Fractional Poisson process: long-range dependence and applications in ruin theory

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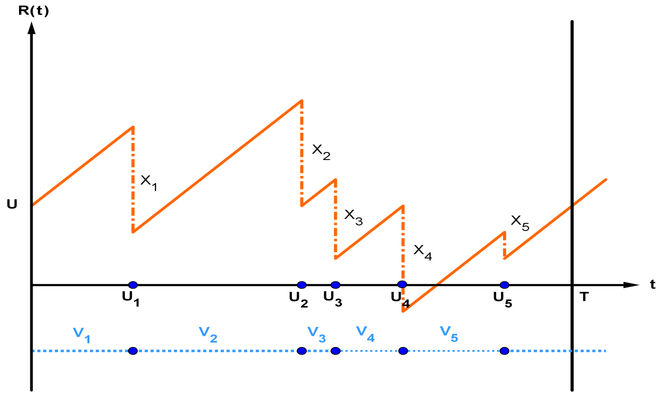
Outline

1. Introduction
2. Fractional Poisson process
3. Direct applications in ruin theory

Risk process : Insurance company's reserve evolution

$$R(t) = u + ct - \sum_{i=1}^{N(t)} X_i \ .$$



Classical assumptions

$$R(t) = u + ct - \sum_{i=1}^{N(t)} X_i \ ,$$

where

- $(N(t))_{t \geq 0}$: Poisson process with parameter $\lambda > 0$.

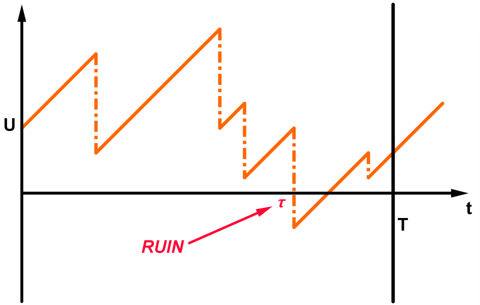
  $\hookrightarrow$ Claim inter-occurrence times $(V_i)_{i \geq 1}$: sequence of independent and exponentially distributed with parameter λ random variables.

- Claim amounts $(X_i)_{i \geq 1}$: sequence of independent and identically distributed positive random variables.

- $(X_i)_{i \geq 1}$ is independent from $(V_i)_{i \geq 1}$.

Remark : by convention, $\sum_{i=1}^{N(t)} X_i = 0$ if $N(t) = 0$.

Classical problems



- Finite-time ruin probability:

$$\psi(u,T) = P(\exists\tau \in [0,T]\ ,\ R(\tau) < 0 | R(0) = u),$$

- and infinite-time ruin probability:

$$\psi(u) = \lim_{T\to\infty} \psi(u,T).$$

Light-tailed vs Heavy-tailed

Light-tailed	Heavy-tailed
A random variable X is said light-tailed if $$\exists r > 0\,,\ \mathbf{E}\left[\mathrm{e}^{rX}\right] < +\infty\,.$$ Examples : exponential, gamma, Weibull with shape parameter greater than 1.	A random variable X is said heavy-tailed if $$\forall r > 0\,,\ \mathbf{E}\left[\mathrm{e}^{rX}\right] = +\infty\,.$$ Examples : lognormal, Pareto, Burr, Weibull with shape parameter less than 1.

Subexponential distribution

A distribution $K \in \mathbb{R}_+$ is said to be subexponential if, with $\overline{K} = 1 - K$,

$$\lim_{x\to\infty} \frac{\overline{K*K}(x)}{\overline{K}(x)} = 2\,.$$

We denote $K \in \mathcal{S}$.

In particular, if $X_1, \ldots, X_n$ are i.i.d. with distribution K, then

$$\mathbf{P}(X_1 + \ldots + X_n > x) \sim \mathbf{P}(\max(X_1, \ldots, X_n) > x) \sim n\bar{K}(x)\,,\; x \to \infty\,.$$

"Principle of a single big jump"

Examples : Log-normal, Pareto, Burr,...

Regularly varying distribution

A distribution $K \in \mathbb{R}_+$ is said to be regularly varying with index $\alpha \geq 0$ if, with $\overline{K} = 1 - K$,

$$\lim_{x \to \infty} \frac{\overline{K(tx)}}{\overline{K}(x)} = t^{-\alpha}.$$

We denote $K \in \mathbb{R}_{-\alpha}$.

