Asymptotics for IBNR/infinite queue processes

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Innovative Research in Mathematical Finance, CIRM,
5th September 2018.

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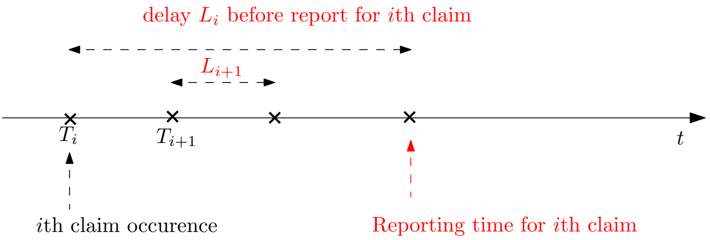
Incurred but not reported claims

Modelling of a situation where incoming claims are reported with some delay :

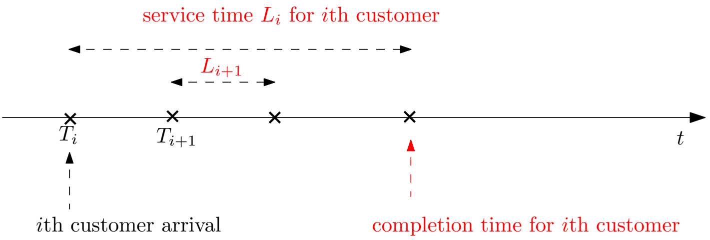


Number of non reported claims at time t :

$$Z(t) = \sum_{i=1}^{\infty} \mathbf{1}_{[T_i \leq t < T_i + L_i]}$$

Queueing point of view

Link with $G/G/\infty$ queues :



With this point of view, $Z(t) = \sum_{i=1}^{\infty} \mathbf{1}_{[T_i \leq t < T_i + L_i]}$ is the number of customers in the queue at time t.

Extension for model in dim. 1

$$Z(t) = Z^{\delta}(t) = \sum_{i=1}^{\infty} \mathbf{1}_{[T_i \leq t < T_i + L_i]} X_i e^{-\delta(T_i + L_i)},$$

- $\{\tau_i := T_{i+1} - T_i, \ i \in \mathbb{N}\}$ i.i.d. interclaim times, $\{L_i, \ i \in \mathbb{N}\}$ i.i.d. delay times,
- $\{X_i, \ i \in \mathbb{N}\}$ **i.i.d.** batch sizes,
- $\delta \geq 0$ discount rate.

Model in dim. k

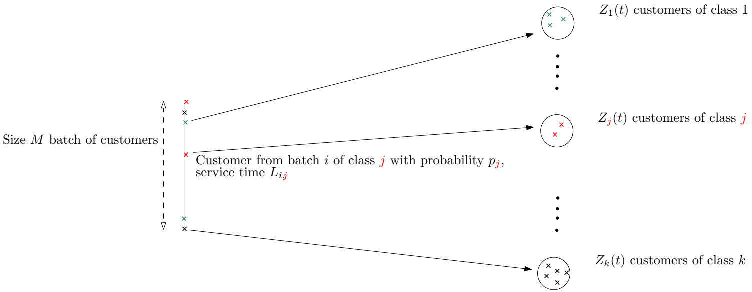
k branches $Z(t) = (Z_1(t), \dots, Z_k(t)) = Z^\delta(t)$, $t \geq 0$, where

$$Z_j(t) = Z_j^\delta(t) := \sum_{i=1}^{\infty} \mathbf{1}_{\{T_i \leq t < T_i + L_{i,j}\}} X_{i,j} e^{-\delta(T_i + L_{i,j})}, \quad j \in \{1, \dots, k\}.$$

- $\{\tau_i := T_{i+1} - T_i,\ i \in \mathbb{N}\}$ i.i.d. interclaim times,
- $\{(L_{i,1}, \dots, L_{i,k}),\ i \in \mathbb{N}\}$ i.i.d. delay times, with independent components $L_{i,1}, \dots, L_{i,k}$,
- $\{(X_{i,1}, \dots, X_{i,k}),\ i \in \mathbb{N}\}$ i.i.d. batch sizes, with *correlated* components $X_{i,1}, \dots, X_{i,k}$ and generic distribution of r.v. $X = (X_1, \dots, X_k)$.

Queueing interpretation

$\delta = 0$, Batches of sizes $(X_{i,1}, ..., X_{i,k}) \sim \mathcal{M}(M, p_1, ..., p_k)$ for some $M \in \mathbb{N}^*$,
Batch j of size $X_{i,j}$ with customers with same service time L_{ij},
$\Longrightarrow Z_1(t)$,..., $Z_k(t)$ are *correlated* $G/G/\infty$ queues.



Example $M = 1$: an arriving customer is sent to queue $Z_j(t)$ with probability p_j.

