Beyond the Kabanov financial market model with proportional transaction costs.

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The beginning

-Y. Kabanov and M. Safarian. On Leland's strategy of option pricing with transaction costs. Finance and Stochastics, 1, 3, 1997, 239-250.

-Y. Kabanov. Hedging and liquidation under transaction costs in currency markets. Finance and Stochastics, 3, 1999, 2, 237-248.

-F. Delbaen, Y. Kabanov. and E. Valkeila. Hedging under transaction costs in currency markets : a discrete-time model. Mathematical Finance, 12, 2002, 1, 45-61.

-G. Last Y. Kabanov. Hedging under transaction costs in currency markets : a continuous time model. Mathematical Finance, 12, 2002, 1, 63-70.

-Y. Kabanov. The Harrison-Pliska arbitrage pricing theorem under transaction costs. Journal of Mathematical Economics, 35 (2001), 2, 185-196.

This papers is followed by 17 other articles by Y. Kabanov on transaction costs and last but not least, the famous book :

Figure: Y. Kabanov and M. Safarian. Markets with Transaction Costs. Mathematical Theory. Springer-Verlag, 2009.

More than 1000 citations (Google Scholar) of Kabanov articles on transaction costs. Articles on the Kabanov model by B. Bouchard, L. Campi, E. Lépinette, J. Muhle Karbe, M. Rasonyi, B. Rudloff, Soner M, Schachermayer, Striker C., N. Touzi among others....

A general model derived from the Kabanov model

Let $(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, \mathrm{P})$ be a complete stochastic basis. The Kabanov model is defined by a sequence of random closed sets $(G_t)_{t\in[0,T]}$ in $\mathbf{R}^d$ such that

$$\mathrm{Graph}\, G_t = \{(\omega, x) \in \Omega \times \mathbf{R}^d :\ x \in G_t(\omega)\} \in \mathcal{F}_t \times \mathcal{B}(\mathbf{R}^d), \quad t \in [0, T].$$

At time t, G_t is interpreted as the set of all solvent portfolio positions in the d assets of consideration expressed in physical units, that can be liquidated without any debt.

The solvency set G_t may be associated to a liquidation value process, e.g. the maximal amount of money expressed in the first asset you can get when liquidating the portfolio position $x \in \mathbf{R}^d$:

$$\mathbf{L}_t(x) := \sup\{\alpha : \; x - \alpha e_1 \in G_t\}, \quad e_1 = (1, 0, \cdots, 0) \in \mathbf{R}^d.$$

Under mild conditions, $x - \mathbf{L}_t(x)e_1 \in G_t$ so that $x = (x - \mathbf{L}_t(x)e_1) + \mathbf{L}_t(x)e_1$ is changed into $\mathbf{L}_t(x)e_1$ once the position $x - \mathbf{L}_t(x)e_1$ is liquidated. We may show that

$$G_t(\omega) = \{x \in \mathbf{R}^d : \; \mathbf{L}_t(x) \geq 0\}.$$

Portfolio processes

A discrete-time portfolio process, expressed in physical units, is an adapted process in $\mathbf{R}^d$, with initial capital value $V_{-1} \in \mathbf{R}^d$, and satisfying :

$$V_t - V_{t-1} \in -G_t, \quad t = 0, \cdots, T.$$

Interpretation : $V_{t-1} = (V_{t-1} - V_t) + V_t$ is changed into V_t at time t as we may liquidate $V_{t-1} - V_t \in G_t$.

In continuous time, a portfolio process is of bounded variations $\|V\|$ and its Radon-Nikodym derivative $\dot{V} := \frac{dV}{d\|V\|}$ satisfies

$$\dot{V}_t \in -G_t, \quad t \in [0, T].$$

The Kabanov model with proportional transaction costs

The model is composed of d risky assets that can be directly exchanged with each other paying transaction costs defined by an adapted matrix-valued process $(\lambda_t^{i,j})_{i,j=1,\cdots,d}$ where $\lambda_t^{i,j}$ is the transaction cost rate to exchange asset $\#i$ into $\#j$.

