Time Consistency of the Mean-Risk Problem

Birgit Rudloff

Vienna University of Economics and Business

joint work with Gabriela Kováčová

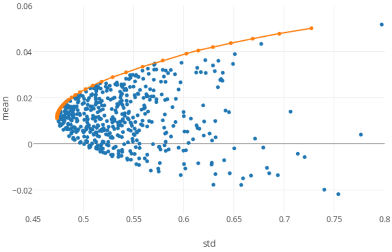
Happy Birthday, Yuri!

Mean-Risk Problem: Motivation

Portfolio Selection

Markowitz (1952)

- expected return (**mean**) is desirable, **risk** is undesirable
- portfolios not dominated in the mean-risk sense are **efficient**
- static mean-variance problem
- extentions: multiple time periods, various measurements of risk



Mean-Risk via Scalarization

- Maximize mean under a risk constraint, or vice versa

$$\max \mathbb{E}[v_T] \ \text{ s.t. } \ \rho(v_T) \leq r$$

  - mean-variance: Merton (1972); Li, Ng (2000)
  - mean-CVaR: Bäuerle, Mundt (2009)

Mean-Risk via Scalarization

- Maximize mean under a risk constraint, or vice versa

$$\max \mathbb{E}[v_T] \;\text{ s.t. }\; \rho(v_T) \leq r$$

  - mean-variance: Merton (1972); Li, Ng (2000)
  - mean-CVaR: Bäuerle, Mundt (2009)

- Specify a risk aversion λ

$$\max\, (1-\lambda)\mathbb{E}[v_T] - \lambda\rho(v_T)$$

  - mean-variance: Li, Ng (2000); Zhou, Li (2000)
  - mean-CVaR: Rudloff, Street, Valladão (2014)

Mean-Risk via Scalarization

- Maximize mean under a risk constraint, or vice versa

$$\max \mathbb{E}[v_T] \text{ s.t. } \rho(v_T) \leq r$$

  - mean-variance: Merton (1972); Li, Ng (2000)
  - mean-CVaR: Bäuerle, Mundt (2009)

- Specify a risk aversion λ

$$\max\ (1-\lambda)\mathbb{E}[v_T] - \lambda\rho(v_T)$$

  - mean-variance: Li, Ng (2000); Zhou, Li (2000)
  - mean-CVaR: Rudloff, Street, Valladão (2014)

- Dynamic problems - **time inconsistency**

Some Approaches to Time Inconsistency

- Pre-commitment
  - Li, Ng (2000); Zhou, Li (2000)
- Game-theoretic approach
  - Björk, Murgoci, Zhou (2014)
- Time-varying risk aversion
  - Basak, Chabakauri (2010); Cui, Li, Wang, Zhu (2012); Björk, Murgoci, Zhou (2014); Karnam, Ma, Zhang (2016)
- Certainty equivalent w.r.t. a time consistent dynamic utility
  - Rudloff, Street, Valladão (2014)

Some Approaches to Time Inconsistency

- Pre-commitment
  - Li, Ng (2000); Zhou, Li (2000)
- Game-theoretic approach
  - Björk, Murgoci, Zhou (2014)
- Time-varying risk aversion
  - Basak, Chabakauri (2010); Cui, Li, Wang, Zhu (2012); Björk, Murgoci, Zhou (2014); Karnam, Ma, Zhang (2016)
- Certainty equivalent w.r.t. a time consistent dynamic utility
  - Rudloff, Street, Valladão (2014)

- Here: Consider the mean-risk as a **vector optimization problem**

Vector Optimization

Vector Optimization Problem

- Image space: a partially ordered vector space $(Y, \leq_C)$
  - An ordering cone $C \subseteq Y$: $y_1 \leq_C y_2 \Leftrightarrow y_2 \in y_1 + C$

$$\min\ f : X \to Y \text{ w.r.t. } \leq_C \text{ over } S \subseteq X \qquad \text{(VOP)}$$

Vector Optimization Problem

- Image space: a partially ordered vector space $(Y, \leq_C)$
  - An ordering cone $C \subseteq Y$: $y_1 \leq_C y_2 \Leftrightarrow y_2 \in y_1 + C$

$$\min \; f : X \to Y \text{ w.r.t. } \leq_C \text{ over } S \subseteq X \qquad \text{(VOP)}$$

- Image of the feasible set $f[S] := \{f(x) \mid x \in S\}$
- **Upper image**

$$\mathcal{P} := \text{cl}\;(f[S] + C)$$

Vector Optimization Problem

- Image space: a partially ordered vector space $(Y, \leq_C)$
  - An ordering cone $C \subseteq Y$: $y_1 \leq_C y_2 \Leftrightarrow y_2 \in y_1 + C$

$$\min \ f : X \to Y \text{ w.r.t. } \leq_C \text{ over } S \subseteq X \qquad \text{(VOP)}$$

- Image of the feasible set $f[S] := \{f(x) \mid x \in S\}$
- **Upper image**

$$\mathcal{P} := \text{cl } (f[S] + C)$$

- A **minimizer** of the (VOP) is $\bar{x} \in S$ such that

$$\left(\{f(\bar{x})\} - C \backslash \{0\}\right) \cap f[S] = \emptyset$$

- A **weak minimizer** of the (VOP) is $\bar{x} \in S$ such that

$$\left(\{f(\bar{x})\} - \text{int } C\right) \cap f[S] = \emptyset$$

Mean-Risk as a VOP

Market and Feasible Portfolios

- Discrete time with a finite horizon $\mathbb{T} = \{0, 1, \ldots, T\}$
- Finite $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0,\ldots,T}, \mathbb{P})$

Market and Feasible Portfolios

- Discrete time with a finite horizon $\mathbb{T} = \{0, 1, \ldots, T\}$
- Finite $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0,\ldots,T}, \mathbb{P})$
- d assets with price process $(S_t)_{t=0,\ldots,T}$

