On Approximation of the Backward Stochastic Differential Equation.

Small noise, ergodic diffusion and unknown volatility cases.

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Backward Stochastic Differential Equation

Problem: We are given a stochastic differential equation (called *forward*)

$$\mathrm{d}X_t = b(t, X_t)\,\mathrm{d}t + a(t, X_t)\,\mathrm{d}W_t, \quad X_0 = x_0,\ 0 \le t \le T,$$

and two functions $f\,(t, x, y, z)$ and $\Phi\,(x)$. We have to construct a couple of processes (Y_t, Z_t) such that the solution of the equation

$$\mathrm{d}Y_t = -f(t, X_t, Y_t, Z_t)\,\mathrm{d}t + Z_t\,\mathrm{d}W_t, \quad Y_0,\ 0 \le t \le T,$$

(called *backward*) has the final value $Y_T = \Phi\,(X_T)$.

For the existence and uniqueness of the solution see Pardoux and Peng (1990). The *Markovian case* considered here was discussed by Pardoux and Peng (1992) and El Karoui & al. (1997).

Solution: Suppose that $u(t,x)$ satisfies the equation

$$\frac{\partial u}{\partial t} + b(t,x)\frac{\partial u}{\partial x} + \frac{1}{2}a(t,x)^2\frac{\partial^2 u}{\partial x^2} = -f\left(t,x,u,a(t,x)\frac{\partial u}{\partial x}\right),$$

with the final condition $u(T,x) = \Phi(x)$. Then if we put $Y_t = u(t,X_t)$, $Z_t = a(t,X_t)\,u'_x(t,X_t)$. Then by Itô's formula

$$\begin{aligned} \mathrm{d}Y_t &= \left[\frac{\partial u}{\partial t}(t,X_t) + b(t,X_t)\frac{\partial u}{\partial x}(t,X_t) + \frac{1}{2}a(t,x)^2\frac{\partial^2 u}{\partial x^2}(t,X_t)\right]\mathrm{d}t \\ &\quad + a(t,X_t)\frac{\partial u}{\partial x}(t,X_t)\,\mathrm{d}W_t \\ &= -f(t,X_t,Y_t,Z_t)\,\mathrm{d}t + Z_t\,\mathrm{d}W_t, \qquad Y_0 = u(0,X_0). \end{aligned}$$

The final value $Y_T = u(T,X_T) = \Phi(X_T)$.

Statistical problems. We consider this problem in the situations, where the forward equation contains some unknown parameter ϑ:

$$\mathrm{d}X_t = b(\vartheta, t, X_t)\,\mathrm{d}t + a(\vartheta, t, X_t)\,\mathrm{d}W_t, \quad X_0 = x_0,\ 0 \leq t \leq T.$$

Then $u = u\,(t, x, \vartheta)$ and the proposed approximations $\hat{Y}_t, \hat{Z}_t$ of the couple Y_t, Z_t are given by the relations

$$\hat{Y}_t = u(t, X_t, \vartheta_t^*), \qquad \hat{Z}_t = u'_x(t, X_t, \vartheta_t^*)\, a(\vartheta_t^*, t, X_t).$$

Here ϑ_t^* is some *good* estimator-process of ϑ with the *small error* of estimation $\mathbf{E}_\vartheta \left(\hat{\vartheta}_t - \vartheta\right)^2$. This provides us the small errors $\mathbf{E}_\vartheta \left(\hat{Y}_t - Y_t\right)^2$ and $\mathbf{E}_\vartheta \left(\hat{Z}_t - Z_t\right)^2$.

$$\vartheta^* = \left(\vartheta_t^*, 0 < t \leq T\right)$$

Main problem: *how to find a good estimator-process* $\vartheta_t^*, 0 < t \leq T$? *Good* means :

- *It depends on observations* $X^t = (X_s, 0 \leq s \leq t)$ *and is stochastic process* $\vartheta^\star = \vartheta_t^\star, 0 < t \leq T$.
- *Easy to calculate for all* $t \in (0, T]$.
- *Asymptpotically efficient for all* $t \in (0, T]$.