In particular, there exists a function $L \in \mathbb{R}_0$ such that

$$\overline{K}(x) = L(x) x^{-\alpha}.$$

Examples : Pareto, Burr,...

Mittag-Leffler distribution

V is Mittag-Leffler distributed with parameters $\lambda > 0$ and $\mathrm{H} \in (0, 1]$ if

$$\mathbf{P}(V > t) = E_{\mathrm{H}}(-\lambda t^{\mathrm{H}})\,, \text{for } t \geq 0$$

where

$$E_{\mathrm{H}}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1 + \mathrm{H}k)}$$

is the Mittag-Leffler function (Γ denotes the Euler's Gamma function) which is defined for any complex number z.

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It holds that

$$\mathbb{E}\left(e^{-\xi V}\right) = \frac{\lambda}{\lambda + \xi^H} \, .$$

Definition 1 : Renewal process

$$N_{\mathrm{H}}(t) = \max\{n \geq 0 \ : \ U_n \leq t\} = \sum_{k \geq 1} \mathbb{1}_{U_k \leq t} \, ,$$

with

- $U_n = \sum_{k=1}^{n} V_k$ for $n \geq 1$;
- and $(V_k)_{k \geq 1}$ are i.i.d. with Mittag-Leffler distribution with parameters $\lambda > 0$ and $\mathrm{H} \in (0, 1]$.

$\Rightarrow$ $(N_{\mathrm{H}}(t))_{t \geq 0}$ is a fractional Poisson process with parameters λ and H.

Definition 2 : Time-changed usual Poisson process

Let :

- $(N(t))_{t\geq 0}$ be a Poisson process with parameter $\lambda > 0$;
- $(E_{\mathrm{H}}(t))_{t\geq 0}$ be the right continuous inverse of a standard H-stable subordinator $(D_{\mathrm{H}}(t))_{t\geq 0}$. (i.e. $E_{\mathrm{H}}(t) = \inf\{r > 0 \ : \ D_{\mathrm{H}}(r) > t\}$ where $\mathbf{E}\left[e^{-sD_{\mathrm{H}}(t)}\right] = \exp(-ts^{\mathrm{H}})$).

$\Rightarrow (N_{\mathrm{H}}(t))_{t\geq 0} := N(E_{\mathrm{H}}(t))_{t\geq 0}$ is a fractional Poisson process with parameters $\lambda > 0$ and $\mathrm{H} \in (0,1]$.

Remark: From Meerschaert et al. (2011), Definition 1 and Definition 2 are equivalent.

First properties

Let $(N_{\mathrm{H}}(t))_{t\geq 0}$ be a fractional Poisson process with parameters $\lambda > 0$ and $\mathrm{H} \in (0,1]$.

We have that

- $(N_1(t))_{t\geq 0}$ is a classical Poisson process with parameter $\lambda > 0$;
- $L_{\mathrm{H}}(\xi) := \mathbf{E}(\exp(-\xi V_1)) = \dfrac{\lambda}{\lambda + \xi^{\mathrm{H}}}$;
- if $\mathrm{H} \in (0,1)$, then $\mathbf{P}(V_1 > t) \sim_{t\to\infty} \dfrac{t^{-\mathrm{H}}}{\lambda\Gamma(1-\mathrm{H})}$. ;
- as a consequence, for $\mathrm{H} \in (0,1)$ the inter-arrival times are regularly varying with parameter H, so heavy-tailed, and with infinite mean ;
- $(N_{\mathrm{H}}(t))_{t\geq 0}$ is light-tailed, i.e. $\mathbf{E}\left[\exp\{\xi N_{\mathrm{H}}(t)\}\right] < \infty$ for any $\xi \in \mathbb{R}$.
- N_{H} is not a second order stationary process.

Long-range dependence

Let $(X_j^{\mathrm{H}})_{j\geq 1}$ be the fractional Poissonian noise, defined for $j \geq 1$ by $X_j^{\mathrm{H}} := N_{\mathrm{H}}(j) - N_{\mathrm{H}}(j-1)$.

Theorem

The fractional Poissonian noise $(X_j^{\mathrm{H}})_{j\geq 1}$ has the long-range dependence property for any $\mathrm{H} \in (0;1)$.

