Samuelson revisited — a new FTAP without a fixed numéraire

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Introduction

A motivating example

- **Question:**
  - Do we really know how to define an **arbitrage-free market?**
  - In very simple examples, this is not so clear after all . . .
- **Example:** $N = 2$ assets (and no bank account), given by

$$\mathbf{S}_t^i = \exp\left(\sigma_i W_t^i + (m_i - \sigma_i^2/2)t\right) \quad \text{for } t \geq 0,\ i = 1, 2,$$

with possibly ρ-correlated Brownian motions W^1, W^2.

- Is this arbitrage-free? In which sense?
- Usually, pass to **discounted prices**. But — which of the two symmetric assets to use here?

A motivating example (cont'd)

- Suppose parameters satisfy $m_2 - m_1 + \sigma_1^2 - \rho\sigma_1\sigma_2 = 0$. Look at $X = \mathbf{S}^2/\mathbf{S}^1$, set $X' = 1/X$. Simple computation shows that

$$X \textbf{ is a nonnegative martingale with } \lim_{t\to\infty} X_t = 0 \ P\textbf{-a.s.}$$

- If we discount prices by $\mathbf{S}^1$, then discounted model $(1, X)$ is **arbitrage-free** because it satisfies NFLVR.
- If we discount prices by $\mathbf{S}^2$, then discounted model $(X', 1)$ is **not arbitrage-free** — we even have $\lim_{t\to\infty} X'_t = +\infty$ P-a.s.
- Is there any reason to choose one of the symmetric assets for discounting? Not really ...
- **So — how do we define "arbitrage-free" here?**

Basic goals

- Start with **general model** for frictionless financial market with N asset prices on **stochastic** interval

$$[\![0, T]\!] = \{(\omega, t) \in \Omega \times [0, \infty) : 0 \leq t \leq T(\omega)\}.$$

  - (This includes models on finite interval $[0, T]$ as well as models on $[0, \infty)$ with infinite horizon.)
- Find economically reasonable **definition for arbitrage-free market** in this setting.
- Give **dual characterisation** in terms of some martingale properties.
- Illustrate results by **examples**.

Framework

Basic setup

- N assets, described by $\mathbb{R}^N$-valued **semimartingale** $\mathbf{S} = (\mathbf{S}_t) = (\mathbf{S}_t^1, \dots, \mathbf{S}_t^N)$, where $\mathbf{S}_t^i$ is time-t price of asset i.
  - **If** there is a riskless asset, it must be part of $\mathbf{S}$. Not assumed in general (see example above).
  - Prices are not discounted by anything.
  - **Special case** *is* **classic setup** *with* $N = 1 + d$ *and* $\mathbf{S} = (1, X)$ *for an* $\mathbb{R}^d$*-valued semimartingale* X *(bank account and risky assets, already discounted).*
  - Later, several (mild) conditions on $\mathbf{S}$ will appear.

- Sometimes, we want (or need) to **change accounting unit** via process ("numéraire") $D = (D_t)$ to new prices $S = \mathbf{S}/D$. Then always assume $D_0 = 1$, $D > 0$ and $D_- > 0$.

Basic setup (cont'd)

- As usual, strategies $\vartheta = (\vartheta_t)$ are **self-financing**, with **wealth**

$$\mathbf{V}_t(\vartheta) = V_t(\vartheta)[\mathbf{S}] := \vartheta_t \cdot \mathbf{S}_t = \vartheta_0 \cdot \mathbf{S}_0 + \vartheta \bullet \mathbf{S}_t = \vartheta_0 \cdot \mathbf{S}_0 + \int_0^t \vartheta \, d\mathbf{S}.$$

  - *In the classic setup with* $\mathbf{S} = (1, X)$, *we can identify* ϑ *with a pair* (v_0, H) *and get wealth in the familiar form as* $\mathbf{V}_t(v_0, H) = v_0 + \int_0^t H \, dX$.

- For **admissibility**, impose that $\mathbf{V}(\vartheta) \geq 0$ and write $\vartheta \in L_+^{\mathrm{sf}}$.