Market and Feasible Portfolios

- Discrete time with a finite horizon $\mathbb{T} = \{0, 1, \ldots, T\}$
- Finite $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0,\ldots,T}, \mathbb{P})$
- d assets with price process $(S_t)_{t=0,\ldots,T}$
- Investor with an initial wealth v_0

Market and Feasible Portfolios

- Discrete time with a finite horizon $\mathbb{T} = \{0, 1, \dots, T\}$
- Finite $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0,\dots,T}, \mathbb{P})$
- d assets with price process $(S_t)_{t=0,\dots,T}$
- Investor with an initial wealth v_0
- Portfolio = trading strategy $(\psi_s)_{s=0,\dots,T-1}$
  - constraints $\psi_s \in \Phi_s$ (e.g. $\psi_s \geq 0$), cond. convex and closed Φ_s

Market and Feasible Portfolios

- Discrete time with a finite horizon $\mathbb{T} = \{0, 1, \dots, T\}$
- Finite $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0,\dots,T}, \mathbb{P})$
- d assets with price process $(S_t)_{t=0,\dots,T}$
- Investor with an initial wealth v_0
- Portfolio = trading strategy $(\psi_s)_{s=0,\dots,T-1}$
  - constraints $\psi_s \in \Phi_s$ (e.g. $\psi_s \geq 0$), cond. convex and closed Φ_s
- portfolio value $v_{s+1} = S_{s+1}^{\mathsf{T}} \psi_s$

Market and Feasible Portfolios

- Discrete time with a finite horizon $\mathbb{T} = \{0, 1, \ldots, T\}$
- Finite $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0,\ldots,T}, \mathbb{P})$
- d assets with price process $(S_t)_{t=0,\ldots,T}$
- Investor with an initial wealth v_0
- Portfolio = trading strategy $(\psi_s)_{s=0,\ldots,T-1}$
    - constraints $\psi_s \in \Phi_s$ (e.g. $\psi_s \geq 0$), cond. convex and closed Φ_s
- portfolio value $v_{s+1} = S_{s+1}^{\mathsf{T}} \psi_s$
- Feasible portfolios at time t with wealth $v_t \in L_t$

$$\Psi_t(v_t) := \Big\{ (\psi_s)_{s=t,\ldots,T-1} \;\Big|\; S_s^{\mathsf{T}} \psi_s = v_s,\ v_{s+1} = S_{s+1}^{\mathsf{T}} \psi_s, \\ \psi_s \in \Phi_s,\ s = t, \ldots, T-1 \Big\}$$

Measuring the Mean and the Risk

- **Conditional expectation** of the terminal value $\mathbb{E}_t(v_T)$

Measuring the Mean and the Risk

- **Conditional expectation** of the terminal value $\mathbb{E}_t(v_T)$
- Value of a **time consistent dynamic convex risk measure** $\rho_t(v_T)$

Measuring the Mean and the Risk

- **Conditional expectation** of the terminal value $\mathbb{E}_t(v_T)$
- Value of a **time consistent dynamic convex risk measure** $\rho_t(v_T)$

  A family of mappings $\rho_t : L_T \to L_t$, such that $\forall v, w \in L_T, \forall \lambda \in L_t$
  - translation invariance: $\rho_t(v + \lambda) = \rho_t(v) - \lambda$,
  - monotonicity: $v \leq w \Rightarrow \rho_t(v) \geq \rho_t(w)$,
  - convexity: $\rho_t(\lambda v + (1 - \lambda)w) \leq \lambda\rho_t(v) + (1 - \lambda)\rho_t(w)$ for $0 \leq \lambda \leq 1$,
  - recursiveness: $\rho_t(v) = \rho_t(-\rho_{t+1}(v))$,

Measuring the Mean and the Risk

- **Conditional expectation** of the terminal value $\mathbb{E}_t(v_T)$
- Value of a **time consistent dynamic convex risk measure** $\rho_t(v_T)$

  A family of mappings $\rho_t : L_T \to L_t$, such that $\forall v, w \in L_T, \forall \lambda \in L_t$
  - translation invariance: $\rho_t(v + \lambda) = \rho_t(v) - \lambda$,
  - monotonicity: $v \leq w \Rightarrow \rho_t(v) \geq \rho_t(w)$,
  - convexity: $\rho_t(\lambda v + (1-\lambda)w) \leq \lambda\rho_t(v) + (1-\lambda)\rho_t(w)$ for $0 \leq \lambda \leq 1$,
  - recursiveness: $\rho_t(v) = \rho_t(-\rho_{t+1}(v))$,

- Given a wealth $v_t \in L_t$ we consider a problem

$$
\begin{aligned}
\min_{(\psi_s)_{s=t,\dots,T-1}} \quad & \begin{pmatrix} -\mathbb{E}_t(v_T) \\ \rho_t(v_T) \end{pmatrix} \text{ w.r.t. } \leq_{L_t(\mathbb{R}^2_+)} \\
\text{s.t.} \quad & S_s^\mathsf{T}\psi_s = v_s, \\
& v_{s+1} = S_{s+1}^\mathsf{T}\psi_s, \\
& \psi_s \in \Phi_s, \quad s = t, \dots, T-1.
\end{aligned}
\qquad (D_t(v_t))
$$

Efficient and Weakly Efficient Portfolios

Definition: Efficient Portfolio

A feasible portfolio $(\psi_s)_{s=t,\dots,T-1} \in \Psi_t(v_t)$ is **efficient at a time t for an investment $\mathbf{v_t}$**, if and only if there exists no feasible portfolio $(\phi_s)_{s=t,\dots,T-1} \in \Psi_t(v_t)$, such that

$$\begin{aligned} \mathbb{E}_t(v_T^{\phi}) &\geq \mathbb{E}_t(v_T^{\psi}), \\ \rho_t(v_T^{\phi}) &\leq \rho_t(v_T^{\psi}), \end{aligned} \tag{1}$$

and at least one of the above is not attained as an equality $\mathbb{P}$-a.s..

