Set-valued risk measures of non-convex portfolios

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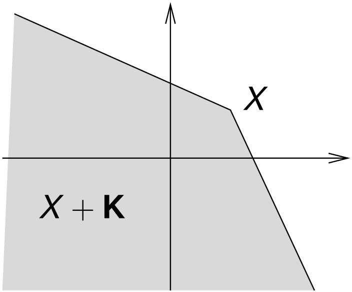
Kabanov's exchange cone model

- X is p-integrable random vector in $\mathbb{R}^d$ (gains on d assets/currencies).
- **K** is a random cone (e.g. generated by bid-ask exchange rates for currencies).
- **K** is the family of portfolios available at price zero.
- **K** describes transaction rules at the time when the gain X is assessed.

- Aim: measure the risk of X taking $\mathbf{K}$ into account.



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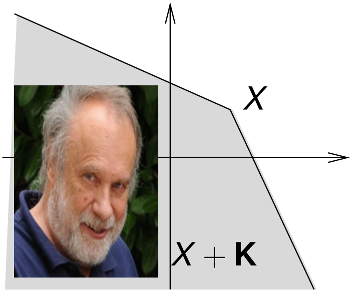


Vector-valued risk

- Tempting to describe the risk of $X \in \mathbb{R}^d$ by a vector $r(X) \in \mathbb{R}^d$ (taking into account the cone K).
- Natural assumptions:
  - $r(X + a) = r(X) - a$,
  - if $X \leq Y$ coordinatewisely, then $r(X) \geq r(Y)$,
  - $r(cX) = cr(X)$,
  - $r(X + Y) \leq r(X) + r(Y)$ coordinatewisely.

Theorem (I.Cascos, IM, 2007)

In this case

$$r(X) = (r_1(X_1), \dots, r_d(X_d)),$$

that is, all vector-valued coherent risk measures marginalise.

Set-valued risk

- Capital $a \in \mathbb{R}^d$ that makes $X + a$ acceptable can be chosen in many incomparable ways.
- So the risk measure is a set in $\mathbb{R}^d$.
- The possible values of risk measures are upper sets, i.e. sets F such that with each x it contains all $y \geq x$ (coordinatewisely).

- Position is acceptable if the risk measure contains the origin.
- Larger set means lower risk.

Previous work

- Hamel & Heyde (2010) and Hamel et al. (2011, 2013) derived dual representations for risk measures of $X + \mathbf{K}$.
- Ben Tahar & Lepinette (2014) and Feinstein & Rudloff (2012,2013) worked out the dynamical setting.
- Molchanov and Cascos (2016): primal representation in terms of selections.
- All previous work in the convex setting.

Set-valued portfolio

- Portfolio $\mathbf{X}$ is a random convex closed set such that $\mathbf{X} = \mathbf{X} + \mathbb{R}^d_-$ (lower set).
- Risk $\mathrm{R}(\mathbf{X})$ is an upper set; utility is a lower set.
- $\mathbf{X}$ is not necessarily convex.

Why set-valued portfolios?

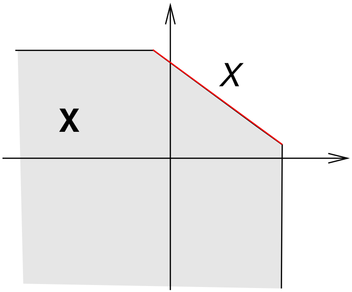
- $\mathbf{X}$ incorporates rather general liquidity restrictions.
- In the simplest case, $\mathbf{X} = X + \mathbf{K}$ for a cone $\mathbf{K}$ (Kabanov's model).
- In the dynamical setting the value of risk measure is a random set and it is used as the argument of another risk measure
  - numerical case $r_t(-r_s(X))$
  - set-valued setting $\mathrm{R}_t(-\mathrm{R}_s(\mathbf{X}))$, where

$$-F = \{-x : \; x \in F\}$$

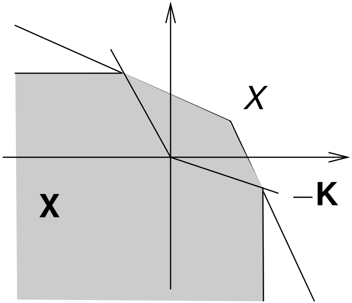
  denotes the central symmetric set to $F \subset \mathbb{R}^d$.