Long-range dependence

Renewal theory

As a renewal process $(N_t)_{t\geq 0}$ has the property of long-range dependence if

- $\limsup_{t\to\infty} \frac{\mathrm{Var}(N_t)}{t} = \infty$
- it is stationary.

A similar point of view (Maheshwari and Vellaisamy)

A process $(X_t)_{t\geq 0}$ is said to have the LRD propoerty if there exists $d \in (0,1)$ (SRD if $d \in (1,2)$) such that

$$\mathrm{Corr}(X_s, X_t) \sim_{t\to\infty} c(s)t^{-d} \ .$$

They proved that the fPn satisfies $\mathrm{Corr}(X_s, X_t) \sim t^{-\frac{1}{2}(3-H)}$ and thus it has short range dependence.

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We consider the randow walk

$$X_j = \sum_{k=1}^{j} Z_k$$

with an i.i.d. sequence $(Z_k)_{k\geq 1}$ with zero mean and variance 1.

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- The process $(X_j)_{j\geq 1}$ is not a stationary process
- It should be SRD because

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But the random walk is well known to produce long leads and rare sign of changes despite being "fair".

Definition (Heyde and Yang (1997))

A process $(X_m)_{m\geq 1}$ (not necessarily stationary) has the property of long-range dependence if the block mean process

$$Y_t^{(m)} = \frac{\sum_{j=tm-m+1}^{j=tm} X_j}{\sum_{j=tm-m+1}^{j=tm} \mathrm{Var}(X_j)}$$

defined for an integer $t \geq 1$ satisfies

$$\lim_{m\to\infty} \left(\sum_{j=tm-m+1}^{j=tm} \mathrm{Var}(X_j) \right) \mathrm{Var}\left(Y_t^{(m)} \right) = +\infty \ .$$

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The random walk $X_j = \sum_{k=1}^{j} Z_k$ is LRD

$$\frac{\mathrm{Var}\left(\sum_{j=tm-m+1}^{tm} X_j\right)}{\sum_{j=tm-m+1}^{tm} \mathbb{E}(X_j^2)} \sim c(t) \times m \ .$$

Applications

- Storm origins, raindrop release and arrival on the ground, alluvial events, earthquakes : see Benson et al. (2007) for more details.

  Example : Raindrop sizes for timescales greater than tens to hundreds of seconds : Lavergnat and Gole (1998) with $\mathrm{H} = 0.68$.

- Self-similarity of web traffic : Resnick (2000) with $\mathrm{H} = 0.66$.

In ruin theory

$$R(t) = u + ct - \sum_{i=1}^{N_{\mathrm{H}}(t)} X_i \ ,$$

where

- $(N_{\mathrm{H}}(t))_{t\geq 0}$: fractional Poisson process with parameters $\lambda > 0$ and $H \in (0,1)$.

  $\hookrightarrow$ Claim inter-occurrence times $(V_i)_{i\geq 1}$: sequence of independent and Mittag-Leffler distributed with parameter λ and $H \in (0,1)$ random variables.

- Claim amounts $(X_i)_{i\geq 1}$: sequence of independent and identically distributed positive random variables.

- $(X_i)_{i\geq 1}$ is independent from $(V_i)_{i\geq 1}$.

Remark : by convention, $\sum_{i=1}^{N_{\mathrm{H}}(t)} X_i = 0$ if $N_{\mathrm{H}}(t) = 0$.

With exponential claim amounts (1) : $X_1 \sim \mathcal{E}(\mu)$

Proposition

The distribution of the ruin time τ has a density p_τ given by

$$p_\tau(t) = e^{-\mu(u+ct)} \sum_{n=0}^{\infty} \frac{\mu^n (u+ct)^{n-1}}{n!} \left(u + \frac{ct}{n+1} \right) f_{\mathrm{H}}^{*(n+1)}(t) \ , \qquad (1)$$

where f_{H}^{*n} denotes the $n-$fold convolution of the function f_{H} defined by for $t \geq 0$ by

$$f_{\mathrm{H}}(t) = ut^{\mathrm{H}-1} E_{\mathrm{H},\mathrm{H}}(-\lambda t^{\mathrm{H}}) \qquad (2)$$

where

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}$$

is the generalized two-parameter Mittag-Leffler function.