Efficient and Weakly Efficient Portfolios

Definition: Efficient Portfolio

A feasible portfolio $(\psi_s)_{s=t,\dots,T-1} \in \Psi_t(v_t)$ is **efficient at a time t for an investment $\mathbf{v_t}$**, if and only if there exists no feasible portfolio $(\phi_s)_{s=t,\dots,T-1} \in \Psi_t(v_t)$, such that

$$\begin{aligned} \mathbb{E}_t(v_T^{\phi}) &\geq \mathbb{E}_t(v_T^{\psi}), \\ \rho_t(v_T^{\phi}) &\leq \rho_t(v_T^{\psi}), \end{aligned} \tag{1}$$

and at least one of the above is not attained as an equality $\mathbb{P}$-a.s..

Portfolio $(\psi_s)_{s=t,\dots,T-1} \in \Psi_t(v_t)$ is **weakly efficient at a time t for an investment $\mathbf{v_t}$** if both inequalities (1) are strict in every state of the world $\omega \in \Omega$.

Efficient and Weakly Efficient Portfolios

Definition: Efficient Portfolio

A feasible portfolio $(\psi_s)_{s=t,\dots,T-1} \in \Psi_t(v_t)$ is **efficient at a time t for an investment $\mathbf{v_t}$**, if and only if there exists no feasible portfolio $(\phi_s)_{s=t,\dots,T-1} \in \Psi_t(v_t)$, such that

$$\begin{aligned} \mathbb{E}_t(v_T^{\phi}) &\geq \mathbb{E}_t(v_T^{\psi}), \\ \rho_t(v_T^{\phi}) &\leq \rho_t(v_T^{\psi}), \end{aligned} \tag{1}$$

and at least one of the above is not attained as an equality $\mathbb{P}$-a.s..

Portfolio $(\psi_s)_{s=t,\dots,T-1} \in \Psi_t(v_t)$ is **weakly efficient at a time t for an investment $\mathbf{v_t}$** if both inequalities (1) are strict in every state of the world $\omega \in \Omega$.

- $(\psi_s)_{s=t,\dots,T-1}$ is a (weakly) efficient portfolio $\Leftrightarrow$ it is a (weak) minimizer of the problem $D_t(v_t)$

Time Consistency and a Set-Valued Bellman's Principle

Mean-Risk and Time Consistency

- Recall: Scalar mean-risk (for a fixed risk aversion) is time inconsistent

Mean-Risk and Time Consistency

- Recall: Scalar mean-risk (for a fixed risk aversion) is time inconsistent
- For a mean-risk VOP we obtain a **time consistency in the sense of weak minimizers** (weakly efficient portfolios)

Theorem

The family of mean-risk problems $\mathcal{D}$ has the following property,

$$(\psi_s)_{s=t,\dots,T-1} \text{ being a } \textit{weak minimizer} \text{ of } D_t(v_t)$$
$$\text{implies}$$
$$(\psi_s)_{s=t+1,\dots,T-1} \text{ being a } \textit{weak minimizer} \text{ of } D_{t+1}\left(S_{t+1}^{\mathsf{T}}\psi_t\right).$$

Mean-Risk and Time Consistency

- Recall: Scalar mean-risk (for a fixed risk aversion) is time inconsistent
- For a mean-risk VOP we obtain a **time consistency in the sense of weak minimizers** (weakly efficient portfolios)

Theorem

The family of mean-risk problems $\mathcal{D}$ has the following property,

$$(\psi_s)_{s=t,\dots,T-1} \text{ being a } \textit{weak minimizer} \text{ of } D_t(v_t)$$
$$\text{implies}$$
$$(\psi_s)_{s=t+1,\dots,T-1} \text{ being a } \textit{weak minimizer} \text{ of } D_{t+1}\left(S_{t+1}^{\mathsf{T}}\psi_t\right).$$

- A weaker notion than the scalar time consistency

Scalar Time Consistency and Bellman's Principle

Recall:

- scalar problem with **value function** $J_t(v_t)$

$$
\begin{aligned}
J_t(v_t) := \inf_{u_t,\dots,u_{T-1}} \quad & \mathbb{E}_t \left[\sum_{s=t}^{T-1} f_s(v_s, u_s, z_s) + f_T(v_T) \right] \\
s.t. \quad & v_{s+1} = g_s(v_s, u_s, z_s), \\
& u_s \in U_s(v_s), \;\; s = t, \dots, T-1
\end{aligned}
$$

Scalar Time Consistency and Bellman's Principle

Recall:

- scalar problem with **value function** $J_t(v_t)$

$$
\begin{aligned}
J_t(v_t) := \inf_{u_t,\ldots,u_{T-1}} \quad & \mathbb{E}_t \left[\sum_{s=t}^{T-1} f_s(v_s, u_s, z_s) + f_T(v_T) \right] \\
s.t. \quad & v_{s+1} = g_s(v_s, u_s, z_s), \\
& u_s \in U_s(v_s), \quad s = t, \ldots, T-1
\end{aligned}
$$

- ... is time consistent if **Bellman's equation** is satisfied,

$$
\begin{aligned}
J_t(v_t) \; &= \inf_{u_t \in U_t(v_t)} \mathbb{E}_t \left[f_t(v_t, u_t, z_t) + J_{t+1}(g_t(v_t, u_t, z_t)) \right], \\
J_T(v_T) :=& \; f_T(v_T).
\end{aligned}
$$

Scalar Time Consistency and Bellman's Principle

Recall:

- scalar problem with **value function** $J_t(v_t)$

$$\begin{aligned} J_t(v_t) := \inf_{u_t,\dots,u_{T-1}} \quad & \mathbb{E}_t \left[\sum_{s=t}^{T-1} f_s(v_s,u_s,z_s) + f_T(v_T) \right] \\ s.t. \quad & v_{s+1} = g_s(v_s,u_s,z_s), \\ & u_s \in U_s(v_s), \;\; s = t,\dots,T-1 \end{aligned}$$