Known results

(Non discounted) IBNR process in dim 1**/ Infinite queue,** i.e.

$$Z(t) = \sum_{i=1}^{\infty} \mathbf{1}_{[T_i \le t < T_i + L_i]} X_i$$

$\longrightarrow$ Distribution available in Takács (1962) for exponential interclaims or delays and $X_i = 1$, in Willmot & Drekic (2002/2009), Guo & al (2014), Landriault & al (2014/2016), when interclaims are Matrix Exponential distributed.

Discounted IBNR process in dim k**,** i.e.

$$Z_j(t) = \sum_{i=1}^{\infty} \mathbf{1}_{[T_i \le t < T_i + L_{ij}]} X_{ij} e^{-\delta(T_i + L_{ij})}, \quad j = 1, ..., k.$$

$\longrightarrow$ Recursive renewal equation for joint moments in Woo (2016) .

Known results and objective of talk

In general : compact expression for either distribution (LT or cdf) or moments are not available.

$\Longrightarrow$ **Goal :** We define $\boxed{\tilde{Z}(t) = \tilde{Z}^{\delta}(t) := e^{\delta t} Z(t)}$

- **Asymptotics** for joint moments or **Convergence in distribution** as $t \to \infty$ for the k dimensional process $\tilde{Z}(t)$ for light tailed delays and *i.i.d.* X_i's,
- **Extreme behaviour/Convergence in distribution** for the 1 dimensional process when arrivals are Poisson and $(X_i)_{i \in \mathbb{N}}$ is a finite *Markov chain*.

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Assumptions

Recall that

$$Z_j(t) = Z_j^\delta(t) := \sum_{i=1}^{\infty} \mathbf{1}_{\{T_i \le t < T_i + L_{i,j}\}} X_{i,j} e^{-\delta(T_i + L_{i,j})}, \ j \in \{1, ..., k\},$$

and $\tilde{Z}(t) = \tilde{Z}^\delta(t) := e^{\delta t} Z(t)$.

Main assumptions :

- $X = (X_1, ..., X_k)$ admits joined moments of all order,
- density f of τ_1 is bounded and light tailed.

Notation

We let, for all $n = (n_1, ..., n_k) \in \mathbb{N}^k$ and $s = (s_1, ..., s_k) \in \mathbb{R}^k$,

$$\begin{aligned} \eta_n &:= \textstyle\sum_{j=1}^{k} n_j, \\ \tilde{M}_n(t) = \tilde{M}_n(t,\delta) &:= \mathbb{E}\left[\textstyle\prod_{j=1}^{k} \left(\tilde{Z}_j(t)\right)^{n_j}\right], \\ \psi(s,t) &= \mathbb{E}\left[e^{<s,Z(t)>}\right], \quad \tilde{\psi}(s,t) = \mathbb{E}\left[e^{<s,\tilde{Z}(t)>}\right]. \end{aligned}$$

And we define the *partial order* on $\mathbb{N}^k$:

$$\ell = (\ell_1, ..., \ell_k) < n = (n_1, ..., n_k) \iff \ell_j \le n_j,\ j = 1, ..., k,\ \text{and } \exists j_0,\ \ell_{j_0} < n_{j_0}.$$

Renewal equation for $t \mapsto \tilde{M}_n(t)$

Theorem (Woo (2016))

For all $n \in \mathbb{N}^k$, $t \mapsto \tilde{M}_n(t)$ satisfies the renewal equation

$$\tilde{M}_n(t) = \tilde{b}_n(t) + \tilde{M}_n \star F(t), \qquad t \geq 0, \tag{1}$$

where F is the cdf of τ_1, and

$$\tilde{b}_n(t) := \sum_{\ell < n} \binom{n_1}{\ell_1} \cdots \binom{n_k}{\ell_k} \mathbb{E}\left[\prod_{j=1}^{k} X_j^{n_j - \ell_j}\right] \mathbb{E}\left[\tilde{M}_\ell(t-\tau_1)\Pi_{n,\ell}(t-\tau_1).\mathbf{1}_{[\tau_1 < t]}\right]$$

for some explicit $\Pi_{n,\ell}(.)$.

Explicit expression (sort of...)

Solving (1) :

$$\tilde{M}_n(t) = \sum_{j=0}^{\infty} F^{\star(j)} \star \tilde{b}_n(t)$$

$\longrightarrow$ need to truncate sum with many integrals,

$\longrightarrow$ $\tilde{b}_n(.)$ depends on $\tilde{M}_\ell(.)$ for $\ell < n$.

$\Longrightarrow$ Hardly tractable in practice.