<u>*Proof*</u> : Direct application of Borovkov and Dickson (2008), since it is a Sparre-Andersen process.

With exponential claim amounts (2) : $X_1 \sim \mathcal{E}(\mu)$

Proposition

For any $x > 0$ it holds that

$$\xi \int_0^{\infty} e^{-\xi t} \psi(u,t) dt = 1 - y(\xi) \exp\Big\{ - u\mu\big(1 - y(\xi)\big)\Big\} \ , \quad \xi > 0$$

where $y(\xi)$ is the unique solution of the equation

$$y(\xi) = \frac{\lambda}{\lambda + \Big(\xi + c\mu(1 - y(\xi))\Big)^{\mathrm{H}}} \ , \quad \xi > 0. \tag{3}$$

<u>*Proof*</u> : Direct application of Theorem 1 in Malinovskii (1998).

With exponential claim amounts (3) : $X_1 \sim \mathcal{E}(\mu)$

Proposition

Under the assumptions of this section, we have

$$\psi(u) = \left(1 - \frac{\gamma}{\mu}\right) e^{-\gamma u},$$

where $\gamma > 0$ *is the unique solution of*

$$\gamma^{\mathrm{H}} - \mu\gamma^{\mathrm{H}-1} + \frac{\lambda}{c^{\mathrm{H}}} = 0\,. \tag{4}$$

Proof : Direct application of Theorem VI.2.2 in Asmussen and Albrecher (2010).

With heavy-tailed claim amounts (1)

Proposition

If the distribution F of the claim sizes is sub-exponential, then

$$\psi(u,t) \sim \frac{\lambda t^{\mathrm{H}}\,\overline{F}(u)}{\Gamma(1+\mathrm{H})}$$

as u goes to $+\infty$.

Proof :

$$\mathbf{P}\left(\sum_{i=1}^{N_{\mathrm{H}}(t)} X_i > u + ct\right) \le \psi(u,t) \le \mathbf{P}\left(\sum_{i=1}^{N_{\mathrm{H}}(t)} X_i > u\right) ;$$

$$\mathbf{P}\left(\sum_{i=1}^{N_{\mathrm{H}}(t)} X_i > u + ct\right) \sim \mathbf{P}\left(\sum_{i=1}^{N_{\mathrm{H}}(t)} X_i > u\right) \sim \mathbf{E}(N_{\mathrm{H}}(t))\,\overline{F}(u) ;$$

and from Lageras (2005) : $\mathbf{E}(N_{\mathrm{H}}(t)) = \dfrac{\lambda t^{\mathrm{H}}}{\Gamma(1+\mathrm{H})}$.

With heavy-tailed claim amounts (2) (In progress...)

Since $(N_{\mathrm{H}}(t))_{t\geq 0}$ is a renewal process, a random walk can be easily exhibited :

$$S_0 = 0\,,\; S_n = (X_1 - cV_1) + \cdots + (X_n - cV_n)\,.$$

With

$$M = \sup\{S_n,\; n \geq 0\}\,,$$

we have, for $u > 0$,

$$\psi(u) = \mathbf{P}(M > u)\,.$$

So from Denisov et al. (2004), we get :

With heavy-tailed claim amounts (3) (In progress...)

Proposition

Assume that $\mathbf{P}(X_1 > x) = L(x)x^{-\alpha}$ *for some slowly regularly varying function* L *and* $\alpha > 0$ *(so* $X_1 \in \mathcal{R}_{-\alpha}$*).*

- *If* $\alpha > \mathrm{H}$ *then*

$$\psi(u) \sim \frac{\lambda\Gamma(\alpha - \mathrm{H})}{c^{\mathrm{H}}\Gamma(\alpha)} u^{-\alpha+\mathrm{H}} L(u) \quad u \to \infty\,.$$

- *If* $\alpha = \mathrm{H}$ *and* $\int_0^{+\infty} \frac{L(t)}{t} dt < +\infty$ *then*

$$\psi(u) \sim \frac{\lambda}{c^{\mathrm{H}}\Gamma(\mathrm{H})} \int_u^{+\infty} \frac{L(t)}{t} dt\,, \quad u \to \infty\,.$$

Thank you for your attention !