- Extend all processes to $[0, \infty)$ by keeping them constant after T. [Small technical detail about strategies ϑ ...]

Possible conditions on **S**

- Consider **market portfolio** $\mathbf{1} = (1, \dots, 1) \in L^{\mathrm{sf}}$ of holding one unit of each asset.
- More generally, can consider **"reference portfolio"** $\eta \in L^{\mathrm{sf}}$.
- **(C1)** $\exists\ \eta^* \in L^{\mathrm{sf}}$ satisfying

$$0 < \inf_{t \geq 0} \mathbf{V}_t(\eta^*) \leq \sup_{t \geq 0} \mathbf{V}_t(\eta^*) < \infty \quad P\text{-a.s.}$$

- **(C2)** Market portfolio satisfies

$$0 < \inf_{\mathbf{t \geq 0}} \mathbf{V}_t(\mathbf{1}) \leq \sup_{\mathbf{t \geq 0}} \mathbf{V}_t(\mathbf{1}) < \infty \quad P\text{-a.s.}$$

  - Clearly (C2) implies (C1).
  - Equally clearly, both are highly restrictive — just think of GBM model on $[0, \infty)$ from initial example.

Possible conditions on $\mathbf{S}$ (cont'd)

- **(C2′)** Market portfolio $\mathbf{1} = (1, \dots, 1) \in L^{\mathrm{sf}}$ satisfies, for all $T \in (0, \infty)$,

$$0 < \inf_{\mathbf{0 \le t \le T}} \mathbf{V}_t(\mathbf{1}) \le \sup_{\mathbf{0 \le t \le T}} \mathbf{V}_t(\mathbf{1}) < \infty \quad P\text{-a.s.}$$

- **(C3)** $\mathbf{S} \ge 0$ P-a.s.
- **Equivalent formulation of (C2′):** Total **market value** $\mathbf{V}(\mathbf{1}) = \mathbf{1} \cdot \mathbf{S} = \sum_{i=1}^{N} \mathbf{S}^i$ satisfies $\mathbf{1} \cdot \mathbf{S} > 0$ and $\mathbf{1} \cdot \mathbf{S}_- > 0$ on $[0, \infty)$ (*uniformly on compact intervals*, but not necessarily uniformly over $t \ge 0$).
  - Condition (C2′) looks reasonable. We cannot work with it (at least not yet ...) without also having (C3).
  - *Both (C2′) and (C3) are always satisfied in the classic setup* $\mathbf{S} = (1, X)$ *if* $X \ge 0$.

Key idea for definitions

- **Basic idea:** a market deserves to be called **"arbitrage-free"** if it is inherently stable — total inactivity in trading cannot be improved.
- Put differently: the strategy $\vartheta \equiv 0$ of doing nothing cannot be beaten by another strategy — it is **"maximal"** in some sense.
- **Key question:** What is a good concept of a strategy being maximal?

Strong maximality for **S**

- **Classic concept:** strategy $\vartheta \in L_+^{\mathrm{sf}}$ is **strongly maximal (sm) for S** in L_+^{sf} if there is no (nontrivial payoff) $f \in L_+^0 \setminus \{0\}$ such that for every $\epsilon > 0$, there is $\hat{\vartheta}^\epsilon \in L_+^{\mathrm{sf}}$ with
  - $V_0(\hat{\vartheta}^\epsilon)[\mathbf{S}] \le V_0(\vartheta)[\mathbf{S}] + \epsilon$,
  - $\liminf_{t\to\infty} V_t(\hat{\vartheta}^\epsilon - \vartheta)[\mathbf{S}] \ge f$ P-a.s.
- *(If we add to ϑ some nontrivial payoff at ∞, total time-0 superreplication price must exceed time-0 value of ϑ.)*
- (This is familiar concept used in similar forms by several authors.)