The MLE $\hat\vartheta_t$ defined by

$$V\left(\hat\vartheta_t, X^t\right) = \sup_{\vartheta \in \Theta} V\left(\vartheta, X^t\right)$$

can not be used as *Good* because in non linear case to solve this equation for all $t \in (0, T]$ is a difficult problem.

As *Forward Equations* we consider three diffusion processes:

- Diffusion process with *small noise* $(\varepsilon \to 0)$

$$
\begin{aligned}
A: \qquad & \mathrm{d}X_t = S\left(\vartheta, t, X_t\right)\mathrm{d}t + \varepsilon\sigma\left(t, X_t\right)\,\mathrm{d}W_t, \quad x_0,\ 0 \le t \le T,\\
B: \qquad & \mathrm{d}X_t = -a\left(\vartheta, t\right)X_t\mathrm{d}t + \varepsilon b\left(\vartheta, t\right)\mathrm{d}V_t, \quad x_0 \neq 0,\\
& \mathrm{d}R_t = A\left(\vartheta, t\right)X_t\mathrm{d}t + \varepsilon\sigma\left(t\right)\mathrm{d}W_t, \quad R_0 = 0, \quad 0 \le t \le T.
\end{aligned}
$$

- Discrete time observations $X^n = (X_{t_0}, X_{t_1}, \ldots X_{t_n})$, $t_i = i\frac{T}{n}$ of the process $(n \to \infty)$

$$
\mathrm{d}X_t = S\left(t, X_t\right)\mathrm{d}t + \sigma\left(\vartheta, t, X_t\right)\,\mathrm{d}W_t, \quad X_0,\ 0 \le t \le T.
$$

- Ergodic diffusion process $(T \to \infty)$

$$
\mathrm{d}X_t = S\left(\vartheta, X_t\right)\mathrm{d}t + \sigma\left(X_t\right)\,\mathrm{d}W_t, \quad X_0,\ 0 \le t \le T.
$$

We propose estimator-processes ϑ^* such that approximations of BSDE $\hat{Y}_t = u(t, X_t, \vartheta^*_t) \to Y_t$ have minimal errors $\mathbf{E}_\vartheta\left(\hat{Y}_t - Y_t\right)^2$.

Small noise asymptotics. Case A.

(joint work with L.Zhou)

The observed diffusion process (forward) is

$$\mathrm{d}X_t = S\left(\vartheta, t, X_t\right)\mathrm{d}t + \varepsilon\sigma\left(t, X_t\right)\,\mathrm{d}W_t, \quad X_0,\ 0 \leq t \leq T$$

where $\vartheta \in \Theta = (\alpha, \beta)$ is unknown parameter. We are given two functions $f\left(t, x, y, z\right)$, $\Phi\left(x\right)$ and we have to find a couple of stochastic processes $\left(\hat{X}_t, \hat{Z}_t, 0 \leq t \leq T\right)$ which approximates well the solution of the BSDE

$$\mathrm{d}Y_t = -f\left(t, X_t, Y_t, Z_t\right)\,\mathrm{d}t + Z_t\,\mathrm{d}W_t, \qquad Y_0, \quad 0 \leq t \leq T$$

satisfying the condition $Y_T = \Phi\left(X_T\right)$. The functions $S\left(\cdot\right)$ and $\sigma\left(\cdot\right)$ are known and smooth. We have to minimize the errors

$$\mathbf{E}_\vartheta\left(\hat{Y}_t - Y_t\right)^2 \to \min, \qquad \mathbf{E}_\vartheta\left(\hat{Z}_t - Z_t\right)^2 \to \min$$

as $\varepsilon \to 0$.