- ... is time consistent if **Bellman's equation** is satisfied,

$$\begin{aligned} J_t(v_t) \;&=\; \inf_{u_t \in U_t(v_t)} \mathbb{E}_t \left[f_t(v_t,u_t,z_t) + J_{t+1}(g_t(v_t,u_t,z_t)) \right], \\ J_T(v_T) &:=\; f_T(v_T). \end{aligned}$$

- The Bellman's equation provides a possibility to solve the problem recursively

Scalar Time Consistency and Bellman's Principle

Recall:

- scalar problem with **value function** $J_t(v_t)$

$$
\begin{aligned}
J_t(v_t) := \inf_{u_t,\dots,u_{T-1}} \quad & \mathbb{E}_t \left[\sum_{s=t}^{T-1} f_s(v_s, u_s, z_s) + f_T(v_T) \right] \\
s.t. \quad & v_{s+1} = g_s(v_s, u_s, z_s), \\
& u_s \in U_s(v_s), \;\; s = t, \dots, T-1
\end{aligned}
$$

- ... is time consistent if **Bellman's equation** is satisfied,

$$
\begin{aligned}
J_t(v_t) \; &= \inf_{u_t \in U_t(v_t)} \mathbb{E}_t \left[f_t(v_t, u_t, z_t) + J_{t+1}(g_t(v_t, u_t, z_t)) \right], \\
J_T(v_T) &:= \; f_T(v_T).
\end{aligned}
$$

- The Bellman's equation provides a possibility to solve the problem recursively
- Is there a similar recursive relation for a vector optimization mean-risk problem?

Scalar Time Consistency and Bellman's Principle

Recall:

- scalar problem with **value function** $J_t(v_t)$

$$\begin{aligned} J_t(v_t) := \inf_{u_t,\dots,u_{T-1}} \quad & \mathbb{E}_t \left[\sum_{s=t}^{T-1} f_s(v_s, u_s, z_s) + f_T(v_T) \right] \\ s.t. \quad & v_{s+1} = g_s(v_s, u_s, z_s), \\ & u_s \in U_s(v_s), \;\; s = t, \dots, T-1 \end{aligned}$$

- ... is time consistent if **Bellman's equation** is satisfied,

$$\begin{aligned} J_t(v_t) \; &= \inf_{u_t \in U_t(v_t)} \mathbb{E}_t \left[f_t(v_t, u_t, z_t) + J_{t+1}(g_t(v_t, u_t, z_t)) \right], \\ J_T(v_T) &:= \; f_T(v_T). \end{aligned}$$

- The Bellman's equation provides a possibility to solve the problem recursively
- Is there a similar recursive relation for a vector optimization mean-risk problem? What is the value function???

Value Function for a VOP

$$\min \; f : X \to Y \text{ w.r.t. } \leq_C \; \text{ over } S \subseteq X \qquad \text{(VOP)}$$

Value Function for a VOP

$$\min \; f : X \to Y \text{ w.r.t. } \leq_C \; \text{ over } S \subseteq X \qquad \text{(VOP)}$$

- Value function should be an infimum - in what sense for a VOP?

Value Function for a VOP

$$\min\ f : X \to Y \text{ w.r.t. } \leq_C \ \text{ over } S \subseteq X \qquad \text{(VOP)}$$

- Value function should be an infimum - in what sense for a VOP?
- Infimum in the classical sense of vector ordering is unsuitable

Value Function for a VOP

$$\min \ f : X \to Y \text{ w.r.t. } \leq_C \ \text{ over } S \subseteq X \qquad \text{(VOP)}$$

- Value function should be an infimum - in what sense for a VOP?
- Infimum in the classical sense of vector ordering is unsuitable
- **Set optimization approach** provides a candidate

Value Function for a VOP

$$\min\ f : X \to Y \text{ w.r.t. } \leq_C \text{ over } S \subseteq X \qquad \text{(VOP)}$$

- Value function should be an infimum - in what sense for a VOP?
- Infimum in the classical sense of vector ordering is unsuitable
- **Set optimization approach** provides a candidate
  - space of closed upper sets $\mathbb{F}(Y, C) = \{A \subseteq Y \mid \mathrm{cl}(A + C) = A\}$ with $\supseteq$ is a partially ordered conlinear space and a complete lattice with infimum

$$\inf_{(\mathbb{F}, \supseteq)} \mathbb{A} = \mathrm{cl} \bigcup_{A \in \mathbb{A}} A$$

Value Function for a VOP

$$\min\ f : X \to Y \text{ w.r.t. } \leq_C \text{ over } S \subseteq X \qquad \text{(VOP)}$$

- Value function should be an infimum - in what sense for a VOP?
- Infimum in the classical sense of vector ordering is unsuitable
- **Set optimization approach** provides a candidate
  - space of closed upper sets $\mathbb{F}(Y, C) = \{A \subseteq Y \mid \text{cl}(A + C) = A\}$ with $\supseteq$ is a partially ordered conlinear space and a complete lattice with infimum

$$\inf_{(\mathbb{F},\supseteq)} \mathbb{A} = \text{cl} \bigcup_{A \in \mathbb{A}} A$$

  - consider a set-extention of the problem with $F(x) := f(x) + C$,

$$\min\ F : X \to \mathbb{F}(Y, C) \text{ w.r.t. } \supseteq \text{ over } S \subseteq X$$

Value Function for a VOP

$$\min\ f : X \to Y \text{ w.r.t. } \leq_C \text{ over } S \subseteq X \qquad \text{(VOP)}$$

- Value function should be an infimum - in what sense for a VOP?
- Infimum in the classical sense of vector ordering is unsuitable
- **Set optimization approach** provides a candidate
  - space of closed upper sets $\mathbb{F}(Y, C) = \{A \subseteq Y \mid \text{cl}(A + C) = A\}$ with $\supseteq$ is a partially ordered conlinear space and a complete lattice with infimum
  $$\inf_{(\mathbb{F},\supseteq)} \mathbb{A} = \text{cl} \bigcup_{A \in \mathbb{A}} A$$
  - consider a set-extention of the problem with $F(x) := f(x) + C$,
  $$\min\ F : X \to \mathbb{F}(Y, C) \text{ w.r.t. } \supseteq \text{ over } S \subseteq X$$
  - its set-valued infimum is the **upper image**,
  $$\inf_{x \in S} F(x) = \text{cl} \bigcup_{x \in S} f(x) + C = \mathcal{P}$$