(Real) Explicit expression, Poisson arrivals

$$\text{Let } M^*_{t,X}(s) = \mathbb{E}\left[\exp\left(\sum_{j=1}^{k} s_j e^{-\delta L_j} X_{ij} \mathbf{1}_{[L_j>t]}\right)\right],\ t \geq 0,\ s \in \mathbb{R}^k.$$

Proposition

If $\tau_1 \sim \mathcal{E}(\lambda)$ then one has the following expression

$$\tilde{\psi}(s,t) = \exp\left[\lambda \int_0^t \left(M^*_{v,X}(e^{\delta v} s) - 1\right) dv\right], \quad t \geq 0,\ s \in \mathbb{R}^k,$$

and the mgf of $Z(t)$ is obtained explicitly by $\psi(s,t) = \tilde{\psi}(e^{-\delta t} s, t)$.

Asymptotics and limiting distribution

Theorem (R., Woo (2016))

For all $n \in \mathbb{N}^k$ *:*

$$\tilde{M}_n(t) \stackrel{t\to\infty}{\longrightarrow} \chi_n, \qquad t \to \infty, \tag{2}$$

where $\chi_n = \chi_n(\delta) := \dfrac{\int_0^\infty \tilde{b}_n(t)dt}{\mathbb{E}[\tau_1]}$. *Besides, if* $||X|| \leq M$ *constant or if* X *is New Better than Used :*

$$e^{\delta t} Z(t) \stackrel{\mathcal{D}}{\longrightarrow} \mathcal{Z}_\infty, \quad t \to \infty, \tag{3}$$

where $\mathcal{Z}_\infty = (\mathcal{Z}_{\infty,1}, \ldots, \mathcal{Z}_{\infty,k}) = \mathcal{Z}_\infty(\delta)$ *is a light tailed vector valued rv with the joint moments*

$$\mathbb{E}\left[\prod_{i=1}^k \mathcal{Z}_{\infty,i}{}^{n_i}\right] = \chi_n = \chi_n(\delta), \quad n = (n_1, \ldots, n_k) \in \mathbb{N}^k.$$

Hint of Proof

Convergence of moments : (2) is obtained thanks to renewal equation (1) for $\tilde{M}_n(t)$ + Smith's renewal theorem.

Convergence in distribution : (3) is obtained thanks to Convergence of moments (2) + Haviland (1935)'s criterion, then proving convergence of the LT of $e^{\delta t} Z(t)$.

$\square$

Case of exponential delays

χ_n depends on $\tilde{b}_n(.)$, which in turn depends on the $\tilde{M}_\ell(t)$ for $\ell < n$ $\Longrightarrow$ no explicit expression in general.

One particular case :

Theorem (Exponential delays)

Suppose that delays $L = (L_1, ..., L_k)$ verifies $L_j \sim \mathcal{E}(\mu)$ for all $j = 1, ..., k$.
Then the χ_n's, $n \in \mathbb{N}^k$, have an explicit expression, computable recursively in function of LT of τ_1, μ, the joint moments of $X = (X_1, ..., X_k)$.

First two moments of the workload $D(t) := \sum_{i=1}^{\infty}(T_i + L_i - t)\mathbf{1}_{\{T_i \leq t < T_i + L_i\}}$ are also available.

Queueing point of view : $\delta = 0$

When $\delta = 0$, back to example of batches of sizes $(X_{i,1}, ..., X_{i,k}) \sim \mathcal{M}(M, p_1, ..., p_k)$ for some $M \in \mathbb{N}^*$, and service times L_{ij} for customers of batch j of size $X_{i,j}$. Then :

- $Z_1(t)$,..., $Z_k(t)$ queue sizes of *correlated* $G/G/\infty$ queues,
- $(Z_1(t), ..., Z_k(t)) \xrightarrow{\mathcal{D}}_{t\to\infty} \mathcal{Z}_\infty = (\mathcal{Z}_{\infty,1}, \dots, \mathcal{Z}_{\infty,k})$ **stationary regime** of the queues.

Besides, when service times are $\mathcal{E}(\mu)$ then we get k (correlated) $G/M/\infty$ queues, and the distribution of $\mathcal{Z}_\infty = (\mathcal{Z}_{\infty,1}, \dots, \mathcal{Z}_{\infty,k})$ is **explicit**.

Content

Notation

Put $k = 1$. Recall that

$$Z(t) = \sum_{i=1}^{\infty} X_i \mathbf{1}_{[T_i \le t < T_i + L_i]},$$

where $\{X_i,\ i \in \mathbb{N}\}$ finite Markov chain with state space $\{0, ..., K\}$, transition matrix P, stationary distribution π.

We define the joint Laplace transform/mgf for $t \ge 0,\ s \le 0$

$$\psi(s,t) := \left[\mathbb{E} \left(e^{sZ(t)} \mathbf{1}_{[X_{N_t} = y]} \middle| \ X_0 = x \right) \right]_{(x,y) \in \{0,...,K\}^2} .$$

where N_t : total number of customers arrived within $[0, t]$. Two issues :

1. How to determine the distribution of $(Z(t), X_{N_t})$ (e.g. $\psi(s,t)$) for some fixed t ?
2. Behaviour as $t \to \infty$?