A portfolio process $W \in \mathbf{R}^d$ expressed in amounts of money hold in each asset satisfies in discrete-time

$$W_t^i - W_{t-1}^i = \sum_{i=1}^{d} T_t^{ji} - \sum_{i=1}^{d}(1+\lambda_t^{ij})T_t^{i,j}, \quad i = 1, \cdots, d$$

where T_t^{ji} is the amount of capital in asset j transferred in asset i, and $\lambda_t^{ij} T_t^{i,j}$ are the transaction costs to pay when transferring the capital $T_t^{i,j}$ from asset i to asset j.

The Kabanov model with proportional transaction costs

Let $S_t^1, \cdots, S_t^d$ be the positive prices of the risky assets. Expressed in physical units, the process $V_t = (W_t^1/S_t^1, \cdots, W_t^d/S_t^d)$ satisfies $V_t - V_{t-1} \in -G_t$ where G_t is the solvency cone

$$G_t = \mathrm{cone}\{(1+\lambda_t^{i,j})e_i - e_j,\ e_i, \quad 1 \le i,j \le d\}.$$

Classical problem : characterizing super-hedging prices of European/American options and related to this, characterizing no-arbitrage conditions as "analog" to the frictionless case.

Main results for the Kabanov model with proportional transaction costs

-In discrete or continuous time, absence of arbitrage opportunity is equivalent to the existence of Consistent Price Systems, i.e. a martingale $Z \in \mathbf{R}^d$ such that $Z_t \in G_t^* \setminus \{0\}$ (weak form of NA, i.e. NFL) or better $Z_t \in \mathrm{int}\, G_t^*$ under a strong form of NA.

See the main results by Grigoriev, Guasonyi, Kabanov , Lépinette, Rasony but also Schachermayer when $d = 2$ with small transaction costs.

-The super-hedging prices may be dually characterized by the consistent price systems.

See the main results by Bouchard-Chassagneux, Campi-Schachermayer, DeVallière-Kabanov-Lépinette and, recently, Dolinsky-Soner.

When the solvency set is no more a (random) cone.

There are many situations where the solvency set is not a cone, in particular in presence of an order book with several bid/ask prices (G is only convex) but also in presence of fixed costs (G is no more convex).

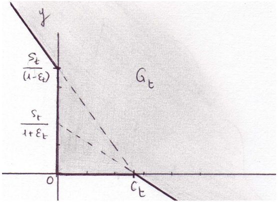

Figure: Solvency set with Bid/Ask prices $S_t/(1+\epsilon_t)$ and $S_t/(1-\epsilon_t)$ and fixed cost c_t per trade.

When the solvency set is no more a (random) cone.

In that case, you may not use the classical tools from convex analysis (e.g. the Hahn-Banach separation theorem) to dually characterize a no-arbitrage condition and deduce super-hedging prices.

A programming dynamic approach based on random preference relations ($x \geq_t y$ iff $x - y \in G_t$) may be used (see papers by Kabanov, Lépinette, Tran) but that seems to be complicated to numerically implement, except when Ω is finite, see Rudloff & Al.

The liquidation value approach : motivation

Let us consider the two-dimensional Kabanov model in discrete-time where the solvency sets are closed convex cones.

If G_t contains $\mathbf{R}^2_+$ and is smaller than a half-plane, it is such that $G_t^* = \mathrm{cone}\left(\{1\} \times [S_t^b, S_t^a]\right)$ for some $\mathcal{F}_t$-measurable random variables S_t^b, S_t^a such that $0 \leq S_t^b \leq S_t^a$, i.e. such a model is a Bid-Ask model. This is the case for a financial market model composed of a bond $B \equiv 1$ and a risky asset with bid and ask prices S^b, S^a.

The liquidation value is $\mathbf{L}_t(x, y) = x + y^+ S_t^b - y^- S_t^a$, $(x, y) \in \mathbf{R}^2$, so that $G_t = \{z \in \mathbf{R}^2 : \ \mathbf{L}_t(z) \geq 0\}$.

