} X_s^2\, ds = \ln \sqrt{\frac{\theta}{\gamma}} + \frac{\gamma-\theta}{2\gamma} - A\Big[1 + O\Big(\Big(1 - \frac{B}{A}\Big) e^{-B}\Big)\Big];$$

$$\frac{(\gamma-\theta)^2}{2} \mathsf{E}_1 \int_0^{\tilde{\tau}(X)} X_s^2\, ds = -\ln \sqrt{\frac{\theta}{\gamma}} - \frac{\gamma-\theta}{2\theta}$$

$$+ B\Big[1 + O\Big(\Big(1 - \frac{A}{B}\Big) e^{A} + B^{-1/2} \exp\Big\{-\frac{\theta B}{\gamma-\theta}\Big\}\Big)\Big].$$

(The proof follows from Lemma 2 and above expressions for functions $f(\theta, B)$ and $f(\gamma, B)$.)

LEMMA 5. If $A \to -\infty$, $B \to \infty$, then

$$\mathsf{P}_1(\tilde{\delta}(X) = 0) = e^{A}(1 + O(e^{-B})),$$
$$\mathsf{P}_0(\tilde{\delta}(X) = 1) = e^{-B}\Big[1 + O\Big(e^{A} + B^{-1/2}\exp\Big\{-\frac{\theta B}{\gamma - \theta}\Big\}\Big)\Big].$$

The following lemma deals with the case when $\gamma, \theta \to \infty$, $\gamma - \theta = c$.

LEMMA 6. Under the assumptions $\gamma, \theta \to \infty$, $\gamma - \theta = c$ we have:

$$\frac{(\gamma-\theta)^2}{2} \mathsf{E}_0 \int_0^{\tilde{\tau}(X)} X_s^2 \, ds = \ln \sqrt{\frac{\theta}{\gamma}} + \frac{\gamma-\theta}{2\gamma} + \frac{Ae^B - Be^A - (A-B)}{e^A - e^B} \Big[1 + O\Big(\gamma^{-5/2} e^{-\gamma\theta/c} \Big) \Big]$$

$$\frac{(\gamma-\theta)^2}{2} \mathsf{E}_1 \int_0^{\tilde{\tau}(X)} X_s^2 \, ds = -\ln \sqrt{\frac{\theta}{\gamma}} - \frac{\gamma-\theta}{2\theta} - \frac{(A-B)e^{A+B} + Be^B - Ae^A}{e^A - e^B} \Big[1 + O\Big(\gamma^{-5/2} e^{-\gamma B/c} \Big) \Big],$$

and

$$\mathsf{P}_0(\tilde{\delta}(X) = 1) = \frac{e^A - 1}{e^A - e^B} \Big[1 + O\Big(\gamma^{-1/2} e^{-\gamma B/c} \Big) \Big],$$

$$\mathsf{P}_1(\tilde{\delta}(X) = 0) = \frac{e^A(1 - e^B)}{e^A - e^B} \Big[1 + O\Big(\gamma^{-3/2} e^{-\gamma B/c} \Big) \Big].$$

PROOF of THEOREMS 1 and 2. If we take $A = \ln \alpha$, $B = -\ln \beta$, then for the plan $(\tilde{\tau}, \tilde{\delta})$ with

$$\tilde{\tau}(X) = \inf\{t \geq 0 \colon \lambda_t(X) \notin (A, B)\}, \qquad \tilde{\delta}(X) = \begin{cases} 1, & \lambda_{\tilde{\tau}(X)} \geq B, \\ 0, & \lambda_{\tilde{\tau}(X)} \leq A, \end{cases}$$

where $\lambda_t(X) = \ln \sqrt{\dfrac{\theta}{\gamma}} + \dfrac{\gamma - \theta}{2}(X_0^2 + X_t^2 - t) + \dfrac{\gamma^2 - \theta^2}{2} \displaystyle\int_0^t X_s^2 \, ds$, we obtain from Lemma 5 that

$$\begin{aligned} \mathsf{P}_0(\tilde{\delta}(X) = 1) &= \beta\Big[1 + O\Big(\alpha + (-\ln \beta)^{-1/2} \beta^{\theta/(\gamma-\theta)}\Big)\Big], \\ \mathsf{P}_1(\tilde{\delta}(X) = 0) &= \alpha(1 + O(\beta)). \end{aligned}$$

Thus $\tilde{\Delta} = (\tilde{\tau}, \tilde{\delta}) \in \Delta_{\alpha,\beta} \quad (\alpha, \beta \to 0)$.

Lemma 1 says that

$$\mathsf{E}_0 \ln \frac{d\mathsf{P}_0}{d\mathsf{P}_1}(\tau, X) \geq \beta \ln \frac{\beta}{1-\alpha} + (1-\beta) \ln \frac{1-\beta}{\alpha}. \qquad (**)$$

If we denote $\ln \frac{d\mathsf{P}_0}{d\mathsf{P}_1}(\tau, X)$ by $\lambda_\tau(X)$, then

$$\lambda_\tau(X) = \ln \sqrt{\frac{\theta}{\gamma}} + (\gamma-\theta)\left(X_0^2 + \int_0^\tau X_s^2\, dB_s\right) + \frac{(\gamma-\theta)^2}{2} \int_0^\tau X_s^2\, ds. \quad (***)$$

Using $(**)$ and $(***)$ we may get for $\mathsf{E}_0 \int_0^\tau X_s^2\, ds$ the bound from below via $\alpha, \beta, \theta, \gamma$. But the previous lemmas give also the expression of $\mathsf{E}_0 \int_0^{\tilde{\tau}} X_s^2\, ds$ via $\alpha, \beta, \theta, \gamma$. This provides a possibility to obtain

$$\mathsf{E}_0 \int_0^{\tau(X)} X_s^2\, ds \geq \mathsf{E}_0 \int_0^{\tilde{\tau}(X)} X_s^2\, ds + \begin{cases} O(\beta \ln(\alpha\beta)), & \alpha, \beta \to 0; \\ O\left(\gamma^{-5/2}\left(\frac{\beta}{1-\alpha}\right)^{\gamma/c}\right), & \gamma, \theta \to \infty, \quad \gamma - \theta = c. \end{cases}$$

Similar inequality can be obtained under the measure P_1.