Beyond Kabanov's model

1:1 transactions up to a certain amount

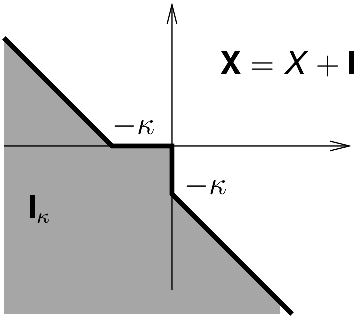


Only transactions that lead to solvent positions and/or disposal of assets are allowed

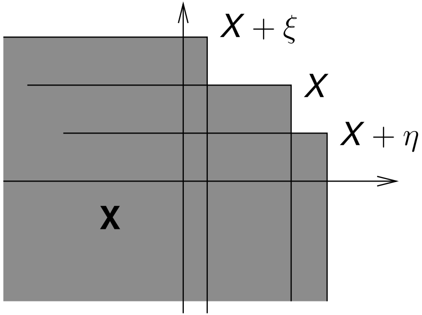


Non-convex portfolio: fixed transaction costs

- In case of each transaction, fixed cost κ is incurred.



Finite set of possible transactions



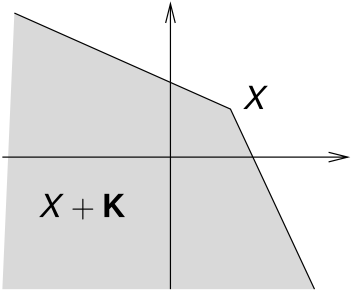
Selections and acceptability

- A random vector $\xi \in \mathbb{R}^d$ is called a selection of $\mathbf{X}$ if $\xi \in \mathbf{X}$ a.s.
  Assume that $\mathbf{X}$ contains at least one p-integrable selection, i.e. the set $L^p(\mathbf{X})$ is not empty, $p \in [1, \infty]$.
- Consider d-tuple $\mathbf{r} = (r_1, \dots, r_d)$ of univariate monetary L^p-risk measures.
- Random set $\mathbf{X}$ is acceptable if it possesses at least one acceptable selection ξ, meaning that

$$\mathbf{r}(\xi) = (r_1(\xi_1), \dots, r_d(\xi_d)) \leq 0\,,$$

  i.e. all individual coordinates of ξ are acceptable.

Example: cones



- $\mathbf{X} = X + \mathbf{K}$ has an acceptable selections exactly if there exists a transfer (selection) $\eta \in \mathbf{K}$ such that $\xi = X + \eta$ is acceptable.

Equivalent definition

- Let $\mathcal{A} \subset L^p(\mathbb{R}^d)$ be a family of acceptable random vectors in $\mathbb{R}^d$.
- Then portfolio **X** is acceptable iff $L^0(\mathbf{X}, \mathfrak{F}) \cap \mathcal{A} \neq \emptyset$, equivalently,

$$L^0(-\mathbf{X}, \mathfrak{F}) + \mathcal{A} \ni 0.$$

Selection risk measure

- The selection risk measure $\mathsf{R}(\mathbf{X})$ is the topological closure of the set

$$\{a \in \mathbb{R}^d : \ \mathbf{X} + a \text{ is acceptable}\} .$$

- Equivalently,

$$\mathsf{R}(\mathbf{X}) = \mathrm{cl} \bigcup_{\xi \in L^p(\mathbf{X})} \left(\mathbf{r}(\xi) + \mathbb{R}^d_+\right) .$$

- This is the primal representation of the risk measure.
- In other words, the risk is the closed union of risks determined by singletons chosen from $\mathbf{X}$.