The liquidation value approach : motivation

Consider $\mathcal{A}_0^T = \sum_{u=0}^T L^0(-G_u, \mathcal{F}_u)$ the set of all terminal portfolio values starting from $0 \in \mathbf{R}^d$. The Grigoriev theorem is the following for $d = 2$.

Theorem

The following are equivalent :
(1) $\mathcal{A}_0^T \cap L^0(\mathbf{R}_+^2) = \{0\}$ *(NA).*
(2) $\overline{\mathcal{A}_0^T} \cap L^0(\mathbf{R}_+^2) = \{0\}$.
(3) There exists a consistent price system in $G^* \setminus \{0\}$.

Actually, there is a counterexample by Y. Kabanov showing that $\mathcal{A}_0^T$ may be not closed even if (NA) holds (see Section 3.2.4., Yuri's book). Note that closedness is required to dually characterize the super hedging prices.

The natural question is whether the same holds when we consider the set of all terminal liquidation values

$$\mathcal{L}_0^T := \{\mathbf{L}_T(V) : \ V \in \mathcal{A}_0^T\}$$

instead of $\mathcal{A}_0^T$. Surprisingly, this approach has not been really studied.

The Dalang–Morton–Willinger version of the fundamental theorem of asset pricing for the Bid-Ask financial market model. The case $d = 2$, by E. Lépinette and J. Zhao.

Theorem

Let $d = 2$. Suppose that Condition (E) holds if $T \geq 3$. The following conditions are equivalent to (NA) :

2. $\mathcal{L}_0^T$ *is closed in probability and* $\mathcal{L}_0^T \cap L^0(\mathbf{R}_+, \mathcal{F}_T) = \{0\}$.
3. *There exists* $Q \sim \mathrm{P}$ *with* $dQ/d\mathrm{P} \in L^\infty$ *such that* $\mathbb{E}_Q \mathbf{L}_T(V) \leq 0$ *for all* $\mathbf{L}_T(V) \in \mathcal{L}_0^T \cap L^1(\mathrm{P})$.
4. *There exists* $Q \sim \mathrm{P}$ *with* $dQ/d\mathrm{P} \in L^\infty$ *such that for all* $t \leq T - 1$, $\mathbb{E}_Q(S^a_{t+1}|\mathcal{F}_t) \geq S^b_t$ *and* $\mathbb{E}_Q(S^b_{t+1}|\mathcal{F}_t) \leq S^a_t$.
5. *There exists* $Q \sim \mathrm{P}$ *with* $dQ/d\mathrm{P} \in L^\infty$ *and a* Q*-martingale* $\tilde{S}$ *such that* $\tilde{S} \in [S^b, S^a]$.

The super hedging problem

In the following, we denote by $\mathcal{M}^\infty(P)$ the set of all $Q \sim \mathrm{P}$ such that $dQ/d\mathrm{P} \in L^\infty$ and $\mathbb{E}_Q \mathbf{L}_T(V) \leq 0$ for all $\mathbf{L}_T(V) \in \mathcal{L}_0^T$.

For any contingent claim $\xi \in L^1(\mathbf{R}, \mathcal{F}_T, \mathrm{P})$, we define Γ_ξ as the set of all initial endowments we need to start a portfolio process whose terminal liquidation value coincides with ξ, i.e.

$$\Gamma_\xi := \{x \in \mathbf{R} : \ \exists V \in \mathcal{A}_0^T : \ \mathbf{L}_T(xe_1 + V_T) \geq \xi\}.$$

Corollary

Suppose that Condition (E) holds if $T \geq 3$. Let us consider a payoff $\xi \in L^0(\mathbf{R}, \mathcal{F}_T)$ satisfying $\mathbb{E}_{\mathrm{P}}|\xi| < \infty$. Then, under Condition (NA), $\Gamma_\xi = [\sup_{Q \in \mathcal{M}^\infty(P)} \mathbb{E}_Q \xi, \infty)$.

Question

Is it possible to extend this result to $d \geq 3$ and to relax our condition (E) ? If yes, this means that we reconciliate the results with or without friction.

Thank you for your attention !