Solution: Let us introduce a family of functions

$$\mathcal{U} = \left\{ \left(u(t,x,\vartheta), t \in [0,T], x \in \mathbb{R} \right), \vartheta \in \Theta \right\}$$

such that for all $\vartheta \in \Theta$ the function $u(t,x,\vartheta)$ satisfies the equation

$$\frac{\partial u}{\partial t} + S(\vartheta,t,x)\frac{\partial u}{\partial x} + \frac{\varepsilon^2\sigma(t,x)^2}{2}\frac{\partial^2 u}{\partial x^2} = -f\left(t,x,u,\varepsilon\sigma(x)\frac{\partial u}{\partial x}\right)$$

and condition $u(T,x,\vartheta) = \Phi(x)$. If we put $Y_t = u(t,X_t,\vartheta)$, then by Itô's formula we obtain BSDE with $Z_t = \varepsilon\sigma(t,X_t)\, u'_x(t,X_t,\vartheta)$. As we do not know the value ϑ we propose first to estimate it using some estimator $\vartheta^\star_\varepsilon$ and then to put

$$\hat{Y}_t = u\left(t,X_t,\vartheta^\star_\varepsilon\right), \qquad \hat{Z}_t = \varepsilon\sigma(t,X_t)\, u'_x\left(t,X_t,\vartheta^\star_\varepsilon\right)$$

Construction of the Estimator: Remind the MLE for this model. Introduce the LR function

$$L\left(\vartheta,X^t\right)=\exp\left\{\int_0^t\frac{S\left(\vartheta,s,X_s\right)}{\varepsilon^2\,\sigma\left(s,X_s\right)^2}\,\mathrm{d}X_s-\int_0^t\frac{S\left(\vartheta,s,X_s\right)^2}{2\,\varepsilon^2\,\sigma\left(s,X_s\right)^2}\,\mathrm{d}s\right\}$$

and define the MLE $\hat\vartheta_{t,\varepsilon}$ by the equation

$$L\left(\hat\vartheta_{t,\varepsilon},X^t\right)=\sup_{\vartheta\in\Theta}L\left(\vartheta,X^t\right).$$

It is known that $\varepsilon^{-1}\left(\hat\vartheta_{t,\varepsilon}-\vartheta_0\right)\Longrightarrow\mathcal{N}\left(0,\mathbb{I}_t\left(\vartheta,x^t\right)^{-1}\right)$, but to use it for $\bar Y_t=u\left(t,X_t,\hat\vartheta_{t,\varepsilon}\right)$ can be computantionally difficult problem. Here

$$\mathbb{I}_t\left(\vartheta,x^t\left(\vartheta\right)\right)=\int_0^t\frac{\dot S\left(\vartheta,s,x_s\left(\vartheta\right)\right)^2}{\sigma\left(s,x_s\left(\vartheta\right)\right)^2}\,\mathrm{d}s$$

Our goal to construct an estimator-process $\vartheta_t^\star$ with the same asymptotics for all $t \in (0,T]$. Introduce a family of functions $\left\{\left(x_s\left(\vartheta\right), 0 \leq s \leq T\right), \vartheta \in \Theta\right\}$ solution of ODE

$$\frac{\mathrm{d}x_s}{\mathrm{d}s} = S\left(\vartheta, s, x_s\right), \qquad x_0, \quad 0 \leq s \leq T.$$

It is known that X_s converges to $x_s\left(\vartheta\right)$ uniformly in $s \in [0,T]$. Fix some (small) $\tau > 0$ and introduce the MDE $\bar\vartheta_{\tau,\varepsilon}$:

$$\left\|X - x\left(\bar\vartheta_{\tau,\varepsilon}\right)\right\|_\tau^2 = \inf_{\vartheta \in \Theta} \left\|X - x\left(\vartheta\right)\right\|_\tau^2 = \inf_{\vartheta \in \Theta} \int_0^\tau \left[X_t - x_t\left(\vartheta\right)\right]^2 \mathrm{d}t.$$

Suppose that the regularity conditions are fulfilled. Then this estimator is consistent and asymptotically normal

$$\varepsilon^{-1}\left(\bar\vartheta_{\tau,\varepsilon} - \vartheta_0\right) \Longrightarrow \mathcal{N}\left(0, D_\tau\left(\vartheta_0\right)^2\right),$$

where $\mathbb{I}_\tau\left(\vartheta, x^\tau\left(\vartheta\right)\right) \geq D_\tau\left(\vartheta_0\right)^{-2} > 0$ (K. 1994).