Duality-based approach

- Conical setting $\mathbf{X} = X + \mathbf{K}$: Hamel & Heyde (2010) and Hamel et al. (2011, 2013).
- Convex $\mathbf{X}$ is the union of singletons (primal) and the intersection of half-spaces (dual).
- Portfolio $\mathbf{X}$ is acceptable if all half-spaces containing $\mathbf{X}$ are acceptable, that is,

$$\{x \in \mathbb{R}^d : \langle x, u \rangle \leq h(\mathbf{X}, u)\}$$

  is acceptable for all (possibly random) u.
- The acceptability of such half-spaces is derived from the support function $h(\mathbf{X}, u)$.
- However, such approach converts $\mathbf{X}$ to its convex hull.

Fixed points

- If all components of $\mathbf{r}$ are essinf, that is, $\mathbf{r}(\xi) = \mathrm{essinf}\xi$, then

$$\mathsf{R}(\mathbf{X}) = -F_{\mathbf{X}},$$

where

$$F_{\mathbf{X}} = \{x : \mathbf{P}\{x \in \mathbf{X}\} = 1\}$$

denotes the set of fixed points.

Proposition

For any selection risk measure, $-F_{\mathbf{X}} \subseteq \mathsf{R}(\mathbf{X})$.

Selection (Aumann) expectation

- The closure of the set of expectations of all integrable selections
$$\mathbf{E}\mathbf{X} = \mathrm{cl}\{\mathbf{E}\xi : \ \xi \in L^1(\mathbf{X})\}$$
is called the selection (Aumann) expectation of $\mathbf{X}$.
- If $r(\xi) = -\mathbf{E}\xi$, then
$$\mathrm{R}(\mathbf{X}) = -\mathbf{E}\mathbf{X}$$
is a selection risk measure.
- It is linear
$$\mathbf{E}(\mathbf{X} + \mathbf{Y}) = \mathrm{cl}(\mathbf{E}\mathbf{X} + \mathbf{E}\mathbf{Y})$$
and $\mathbf{E}\mathbf{X}$ is always convex on non-atomic probability spaces, no matter if $\mathbf{X}$ is convex or not.
- The primal and dual approaches yield the same set.

Non-linear functions constructed from expectation

- If $\mathbf{X}_1, \dots, \mathbf{X}_n$ are independent copies of $\mathbf{X}$, let
$$\mathrm{R}_\cap(\mathbf{X}) = -\mathbf{E}(\mathbf{X}_1 \cap \cdots \cap \mathbf{X}_n).$$
- It satisfies the property
$$\mathrm{R}_\cap(\mathbf{X} + \mathbf{Y}) \supseteq \mathrm{R}_\cap(\mathbf{X}) + \mathrm{R}_\cap(\mathbf{Y}),$$
however, $0 \in \mathrm{R}_\cap(\mathbf{X})$ does not imply the existence of an acceptable selection with respect to the underlying risk measure (expected minimum).
- The same applies to
$$\mathrm{R}_{\mathcal{Z}}(\mathbf{X}) = \bigcap_{\zeta \in \mathcal{Z}} \mathbf{E}(-\zeta \mathbf{X}),$$
where $\mathcal{Z}$ is a family of q-integrable non-negative random variables with expectation one.

Selection risk measure (reminder)

- The selection risk measure $\mathsf{R}(\mathbf{X})$ is the topological closure of the set

$$\{a \in \mathbb{R}^d : \mathbf{X} + a \text{ is acceptable}\}.$$

- Equivalently,

$$\mathsf{R}(\mathbf{X}) = \mathrm{cl} \bigcup_{\xi \in L^p(\mathbf{X})} \left(\mathbf{r}(\xi) + \mathbb{R}^d_+\right).$$

Main properties

Theorem

The selection risk measure (in the non-convex setting) satisfies the following conditions