A Set-Valued Bellman's Principle for the Mean-Risk

The following recursive form of the upper image (value function) is obtained

Theorem

$$\mathcal{P}_t(v_t) = \mathrm{cl} \left\{ \begin{pmatrix} -\mathbb{E}_t(-x_1) \\ \rho_t(-x_2) \end{pmatrix} \;\middle|\; S_t^{\mathsf{T}} \psi_t = v_t, \psi_t \in \Phi_t, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathcal{P}_{t+1}\left(S_{t+1}^{\mathsf{T}} \psi_t\right) \right\} \tag{B}$$

with

$$\mathcal{P}_T(v_T) = \left\{ \begin{pmatrix} -v_T \\ -v_T \end{pmatrix} \right\} + L_T(\mathbb{R}^2_+).$$

Recursion and a One-Time-Step Problem

- This corresponds to a sequence of one-time-step problem

$$
\begin{aligned}
\min_{\psi_t, x} \quad & \begin{pmatrix} -\mathbb{E}_t(-x_1) \\ \rho_t(-x_2) \end{pmatrix} \text{ w.r.t. } \leq_{L_t(\mathbb{R}^2_+)} \\
\text{s.t.} \quad & S_t^{\mathsf{T}} \psi_t = v_t, \\
& \psi_t \in \Phi_t, \\
& \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathcal{P}_{t+1}\left(S_{t+1}^{\mathsf{T}} \psi_t\right).
\end{aligned}
\qquad (\tilde{D}_t(v_t))
$$

Recursion and a One-Time-Step Problem

- This corresponds to a sequence of one-time-step problem

$$\begin{aligned} \min_{\psi_t, x} \quad & \begin{pmatrix} -\mathbb{E}_t(-x_1) \\ \rho_t(-x_2) \end{pmatrix} \text{ w.r.t. } \leq_{L_t(\mathbb{R}^2_+)} \\ \text{s.t.} \quad & S_t^{\mathsf{T}} \psi_t = v_t, \\ & \psi_t \in \Phi_t, \\ & \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathcal{P}_{t+1}\left(S_{t+1}^{\mathsf{T}} \psi_t\right). \end{aligned} \tag{$\tilde{D}_t(v_t)$}$$

- $\tilde{D}_t(v_t)$ shares the upper image $\mathcal{P}_t(v_t)$ of the mean-risk problem

Recursion and a One-Time-Step Problem

- This corresponds to a sequence of one-time-step problem

$$\begin{aligned} \min_{\psi_t, x} \quad & \begin{pmatrix} -\mathbb{E}_t(-x_1) \\ \rho_t(-x_2) \end{pmatrix} \text{ w.r.t. } \leq_{L_t(\mathbb{R}^2_+)} \\ \text{s.t.} \quad & S_t^\mathsf{T} \psi_t = v_t, \\ & \psi_t \in \Phi_t, \\ & \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathcal{P}_{t+1}\left(S_{t+1}^\mathsf{T} \psi_t\right). \end{aligned} \qquad (\tilde{D}_t(v_t))$$

- $\tilde{D}_t(v_t)$ shares the upper image $\mathcal{P}_t(v_t)$ of the mean-risk problem
- $\tilde{D}_t(v_t)$ is a convex vector optimization problem

Recursion and a One-Time-Step Problem

- $\mathcal{P}_0\left(v_0\right)$ is obtained recursively via a sequence of one-time-step convex VOPs

Recursion and a One-Time-Step Problem

- $\mathcal{P}_0\left(v_0\right)$ is obtained recursively via a sequence of one-time-step convex VOPs
- computation of $\mathcal{P}_t\left(v_t\right)$ uses $\mathcal{P}_{t+1}(S_{t+1}^{\mathsf{T}}\psi_t)$ in constraints $\rightarrow$ a Bellman's principle

Recursion and a One-Time-Step Problem

- $\mathcal{P}_0\,(v_0)$ is obtained recursively via a sequence of one-time-step convex VOPs
- computation of $\mathcal{P}_t\,(v_t)$ uses $\mathcal{P}_{t+1}(S_{t+1}^{\mathsf{T}}\psi_t)$ in constraints $\rightarrow$ a Bellman's principle
- based on a set-optimization notion of infimum $\rightarrow$ set-valued

Recursion and a One-Time-Step Problem

- $\mathcal{P}_0\left(v_0\right)$ is obtained recursively via a sequence of one-time-step convex VOPs
- computation of $\mathcal{P}_t\left(v_t\right)$ uses $\mathcal{P}_{t+1}(S_{t+1}^{\mathsf{T}}\psi_t)$ in constraints → a Bellman's principle
- based on a set-optimization notion of infimum → set-valued
- (B) corresponds to a set-valued infimum with $\Gamma_t(X) := \begin{pmatrix} -\mathbb{E}_t(X_1) \\ \rho_t(X_2) \end{pmatrix}$,

$$\begin{aligned} \mathcal{P}_t\left(v_t\right) &= \inf_{\substack{S_t^{\mathsf{T}}\psi_t = v_t, \\ \psi_t \in \Phi_t}} \Gamma_t(-\mathcal{P}_{t+1}(S_{t+1}^{\mathsf{T}}\psi_t)) \\ &= \inf_{\substack{S_t^{\mathsf{T}}\psi_t = v_t, \\ \psi_t \in \Phi_t}} \inf_{x \in \mathcal{P}_{t+1}(S_{t+1}^{\mathsf{T}}\psi_t)} \Gamma_t\left(-x\right). \end{aligned}$$

Backward Computation of the Efficient Frontier

Implementation of the One-Time-Step Problems

Now assume additionally

Implementation of the One-Time-Step Problems

Now assume additionally

- a coherent risk measure $(\rho_t)_{t=0,\ldots,T-1}$,

Implementation of the One-Time-Step Problems

Now assume additionally

- a coherent risk measure $(\rho_t)_{t=0,\dots,T-1}$,
- short-selling constraints $\psi_s \geq 0$,

Implementation of the One-Time-Step Problems

Now assume additionally

- a coherent risk measure $(\rho_t)_{t=0,\dots,T-1}$,
- short-selling constraints $\psi_s \geq 0$,
- a positive wealth $v_0 > 0$ and prices $S_s > 0$.