Strong index weight maximality

- **Our new concept:** strategy $\vartheta \in L^{\mathrm{sf}}_+$ is **strongly index weight maximal (siwm)** in L^{sf}_+ if
  - there is no $[0,1]$-valued adapted process $\psi = (\psi_t)_{t \geq 0}$ converging P-a.s. to some $\psi_\infty \in L^0_+ \setminus \{0\}$ and such that
  - for every $\epsilon > 0$, there is some $\hat{\vartheta}^\epsilon \in L^{\mathrm{sf}}_+$ with
    - $\mathbf{V}_0(\hat{\vartheta}^\epsilon) \leq \mathbf{V}_0(\vartheta) + \epsilon$,
    - $\liminf_{t\to\infty}(\hat{\vartheta}^\epsilon_t - \vartheta_t - \psi_t \mathbf{1}) \geq 0$ P-a.s.
- *(ψ is long-only portfolio which stabilises over time and produces significant share of market portfolio. If we add to ϑ such a desirable portfolio, total time-0 superreplication cost must exceed time-0 value of ϑ.)*

Comparison of concepts

- **Common property:** maximal strategy can only be improved at nonzero initial cost.
- **Key difference:**
  - for **traditional** concept, improvement is in terms of **wealth**.
  - for **new** concept, improvement is in terms of some **reference strategy** (here, the market portfolio $\mathbf{1}$).
- **Important consequence:** new concept is discounting-invariant:
  - Suppose we **change units** with process (D_t) with $D_0 = 1$ and $D > 0$, $D_- > 0$ on $[0, \infty)$, to get $S = \mathbf{S}/D$.
  - Then **sm for S does not imply sm for** S.
  - But **siwm (for S) is equivalent to siwm (for** S**)**.

Technical comment

- Define **superreplication prices**, for a payoff $f \in L^0_+$ and for a portfolio $\psi_\infty \in L^0_+$, by

$$\pi_s(f) := \inf\Big\{ v_0 \in \mathbb{R} : \exists \hat{\vartheta} \in L^{\mathrm{sf}} \text{ with } \mathbf{V}_0(\hat{\vartheta}) \le v_0 \text{ and } \liminf_{t\to\infty} \mathbf{V}_t(\hat{\vartheta}) \ge f \;\; P\text{-a.s.}\Big\},$$

$$\tilde{\pi}_s(\psi_\infty) := \inf\Big\{ v_0 \in \mathbb{R} : \exists \hat{\vartheta} \in L^{\mathrm{sf}} \text{ with } \mathbf{V}_0(\hat{\vartheta}) \le v_0 \text{ and } \liminf_{t\to\infty} \hat{\vartheta}_t \ge \psi_\infty \mathbf{1} \;\; P\text{-a.s.}\Big\}.$$

- Then we have (under (C2′) and (C3)):
  - $\vartheta \equiv 0$ **sm for S** $\Longleftrightarrow \pi_s(f) > 0$ for any $f \in L^0_+ \setminus \{0\}$,
  - $\vartheta \equiv 0$ **siwm** $\Longleftrightarrow \tilde{\pi}_s(\psi_\infty) > 0$ for any $\psi_\infty \in L^0_+ \setminus \{0\}$.
- But this does not work well for $\vartheta \not\equiv 0$ ...

Results and examples

Main results I

- **Theorem:** Under (restrictive condition) **(C1)** with reference portfolio η^*:

$$0 \in L_+^{\mathrm{sf}} \text{ is } \mathbf{sm\ for\ S}$$

$$\Longleftrightarrow$$

$$\mathbf{S}^{(\eta^*)} = \mathbf{S}/\mathbf{V}(\eta^*) \text{ satisfies } \mathbf{NUPBR}$$

$$\Longleftrightarrow$$

$\exists$ semimartingale D with $0 < \inf_{t \geq 0} D_t \leq \sup_{t \geq 0} D_t < \infty$ P-**a.s.**
such that $S = \mathbf{S}/D$ **is σ-martingale**
(i.e. D is **narrow σ-martingale deflator**).

  - *Extension of* **(Herdegen) FTAP** *to infinite horizon.*
  - But: condition (C1) is much too restrictive ...