Some results (R. & Woo (2017) and (2018))

- In general, $\psi(s,t)$ not computable, but $\mathbb{E}(Z(t))$, $\mathbb{E}(Z(t)^2)$ are available in some cases, for $t \leq +\infty$.

- When $T_{i+1} - T_i \sim \mathcal{E}(\lambda)$ (Poisson arrival with intensity λ) then

$$\partial_t \psi(s,t) = [-\lambda I + \lambda \tilde{Q}(s,t)]\psi(s,t), \quad \psi(s,0) = I,$$

for some (substochastic) matrix $(\tilde{Q}(s,t))_{s\leq 0, t\geq 0}$.
Unfortunately, this matrix ODE does not admit a closed form solution !

Fast arrivals, Slow service in the Poisson arrival case

$\Longrightarrow$ Rescaling approach :

- speed up arrivals $\lambda \longrightarrow \lambda n^\gamma$ for some $\gamma > 0$,
- renormalize transition matrix $P \longrightarrow (1 - 1/n^\gamma)I + P/n^\gamma$,
- suppose that L_j's are fat tailed with index $\alpha \in (0,1)$ and slow down services $L_j \longrightarrow L_j/n$.

How does the corresponding queue $Z^{(n)}(t)$ jointly to corresponding state $X^{(n)}_{N_t^{(n)}}$ behave when n grows large, and $t \in [0,1]$ is fixed ?
$\Longrightarrow$ different behaviour whether $\gamma < \alpha$, $\gamma > \alpha$ or $\gamma = \alpha$.

Fast arrivals, Slow service

Theorem (R. (2018), in progress)

Let $\beta := 1/(1-\alpha)$, $\{\mathcal{X}(t),\ t \in [0,1]\}$ *a continuous time Markov chain with infinitesimal generating matrix* $\lambda(P-I)$ *with* $\mathcal{X}(0) \sim \pi$, $\{\mathcal{X}^\beta(t),\ t \in [0,1]\}$ *a continuous time inhomogeneous Markov chain with infinitesimal generating matrix* $\beta(1-t)^{\beta-1}\lambda(P-I)$ *with* $\mathcal{X}^\beta(0) \sim \pi$. *Let* $t \in [0,1]$. *One has one of the three limiting behaviours as* $n \to \infty$ *:*

- **Slow arrivals :** *If* $\gamma < \alpha$ *then*

$$\mathcal{D}\left(\left(Z^{(n)}(t), X^{(n)}_{N_t^{(n)}}\right)\middle|\ X_0^{(n)}\right) \longrightarrow \mathcal{D}\left(\left(\mathbf{0},\ \mathcal{X}(t)\right)\middle|\ \mathcal{X}(0)\right),$$

$n \to \infty$, *where* $\mathbf{0} = (0, ..., 0) \in \mathbb{R}^k$,

Fast arrivals, Slow service

Theorem (R. (2018), Cont'd)

- **Fast arrivals :** *If* $\gamma > \alpha$ *then, as* $n \to \infty$,

$$\mathcal{D}\left(\left(\frac{Z^{(n)}(t)}{n^{\gamma-\alpha}}, X^{(n)}_{N^{(n)}_t} \right) \middle| \; X^{(n)}_0 \right) \longrightarrow$$

$$\mathcal{D}\left(\left(\beta\lambda \int_{1-t^{1/\beta}}^{1} \mathcal{X}^{\beta}(v)\, dv,\; \mathcal{X}^{\beta}(1) \right) \middle| \; \mathcal{X}^{\beta}(1 - t^{1/\beta}) \right),$$

$n \to \infty$.

Fast arrivals, Slow service

Theorem (R. (2018), Cont'd)

- **Equilibrium :** *If* $\gamma = \alpha$ *then, as* $n \to \infty$,

$$\mathcal{D}\left(\left(Z^{(n)}(t), X^{(n)}_{N^{(n)}_t} \right) \Big| \; X^{(n)}_0 \right) \longrightarrow$$

$$\mathcal{D}\left(\left(\left(\int_{1-t^{1/\beta}}^{1} \mathcal{X}^{\beta}_j(v)\, \nu^{\beta}_j(dv) \right)_{j=1}^{k}, \; \mathcal{X}^{\beta}(1) \right) \Big| \; \mathcal{X}^{\beta}(1 - t^{1/\beta}) \right)$$

$n \to \infty$, *with* $\{\nu^{\beta}_j(t),\ t \geq 0\}$, $j = 1, \ldots, k$, *are* k *independent Poisson processes with same intensity* $\beta\lambda$, *independent from* $\{\mathcal{X}^{\beta}(t),\ t \in [0,1]\}$.

Merci !