Implementation of the One-Time-Step Problems

Now assume additionally

- a coherent risk measure $(\rho_t)_{t=0,\ldots,T-1}$,
- short-selling constraints $\psi_s \geq 0$,
- a positive wealth $v_0 > 0$ and prices $S_s > 0$.

- (Weakly) efficient portfolios and upper images scale, $\mathcal{P}_t(v_t) = v_t \cdot \mathcal{P}_t(\mathbb{1})$.

Implementation of the One-Time-Step Problems

Now assume additionally

- a coherent risk measure $(\rho_t)_{t=0,\dots,T-1}$,
- short-selling constraints $\psi_s \geq 0$,
- a positive wealth $v_0 > 0$ and prices $S_s > 0$.

- (Weakly) efficient portfolios and upper images scale, $\mathcal{P}_t(v_t) = v_t \cdot \mathcal{P}_t(\mathbb{1})$.
- problem $\tilde{D}_t(v_t)$ reduces to

$$\begin{aligned} \min \quad & \begin{pmatrix} -\mathbb{E}_t(-x_1) \\ \rho_t(-x_2) \end{pmatrix} \text{ w.r.t. } \leq_{L_t(\mathbb{R}^2_+)} \\ \text{s.t.} \quad & S_t^\mathsf{T} \psi_t = v_t,\ \psi_t \geq 0, \\ & x \in \left(S_{t+1}^\mathsf{T} \psi_t\right) \cdot \mathcal{P}_{t+1}(\mathbb{1}). \end{aligned}$$

Efficient Trading Strategy

Theorem

For every efficient mean-risk profile $x_0^* \in \mathcal{P}_0(v_0)$ there exists a portfolio $(\psi_s^*)_{s=0,\ldots,T-1}$ **efficient at every time point** t

Efficient Trading Strategy

Theorem

For every efficient mean-risk profile $x_0^* \in \mathcal{P}_0(v_0)$ there exists a portfolio $(\psi_s^*)_{s=0,\dots,T-1}$ **efficient at every time point** t

- stronger than time consistency in the sense of weak minimizers

Efficient Trading Strategy

Theorem

For every efficient mean-risk profile $x_0^* \in \mathcal{P}_0(v_0)$ there exists a portfolio $(\psi_s^*)_{s=0,\dots,T-1}$ **efficient at every time point** t

- stronger than time consistency in the sense of weak minimizers

---

for $t = 0, \dots, T-1$ **do**

  update wealth $v_t^* = S_t^\mathsf{T} \psi_{t-1}^*$

  position ψ_t^* and mean-risk profile x_{t+1}^* come from $I_t(v_t^*, x_t^*)(\omega_t)$ for $\omega_t \in \Omega_t$

$$\begin{aligned} \min \quad & \rho_t(-x_{t+1,2} | \omega_t) \\ \text{s.t.} \quad & S_t^\mathsf{T} \psi_t = v_t^*, \psi_t \geq 0, \\ & x_{t+1} \in (S_{t+1}^\mathsf{T} \psi_t) \cdot \mathcal{P}_{t+1}(\mathbb{1}), \\ & \mathbb{E}_t(x_{t+1,1} | \omega_t) \leq x_{t,1}^*. \end{aligned} \qquad (I_t(v_t^*, x_t^*)(\omega_t))$$

end for

Efficient Trading Strategy

Theorem

For every efficient mean-risk profile $x_0^* \in \mathcal{P}_0(v_0)$ there exists a portfolio $(\psi_s^*)_{s=0,\dots,T-1}$ **efficient at every time point** t

- stronger than time consistency in the sense of weak minimizers

---

1: **for** $t = 0, \dots, T-1$ **do**
2: update wealth $v_t^* = S_t^\mathsf{T} \psi_{t-1}^*$
3: position ψ_t^* and mean-risk profile x_{t+1}^* come from $I_t(v_t^*, x_t^*)(\omega_t)$ for $\omega_t \in \Omega_t$

$$\begin{aligned} \min\ & \rho_t(-x_{t+1,2}|\omega_t) \\ \text{s.t.}\ & S_t^\mathsf{T}\psi_t = v_t^*, \psi_t \geq 0, \\ & x_{t+1} \in (S_{t+1}^\mathsf{T}\psi_t) \cdot \mathcal{P}_{t+1}(\mathbb{1}), \\ & \mathbb{E}_t(x_{t+1,1}|\omega_t) \leq x_{t,1}^*. \end{aligned} \qquad (I_t(v_t^*, x_t^*)(\omega_t))$$

4: **end for**

- solves sequence of scalar convex OPs on a realized path $\omega_0, \omega_1 \dots, \omega_{T-1}$

Efficient Trading Strategy

Theorem

For every efficient mean-risk profile $x_0^* \in \mathcal{P}_0(v_0)$ there exists a portfolio $(\psi_s^*)_{s=0,\dots,T-1}$ **efficient at every time point** t