1. $\mathrm{R}(\mathbf{X}+a)=\mathrm{R}(\mathbf{X})-a$ *for all* $a\in\mathbb{R}^d$ *(cash invariance).*
2. *If* $\mathbf{X}\subset\mathbf{Y}$ *a.s., then* $\mathrm{R}(\mathbf{X})\subset\mathrm{R}(\mathbf{Y})$ *(monotonicity).*
3. *If* $\mathbf{r}$ *is homogeneous, then* $\mathrm{R}(c\mathbf{X})=c\mathrm{R}(\mathbf{X})$ *for all* $c>0$ *(homogeneity).*
4. *If* $\mathbf{r}$ *is convex, then*

$$\mathrm{R}(\lambda\mathbf{X}+(1-\lambda)\mathbf{Y})\supseteq\lambda\mathrm{R}(\mathbf{X})+(1-\lambda)\mathrm{R}(\mathbf{Y})$$

   (superadditivity for inclusion, larger set means lower risk).

Univariate case

- Consider $X \in L^p(\mathbb{R})$, and let $\mathbf{X} = (-\infty, X]$.
- All selections $\xi \in \mathbf{X}$ are dominated by X.
- Then the selection risk measure of $\mathbf{X}$ is

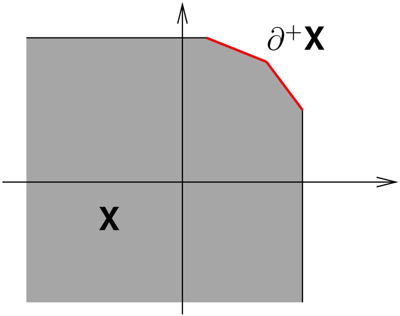
$$\mathrm{R}(\mathbf{X}) = [r(X), \infty).$$

Finding acceptable selections

- $\partial^+ \mathbf{X}$ is the set of Pareto optimal points of $\mathbf{X}$.
- $\mathbf{X}$ is called quasi-bounded if

$$\|\partial^+ \mathbf{X}\| = \sup\{\|x\| : \ x \in \partial^+ \mathbf{X}\}$$

  is finite.
- Selections determining the risk should belong to $\text{cl}\, \partial^+ \mathbf{X}$.



Convexity: expectation

- $\mathrm{R}(\mathbf{X}) = \mathbf{E}(-\mathbf{X})$ is a selection risk measure.
- It takes convex values if the underlying probability space is nonatomic, and then $\mathrm{R}(\mathbf{X}) = \mathrm{R}(\operatorname{conv}\mathbf{X})$.
- This follows from Lyapunov's theorem on ranges of vector-valued measures.
- But this theorem generally does not hold for measures with values in infinite-dimensional Banach spaces.

Convexity: example

Example

If $\mathbf{X} = \{\xi, \eta\} + \mathbb{R}^d_-$, then $\mathrm{R}(\mathbf{X})$ is convex if,

for each $t \in (0, 1)$, there exists $A \in \mathfrak{F}$ such that

$$t\mathbf{r}(\xi) + (1 - t)\mathbf{r}(\eta) \geq \mathbf{r}(\xi \mathbf{1}_A + \eta \mathbf{1}_{A^c}).$$

Convexity: general

- Assume that

$$\mathbf{r}(\xi) = \sup_{\zeta \in L^q(\mathbb{R}^d_+), \mathbf{E}\zeta = 1} \Big(\mathbf{E}(-\zeta\xi) - \alpha(\zeta) \Big), \quad \xi \in L^p(\mathbb{R}^d),$$

  where $\alpha(\zeta) = (\alpha_1(\zeta_1), \dots, \alpha_d(\zeta_d))$ and $\alpha_i : L^q(\mathbb{R}_+) \mapsto (-\infty, \infty]$ are the penalty functions corresponding to the components of $\mathbf{r}$.

Theorem

Assume that the probability space is non-atomic and the penalty function α has all infinite components unless ζ belongs to a finite family. Then $\mathrm{R}(\mathbf{X})$ is convex.

