Robust utility maximization in markets with transaction costs

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based on joint work with
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Outline

1. Introduction
2. Utility functions on the positive real line
3. Utility functions on the whole real line
4. Frictionless case

Literature review

Uncertainty is usually modeled by a family of prior measures $\mathcal{P}$ on the same canonical space. Source of troubles.
E.g. definition of value process simltaneously under each $P \in \mathcal{P}$, compactness substitutes. Approach in e.g. L^0_+ is not feasible.

Existence results in a fairly general class of models are available only in discrete time: Nutz (2016), Blanchard and Carassus (2018), Neufeld and Šikić (2017), Bartl (2017), Bartl et al. (2017).

Formulation of the problem

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, P)$ a filtered probability space.

Θ be a (non-empty) set of parameters (may be infinite dimensional).

There are two assets: a riskless asset $S^0 = 1$ and a risky asset, whose dynamics is unknown.

A family S^θ, $\theta \in \Theta$ of adapted, continuous positive processes.

No condition is imposed on Θ and on S^θ.

Example: The robust Black-Scholes market model

$$dS_t^{(\mu,\sigma)} = S_t^{(\mu,\sigma)}(\mu dt + \sigma dW_t).$$

$$\Theta = \{\theta = (\mu, \sigma) \in \mathbb{R}^2 : \underline{\mu} \leq \mu \leq \overline{\mu},\ \underline{\sigma} \leq \sigma \leq \overline{\sigma}\}.$$

Advantages

This approach

- is particularly tractable and easily implemented when it comes to calibration.
- simplifies technical issues: the canonical setting with problems concerning null events, filtration completion, etc. The measurable selection arguments, the analytic properties, etc..
- Time consistency assumption typical in discrete time. (See Bartl et al. (2017), however.) Not needed here.

Drawbacks and solutions

- Recall: $\sup_H \inf_\theta EU(W_T(\theta, H))$ and $\sup_H \inf_{Q\in\mathcal{P}} E^Q U(W_T(H))$.
- No "abstract" versions.
- Komlós-type arguments on the space L^0 cannot be employed.
- No convexity in θ. Dual problem?
- The candidate dual problem in this setting does not, in general, admit a solution (see Remark 2.3 of Bartl (2017)).
- Work with the primal problem.
- Under proportional transaction costs.
- Work directly with finite variation processes. Guasoni (2002)

A topological space of FV processes

$\mathcal{V}$: the family of non-decreasing, right-continuous functions on $[0, T]$ which are 0 at 0.

Let r_k, $k \in \mathbb{N}$ be an enumeration of $Q := (\mathbb{Q} \cap [0, T]) \cup \{T\}$ with $r_0 = T$. For $f, g \in \mathcal{V}$, define a metric

$$\rho(f, g) := \sum_{k=0}^{\infty} 2^{-k} |f(r_k) - g(r_k)|.$$

Let $\mathbf{V}$ denote the set of $H = (H_0, H^{\uparrow}, H^{\downarrow})$ where $H_t^{\uparrow}, H_t^{\downarrow}$, $t \in [0, T]$ are optional processes, $H^{\uparrow}(\omega), H^{\downarrow}(\omega) \in \mathcal{V}$. H_0: initial transfer.

We equip $\mathbf{V}$ with the metric

$$\varrho(H, G) := E[\rho(H^{\uparrow}, G^{\uparrow}) \wedge 1] + E[\rho(H^{\downarrow}, G^{\downarrow}) \wedge 1], + |H_0 - G_0| \; H, G \in \mathbf{V}.$$

A compactness result

Lemma

Let $H(n) \in \mathbf{V}$, $n \in \mathbb{N}$ be such that

$$\sup_{n\in\mathbb{N}} E^Q[H_T^{\uparrow}(n) + H_T^{\downarrow}(n)] < \infty$$

for some $Q \sim P$. Then there is $H \in \mathbf{V}$ and there are convex weights $\alpha_j^n \geq 0$, $j = n, \dots, M(n)$, $\sum_{j=n}^{M(n)} \alpha_j^n = 1$, $n \in \mathbb{N}$ such that

$$\tilde{H}(n) := \sum_{j=n}^{M(n)} \alpha_j^n H(j)$$

satisfy $\tilde{H}^{\uparrow}(n) \to H^{\uparrow}$ and $\tilde{H}^{\downarrow}(n) \to H^{\downarrow}$, $n \to \infty$ almost surely in $\mathcal{V}$. □

Kabanov (1999)

Trading strategies

- Trading strategies: $H \in \mathbf{V}$.
- Denote: $H^{\uparrow}$ for buying and $H^{\downarrow}$ for selling.
- The position in the risky asset $\phi = H^{\uparrow} - H^{\downarrow}$.
- Assume $\phi_T = 0$.