Main results II

- **Theorem:** Under (restrictive condition) **(C2)** on **1**:

  1) siwm always **implies** sm for **S**.

  2) If we add condition **(C3)** (nonnegativity), sm for **S** also **implies** siwm.

- **Technical core of results.**
- Uses variation of Delbaen/Schachermayer theorem to prove

$$\text{existence of } \lim_{t\to\infty} V_t(\vartheta)[\mathbf{S}^{(\eta^*)}] \text{ for } \vartheta \in L_+^{\mathrm{sf}},$$

  if (AOA condition) 0 is sm for **S** and we have (C1) with η^*.
- **Key trick:** result allows us to pass from original prices **S** to **market weights** $\mu := \mathbf{S}/\sum \mathbf{S}^i$ and back.

Main results III

- **Theorem (FTAP):** Under (good condition) **(C2′) and (C3)**:

$$0 \in L^{\mathrm{sf}}_{+} \text{ is } \textbf{siwm}$$

$$\Longleftrightarrow$$

Market weight process $\mu = \mathbf{S}/\sum \mathbf{S}^i$ satisfies **NUPBR**

$$\Longleftrightarrow$$

$\exists$ semimartingale D with $D > 0$ and $D_{-} > 0$ on $[0, \infty)$ P-a.s.
such that $S = \mathbf{S}/D$ **is σ-martingale**
(i.e. D is **σ-martingale deflator**)
and S *[but perhaps not* **S***]* satisfies (strong) condition **(C2)**.

Comments

- **Terminology:** **S** satisfies **dynamic index weight viability (DIWV)** if **zero strategy** $0 \in L_+^{\mathrm{sf}}$ is **strongly index weight maximal (siwm)** in L_+^{sf}.
- So we have **new FTAP** for **AOA condition DIWV**.
- Structure of result:
  - Primal AOA condition does not depend on chosen accounting units (discounting-invariant).
  - Dual characterisation gives martingale property for prices in **some** accounting units — which **cannot be chosen a priori!**
  - For a general model, **classic absence of arbitrage depends on discounting**, but **our formulation does not**.
  - In the spirit of Samuelson (1965), "**properly anticipated prices fluctuate randomly**" — but the proper discounting is part of the dual characterisation, not an a priori choice!

Examples

- **Example 1:** Model from motivation with $N = 2$ possibly correlated assets given by

$$\mathbf{S}_t^i = \exp\left(\sigma_i W_t^i + (m_i - \sigma_i^2/2)t\right) \quad \text{for } t \geq 0,\ i = 1, 2.$$

- This $\mathbf{S}$ satisfies **DIWV** if and only if

$$m_i - \sigma_i^2 + \rho\sigma_1\sigma_2 = m_{3-i} \quad \text{for } i = 1 \text{ or } i = 2.$$

- Equivalently, one of $\mathbf{S}/\mathbf{S}^1, \mathbf{S}/\mathbf{S}^2$ must be a martingale.

Examples (cont'd)

- **Example 2:** Black–Scholes model given by

$$\mathbf{S}^1_t = \exp(rt) \quad \text{for } t \geq 0,$$
$$\mathbf{S}^2_t = \exp\big(\sigma W_t + (m - \sigma^2/2)t\big) \quad \text{for } t \geq 0,$$

with $\sigma > 0$.

- This $\mathbf{S}$ satisfies **DIWV** if and only if

$$\frac{m-r}{\sigma^2} \in \{0, 1\}.$$

- Equivalently, one of $\mathbf{S}/\mathbf{S}^1, \mathbf{S}/\mathbf{S}^2$ must be a martingale.

What else?

- Many counterexamples for possible, but wrong implications
- Can replace market portfolio $\mathbf{1}$ by another "desirable reference portfolio" η; under suitable (natural) assumptions, DIWV($\mathbf{1}$) and DIWV(η) are then equivalent
- Connection to classic framework and results, including discussion of related literature
- Questions: ... are welcome ...

The end

Thank you for your attention

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