Convexity: deterministic sets

- F is **r**-convex if $\mathrm{R}(F) = -F$.
- This is the case if and only if, for any $x, x' \in F$ and $A \in \mathfrak{F}$, we have

$$\mathbf{r}(x\mathbf{1}_A + x'\mathbf{1}_{A^c}) \in -F.$$

- Intersection of **r**-convex sets is **r**-convex.
- Each lower closed set is convex with respect to the essinf risk measure.
- The set $\mathbf{I}_\kappa$ (fixed transaction costs) is convex with respect to the Average Value-at-Risk at level $\alpha \leq 1/2$.

Law invariance

Theorem

Assume that the probability space is nonatomic and that $\mathbf{r}$ *is Lebesgue continuous, that is, it is continuous on a.s. convergent uniformly p-integrably bounded sequences of random vectors.*

Then $\mathrm{R}(\mathbf{X})$ *is law invariant on portfolios, such that* $\|\partial^{+}\mathbf{X}\|$ *is p-integrable.*

- For convex $\mathbf{X}$ and coherent $\mathbf{r}$, we always have law invariance, see Molchanov & Cascos (2016).

Law invariance: proof

- If $\mathbf{X}$ and $\mathbf{X}'$ share the same distribution, then $\operatorname{cl}\partial^+\mathbf{X}$ and $\operatorname{cl}\partial^+\mathbf{X}'$ are p-integrably bounded and share the same distribution.
- Let $x \in \mathbf{r}(\xi) + \mathbb{R}^d_+$ for $\xi \in L^p(\operatorname{cl}\partial^+\mathbf{X})$.
- Since the weak closures of $L^0(\operatorname{cl}\partial^+\mathbf{X})$ and $L^0(\operatorname{cl}\partial^+\mathbf{X}')$ coincide, there is a sequence $\eta_n \in L^p(\operatorname{cl}\partial^+\mathbf{X}')$ converging weakly to ξ. Then $\|\eta_n\| \leq \|\operatorname{cl}\partial^+\mathbf{X}'\|$, and the latter random variable is integrable.
- Thus, $\{\eta_n, n \geq 1\}$ is relatively compact in $L^1(\mathbb{R}^d)$. By passing to a subsequence, it is possible to assume that $\eta_{n_k} \to \xi$ almost surely.
- The Lebesgue continuity property yields that $\mathbf{r}(\eta_{n_k}) \to \mathbf{r}(\xi)$. Thus, $\mathbf{r}(\xi) \in \mathrm{R}(\operatorname{cl}\partial^+\mathbf{X}')$, since the latter set is closed. Finally, $x \in \mathrm{R}(\mathbf{X}')$ since the latter set is upper.

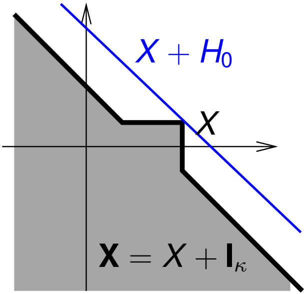
Fixed transaction costs: bounds

- Let $\mathbf{X} = X + \mathbf{I}_\kappa$, where $\mathbf{I}_\kappa = \mathbb{R}^d_- \cup H_{-\kappa}$, and

$$H_t = \{x \in \mathbb{R}^d : \sum_{i=1}^{d} x_i \le t\}, \quad t \in \mathbb{R}.$$

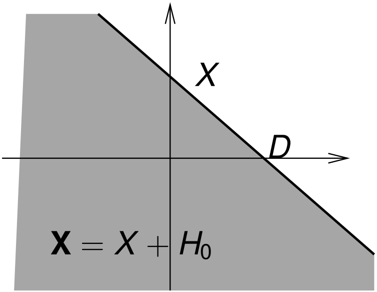
- Then

$$(\mathbf{r}(X) - \mathbf{I}_\kappa) \cup \mathrm{R}(X + H_{-\kappa}) \subset \mathrm{R}(X + \mathbf{I}_\kappa) \subset \mathrm{R}(X + H_0).$$



Selection risk of a half-space: coherent case

Let $D = X_1 + \cdots + X_d$.