The liquidation value is defined by

$$\begin{aligned} W_t^{x,\mathrm{liq}}(\theta, H) &:= x - \int_0^t S_u^{\theta} dH_u^{\uparrow} + \int_0^t (1-\lambda) S_u^{\theta} dH_u^{\downarrow} \\ &+ \phi_t^{+} (1-\lambda) S_t^{\theta} - \phi_t^{-} S_t^{\theta}. \end{aligned} \tag{1}$$

Consistent price systems

Definition

A λ-*consistent price system* (λ-CPS) for S is a pair $(\tilde{S}, Q)$ of a probability measure $Q \sim P$ and a Q local martingale $\tilde{S}$ such that

$$(1-\lambda)S_t \leq \tilde{S}_t \leq S_t, \qquad a.s. \qquad \forall t \in [0,T]. \tag{2}$$

A λ-*strictly consistent price system* (λ-SCPS) is a CPS such that the inequalities are strict in (2).

Utility functions on $\mathbb{R}_+$

Definition (Admissibility)

Let $x > 0$. Denote

$$\mathcal{A}_0^\theta(x) := \{H \in \mathcal{A}^\theta(x) : \; W_t^{x,\mathrm{liq}}(\theta, H) \geq 0 \text{ a.s.}, \phi_T = H_T^\uparrow - H_T^\downarrow = 0\},$$

and $\mathcal{A}(x) = \bigcap_{\theta \in \Theta} \mathcal{A}_0^\theta(x)$.

The first result

Theorem

Let $U : (0, \infty) \to \mathbb{R}$ be a nondecreasing, concave function and $U(\infty) > 0$. Assume that

- *For each $\theta \in \Theta$ and for all $0 < \mu < \lambda$, the price process S^θ admits a μ-CPS.*
- *The dual problem for the model θ, is finite for all $\theta \in \Theta$.*

The robust utility maximization problem admits a solution.

$u(x) < \infty$? Candidate for H^*? Admissibility? Upper semicontinuity?

Nutz (2016): an example with $u(x) < \infty$ but there is no optimizer.
Reason: lack of upper-semicontinuity property in one model.
Condition: $E^P U^+(x + h\Delta S) < \infty, \forall h, P$ from Nutz (2016), Blanchard and Carassus (2018)

Utility functions on $\mathbb{R}$

Assumption

$U : \mathbb{R} \to \mathbb{R}$ is bounded from above, nondecreasing, concave, $U(0) = 0$. Define the convex conjugate by

$$V(y) := \sup_{x \in \mathbb{R}} (U(x) - xy), \qquad y > 0.$$

We also assume that

$$\lim_{x \to -\infty} \frac{U(x)}{x} = \infty, \tag{3}$$

$$\limsup_{y \to \infty} \frac{V(2y)}{V(y)} < \infty. \tag{4}$$

Admissibility

- $X_t \geq -C, \forall t$. Too small when S is non locally bounded.
- $X_t \geq -cW$ where $EU(-\alpha W) > -\infty$. Biagini and Frittelli (2005).
- Supermartingale property. Owen and Žitković (2009)
- $\mathcal{M}_V^{\theta}$: set of probabilities $Q \sim P$ with $EV(dQ/dP) < \infty$.

Admissibility

Define

$$\mathcal{M}_V^\theta := \{Q^\theta : (\tilde{S}^\theta, Q^\theta) \text{ is a consistent price system, } EV(dQ^\theta/dP) < \infty\},$$

$$V^x(\theta, H) = x - \int_0^t S_u^\theta dH_u^\uparrow + \int_0^t (1-\lambda) S_u^\theta dH_u^\downarrow + \phi_t \tilde{S}_t^\theta$$

Definition

We define

$$\mathcal{A}^\theta(x) = \Big\{ H \in \mathbf{V} : \phi_T = 0,\ V^x(\theta, H) \textit{ is a } Q^\theta\textit{-supermartingale for each consistent price system } (\tilde{S}^\theta, Q^\theta) \textit{ such that } Q^\theta \in \mathcal{M}_V^\theta \Big\}$$

and $\mathcal{A}(x) := \bigcap_{\theta \in \Theta} \mathcal{A}^\theta(x)$.