- stronger than time consistency in the sense of weak minimizers

---

for $t = 0, \dots, T-1$ **do**
    update wealth $v_t^* = S_t^\mathsf{T} \psi_{t-1}^*$
    position ψ_t^* and mean-risk profile x_{t+1}^* come from $I_t(v_t^*, x_t^*)(\omega_t)$ for $\omega_t \in \Omega_t$

$$\begin{aligned} \min \quad & \rho_t(-x_{t+1,2}|\omega_t) \\ \text{s.t.} \quad & S_t^\mathsf{T} \psi_t = v_t^*, \psi_t \geq 0, \\ & x_{t+1} \in (S_{t+1}^\mathsf{T} \psi_t) \cdot \mathcal{P}_{t+1}(\mathbb{1}), \\ & \mathbb{E}_t(x_{t+1,1}|\omega_t) \leq x_{t,1}^*. \end{aligned} \qquad (I_t(v_t^*, x_t^*)(\omega_t))$$

end for

- solves sequence of scalar convex OPs on a realized path $\omega_0, \omega_1 \dots, \omega_{T-1}$
- if upper images are polyhedral: only arithmetic operations (instead of solving convex OP)

Moving Scalarization and Risk Aversion

- To an efficient portfolio $(\psi_s)_{s=0,\dots,T-1}$ corresponds a sequence of weights $w_0, \dots, w_{T-1}$, where each $w_t \in L_t(\mathbb{R}^2_+)\backslash\{\mathbb{0}\}$
  - $(\psi_s)_{s=t,\dots,T-1}$ is an optimal solution to

$$\min_{\psi \in \Psi_t(v_t)} -w_{t,1} \cdot \mathbb{E}_t(v_T) + w_{t,2} \cdot \rho_t(v_T)$$

Moving Scalarization and Risk Aversion

- To an efficient portfolio $(\psi_s)_{s=0,\dots,T-1}$ corresponds a sequence of weights $w_0, \dots, w_{T-1}$, where each $w_t \in L_t(\mathbb{R}^2_+) \backslash \{\mathbb{0}\}$
  - $(\psi_s)_{s=t,\dots,T-1}$ is an optimal solution to

$$\min_{\psi \in \Psi_t(v_t)} -w_{t,1} \cdot \mathbb{E}_t(v_T) + w_{t,2} \cdot \rho_t(v_T)$$

  - weights $(w_s)_{s=0,\dots,T-1}$ can be interpreted as a **time varying, state dependent risk aversion**

Moving Scalarization and Risk Aversion

- To an efficient portfolio $(\psi_s)_{s=0,\dots,T-1}$ corresponds a sequence of weights $w_0, \dots, w_{T-1}$, where each $w_t \in L_t(\mathbb{R}^2_+) \setminus \{\mathbb{0}\}$
  - $(\psi_s)_{s=t,\dots,T-1}$ is an optimal solution to

$$\min_{\psi \in \Psi_t(v_t)} -w_{t,1} \cdot \mathbb{E}_t(v_T) + w_{t,2} \cdot \rho_t(v_T)$$

  - weights $(w_s)_{s=0,\dots,T-1}$ can be interpreted as a **time varying, state dependent risk aversion**
- Compare to Björk, Murgoci, Zhou (2014); Karnam, Ma, Zhang (2016)

Moving Scalarization and Risk Aversion

- To an efficient portfolio $(\psi_s)_{s=0,\dots,T-1}$ corresponds a sequence of weights $w_0, \dots, w_{T-1}$, where each $w_t \in L_t(\mathbb{R}^2_+)\backslash\{\mathbb{0}\}$
  - $(\psi_s)_{s=t,\dots,T-1}$ is an optimal solution to

$$\min_{\psi \in \Psi_t(v_t)} -w_{t,1} \cdot \mathbb{E}_t(v_T) + w_{t,2} \cdot \rho_t(v_T)$$

  - weights $(w_s)_{s=0,\dots,T-1}$ can be interpreted as a **time varying, state dependent risk aversion**
- Compare to Björk, Murgoci, Zhou (2014); Karnam, Ma, Zhang (2016)
  - for mean-variance there exists risk aversion c_t making problems

$$\sup_{u \in \mathcal{U}_{[t,T]}} \mathbb{E}_t(X_T^{t,u}) - \frac{1}{2c_t} Var(X_T^{t,u})$$

    time consistent

Two Examples

Examples

- Scalable market setting with i.i.d. asset returns

Examples

- Scalable market setting with i.i.d. asset returns
  - conditionally on the same wealth problems and upper images identical across nodes

Examples

- Scalable market setting with i.i.d. asset returns
  - conditionally on the same wealth problems and upper images identical across nodes
  - T node-wise problems to be solved to compute the efficient frontier

Examples

- Scalable market setting with i.i.d. asset returns
  - conditionally on the same wealth problems and upper images identical across nodes
  - T node-wise problems to be solved to compute the efficient frontier
- Risk measured by a (time consistent version of) a Conditional Value at Risk

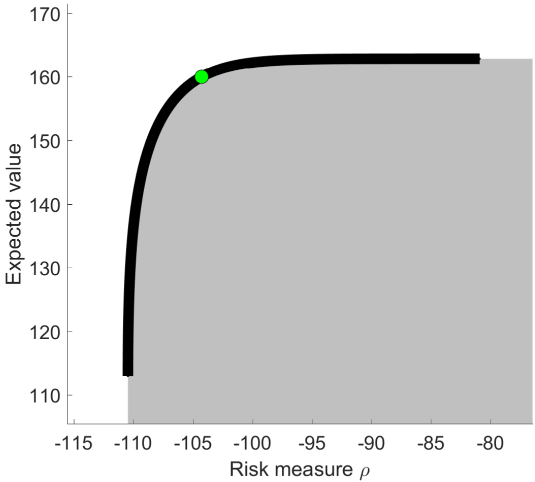
Examples

- Scalable market setting with i.i.d. asset returns
  - conditionally on the same wealth problems and upper images identical across nodes
  - T node-wise problems to be solved to compute the efficient frontier
- Risk measured by a (time consistent version of) a Conditional Value at Risk
  - polyhedral risk measure ⇒ a linear VOP

Examples

- Scalable market setting with i.i.d. asset returns
  - conditionally on the same wealth problems and upper images identical across nodes
  - T node-wise problems to be solved to compute the efficient frontier
- Risk measured by a (time consistent version of) a Conditional Value at Risk
  - polyhedral risk measure $\Rightarrow$ a linear VOP
- `Bensolve` and `Bensolve Tools` used for computations