If $\mathbf{r}$ is coherent, then $\mathrm{R}(X + H_0) = H_{-r(D)}$.

Selection risk of a half-space

Let $D = X_1 + \cdots + X_d$, and let $\mathbf{r} = (r_1, \ldots, r_d)$.

Proposition

i) If all components of $\mathbf{r}$ are identical convex risk measures r, then $\mathrm{R}(X + H_0) = -H_{-dr(D/d)}$.

ii) If one of the components of $\mathbf{r}$ is the negative essential infimum and all other are identical convex risk measures r, then $\mathrm{R}(X + H_0) = -H_{-(d-1)r(\frac{D}{d-1})}$.

iii) If one of components of $\mathbf{r}$ is the negative expectation and all others are identical convex risk measures r such that $r(\xi) \geq -\mathbf{E}\xi$ for all $\xi \in L^1(\mathbb{R})$, then

$$\mathrm{R}(X + H_0) = -H_{-\mathbf{E}D}.$$

Calculating the selection risk

- $\mathbf{X}$ has too many selections.
- If $\mathbf{X} = X + F$ for p-integrable X and deterministic lower closed F, then

$$\mathrm{R}(\mathbf{X}) \supseteq \mathbf{r}(X) + \mathrm{R}(F)$$

  assuming that $\mathbf{r}$ is coherent.

Exact calculation: fixed transaction costs

- Let $\mathbf{X} = \mathbf{I}_\kappa$ (deterministic) in dimension 2, and let $\mathbf{r} = (r_1, r_2)$ have coherent components.
- Then the risk of $\mathbf{I}_\kappa$ is determined by the set

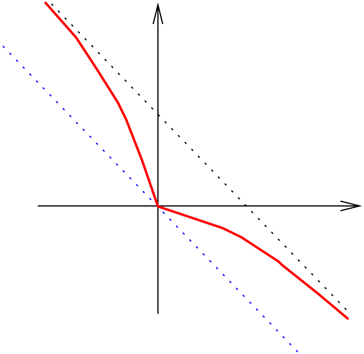
$$B_{\mathbf{r}} = \{(r_1(\mathbf{1}_A), r_2(-\mathbf{1}_A)) : \ A \in \mathfrak{F}\},$$

  where $\mathbf{P}(A)$ is varying between 0 and 1.
- If $\mathbf{r}$ is the Average Value-at-Risk at level $\alpha \leq 1/2$, then

$$\mathrm{R}(\mathbf{I}_\kappa) = -\mathbf{I}_\kappa,$$

  which is the set of fixed points.

Exact calculation: fixed transaction costs, $\alpha > 1/2$

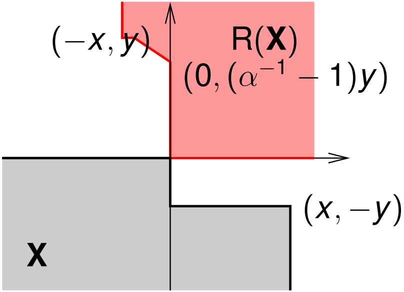


- The set $\mathrm{R}(\mathbf{I}_\kappa)$ becomes $\mathrm{conv}(-\mathbf{I}_\kappa) = \mathbf{E}(-\mathbf{I}_\kappa)$ if $\alpha = 1$.

Two admissible transactions

- Consider the set $\mathbf{X} = M + \mathbb{R}^2_-$, where $M = \{(0,0),(x,-y)\}$ with $x, y > 0$.
- Let $\mathbf{r}$ be the Average Value-at-Risk at level $\alpha > 1/2$.
- Consider selections $\xi = (x,y)\mathbf{1}_A$. Then

$$\mathbf{r}(\xi) = (xr(\mathbf{1}_A), yr(-\mathbf{1}_A)).$$



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