The second result

The optimization problem

$$u(x) = \sup_{H \in \mathcal{A}(x)} \inf_{\theta \in \Theta} EU(W_T^{x,liq}(\theta, H)). \tag{5}$$

Theorem

Let the above Assumption hold, and suppose that for each $\theta \in \Theta$, the price process S^θ admits a λ-SCPS $(Q^\theta, \tilde{S}^\theta)$ such that $Q^\theta \in \mathcal{M}_V^\theta$. Then there exists a solution for the problem (5).

$u(x) < \infty$? Candidate for H^*? Admissibility? Upper semicontinuity?

Orlicz spaces

$\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a *Young function* if it is convex with $\Phi(0) = 0$ and $\lim_{x\to\infty} \Phi(x)/x = \infty$.
The set

$$L^\Phi := \{X \in L^0 : E\Phi(\gamma|X|) < \infty \text{ for some } \gamma > 0\}$$

is a Banach space with the following norm

$$||X||_\Phi := \inf\{\gamma > 0 : X \in \gamma B_\Phi\}$$

where $B_\Phi := \{X \in L^0 : E\Phi(|X|) \leq 1\}$, the unit ball of L^Φ.
Define the conjugate function $\Phi^*(y) := \sup_{x\geq 0}(xy - \Phi(x)), y \in \mathbb{R}_+$. This is also a Young function and $(\Phi^*)^* = \Phi$.
Φ is of class Δ_2 if $\lim_{x\to\infty} \frac{\Phi(2x)}{\Phi(x)} < \infty$.

A compactness result

Lemma (Delbaen, Owari 2018)

Let Φ be a Young function of class Δ_2 and let $\xi_n, n \geq 1$ be a norm-bounded sequence in L^{Φ^*}. Then there are convex weights $\alpha_j^n \geq 0$, $n \leq j \leq M(n)$, $\sum_{j=n}^{M(n)} \alpha_j^n = 1$ such that

$$\xi_n' := \sum_{j=n}^{M(n)} \alpha_j^n \xi_j$$

converges almost surely to some $\xi \in L^{\Phi^*}$ as $n \to \infty$, and $\sup_n |\xi_n'|$ is in L^{Φ^*}. $\square$

Discrete-time frictionless setting

Assume that there is a discrete-time filtration $\mathcal{F}_t$, $t = 0, \ldots, T$.
Trading strategy ϕ: predictable.
Initial capital: c.
Portfolio terminal value: $W_T^c(\phi) := c + \sum_{i=1}^{T} \phi_i (S_i - S_{i-1})$.

Assumptions

There is θ^* such that S^{θ^*} is arbitrage-free and $P(D_t = \mathbb{R}^d) = 1$ for all $t = 1, \dots T$ where D_t is the smallest linear space containing the $\mathcal{F}_{t-1}$-conditional support of S_t.
In plain English: there is no redundant asset.

Sample results

Theorem. There is an optimal strategy if $U(\infty) < \infty$.
Theorem. Under convenient integrability conditions *for all* $\theta \in \Theta$ there is an optimal strategy even if U is unbounded.

The latter result is the first of its kind.

Continuous-time: serious difficulties.

Conclusion

- Looking at the problem under a new angle.
- Existence results.
- Future: duality?

Bartl, D. (2017). Exponential utility maximization under model uncertainty for unbounded endowments. *Preprint*. Retrieved from https://arxiv.org/abs/1610.00999v2.

Bartl, D., Cheridito, P., and Kupper, M. (2017). Robust utility maximization with medial limits. *Preprint, arXiv:1712.07699*.

Blanchard, R. and Carassus, L. (2018). Multiple-priors optimal investment in discrete time for unbounded utility function. *Annals of Applied Probability*. Retrieved from http://www.imstat.org/aap/future_papers.html.

Guasoni, P. (2002). Optimal investment with transaction costs and without semimartingales. *Annals of Applied Probability*, 12(4):1227–1246.

Kabanov, Y. M. (1999). Hedging and liquidation under transaction costs in currency markets. *Finance and Stochastics*, 3(2):237–248.

Neufeld, A. and Šikić, M. (2017). Robust utility maximization in discrete-time markets with friction. *Preprint*. Retrieved from https://arxiv.org/abs/1610.09230v1.

Nutz, M. (2016). Utility maximization under model uncertainty in discrete time. *Mathematical Finance*, 26(2):252–268.

Owen, M. P. and Žitković, G. (2009). Optimal investment with an unbounded random endowment and utility-based pricing. *Mathematical Finance*, 19(1):129–159.

Thank you for your attention!