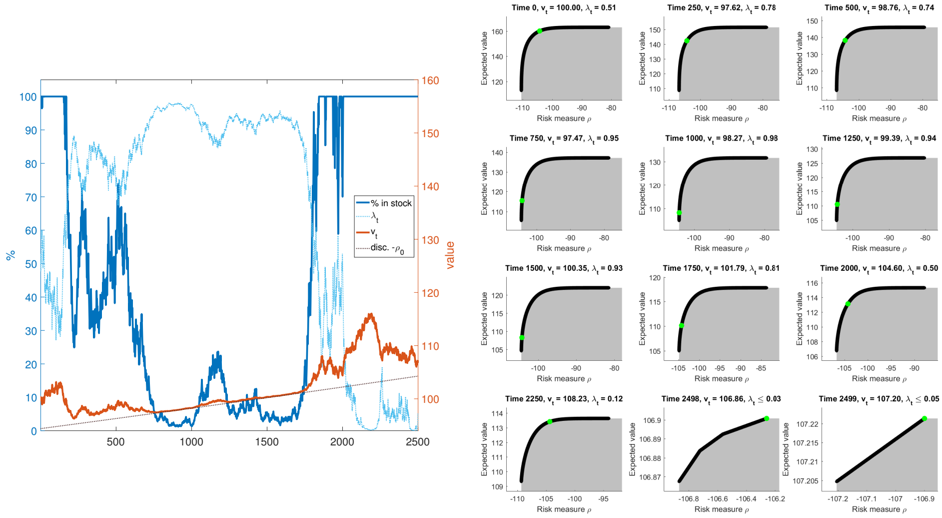
Example 1: Binomial Market Model



- 10 years with 250 trading days, $T = 2\,500$
- $r_B = 1\%$, $\bar{r}_S = 5\%$ p.a., $v_0 = 100$
- Selected mean-risk profile

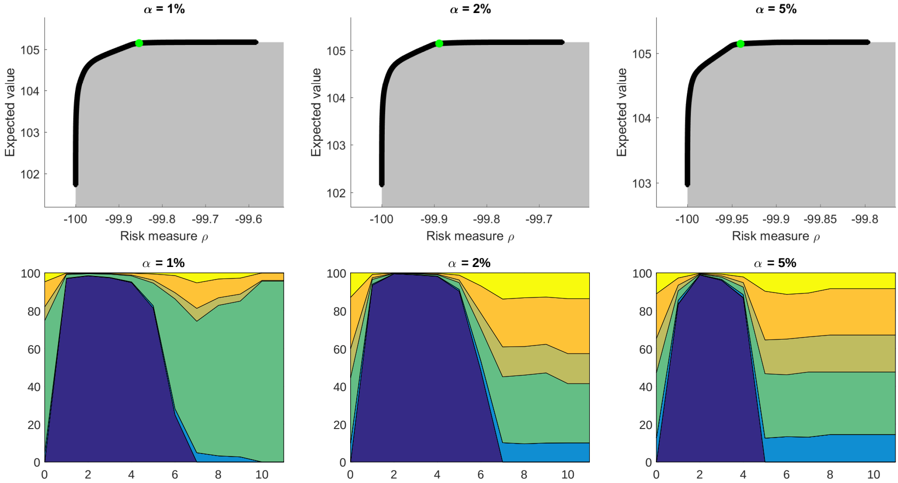
$$\mathbb{E}_0(v_T) = 160$$
$$\rho_0(v_T) = -104.27$$

Example 1: Trading Strategy along a Path



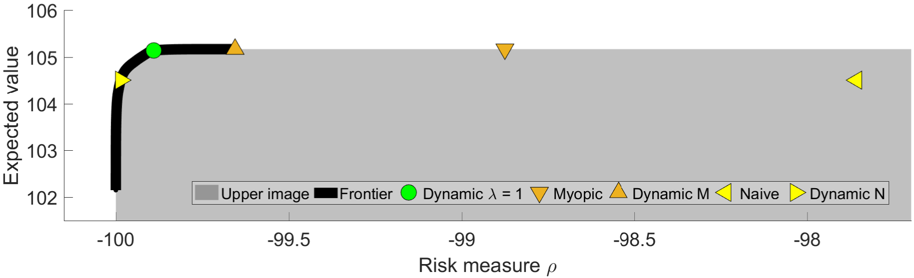
Example 2: Multiple asset classes

- one year of monthly trading, $T = 12$
- 1 bond class ($r_B = 0\%$) 7 stock classes ($\bar{r}_S = 5.17\%$), $v_0 = 100$



Example 2: Dynamic, Myopic and Naive Strategy

- mean-risk profiles of myopic and naive strategy can be computed
- time 0, level $\alpha = 2\%$ of $CVaR$
- dynamic and myopic for risk aversion 0.5



Kováčová G., Rudloff B.: *Time consistency of the mean-risk problem*, submitted for publication, 2018

Karnam Ch., Ma J, Zhang J.: *Dynamic approaches for some time inconsistent problems*, Annals of Applied Probability, 27, 2016

Björk T., Murgoci A., Zhou X.Y.: *Mean-Variance Portfolio Optimization with State-Dependent Risk Aversion*, Mathematical Finance, 2014

Feinstein Z., Rudloff B.: *A Recursive Algorithm for Multivariate Risk Measures and a Set-Valued Bellman's Principle*, Journal of Global Optimization, 2017

Hamel A., Heyde F., Löhne A., Rudloff B., Schrage C.: *Set optimization - a rather short introduction*, Springer Proceedings in Mathematics and Statistics, 2014

Löhne A.: *Vector Optimization with Infimum and Supremum*, Springer-Verlag Berlin Heidelberg, 2011

Löhne A., Rudloff B.: *An algorithm for calculating the set of superhedging portfolios in markets with transaction costs*, International Journal of Theoretical and Applied Finance, 2014