Let us consider $\tau_\varepsilon \rightarrow 0$ but *slowly*, $\tau_\varepsilon = \varepsilon^\delta$, where $\delta \in [0, 2)$. Then, say, the MLE $\hat{\vartheta}_{\tau_\varepsilon}$ is consistent and asymptotically normal but with the *bad rate*

$$\frac{\sqrt{\tau_\varepsilon}}{\varepsilon}\left(\hat{\vartheta}_{\tau_\varepsilon} - \vartheta\right) \Longrightarrow \mathcal{N}\left(0, \frac{\sigma\left(x_0\right)^2}{\dot{S}\left(\vartheta, x_0\right)^2}\right).$$

The similar behavior has the MDE

$$\frac{\sqrt{\tau_\varepsilon}}{\varepsilon}\left(\bar{\vartheta}_{\tau_\varepsilon} - \vartheta\right) \Longrightarrow \mathcal{N}\left(0, D^2\right).$$

The estimators $\hat{\vartheta}_{\tau_\varepsilon}$ and $\bar{\vartheta}_{\tau_\varepsilon}$ are used as preliminary in the construction of asymptotically optimal estimator-process. Then we obtain asymptotically efficient estimation of $Y_t, Z_t, \tau_\varepsilon \leq t \leq T$ even for $\tau_\varepsilon \rightarrow 0$.

Introduce *One-step MLE-process* $\vartheta^\star_{t,\varepsilon}, \tau_\varepsilon \leq t \leq T$

$$\vartheta^\star_{t,\varepsilon} = \bar\vartheta_{\tau_\varepsilon} + \varepsilon \frac{\Delta_t\left(\bar\vartheta_{\tau_\varepsilon}, X^t_{\tau_\varepsilon}\right)}{\mathbb{I}_t\left(\bar\vartheta_{\tau_\varepsilon}, x^t\left(\bar\vartheta_{\tau_\varepsilon}\right)\right)},$$

where

$$\Delta_t\left(\vartheta, X^t_\tau\right) = \int_\tau^t \frac{\dot S\left(\vartheta, s, X_s\right)}{\varepsilon\sigma\left(s, X_s\right)^2}\left[{\rm d}X_s - S\left(\vartheta, s, X_s\right){\rm d}s\right], \qquad t \in \left[\tau_\varepsilon, T\right]$$

and

$$\mathbb{I}_t\left(\vartheta, x^t\left(\vartheta\right)\right) = \int_\tau^t \frac{\dot S\left(\vartheta, s, x_s\left(\vartheta\right)\right)^2}{\sigma\left(s, x_s\left(\vartheta\right)\right)^2}{\rm d}s,$$

$$\mathbb{I}_t\left(\vartheta, X^t\right) = \int_\tau^t \frac{\dot S\left(\vartheta, s, X_s\right)^2}{\sigma\left(s, X_s\right)^2}{\rm d}s.$$

We show that if $\tau_\varepsilon = \varepsilon^\delta, 0 < \delta < 1$, then

$$\varepsilon^{-1}\left(\vartheta^\star_{t,\varepsilon} - \vartheta\right) \Longrightarrow \mathcal{N}\left(0, \mathbb{I}_t\left(\vartheta, x^t\right)^{-1}\right)$$

Introduce the estimators

$$Y_t^\star = u\left(t, X_t, \vartheta_{t,\varepsilon}^\star\right), \qquad Z_t^\star = \varepsilon\sigma\left(t, X_t\right) u_x'\left(t, X_t, \vartheta_{t,\varepsilon}^\star\right)$$

Theorem 1 *Suppose the conditions of regularity hold, then the processes* $Y_t^\star, Z_t^\star, \tau_\varepsilon \leq t \leq T$ *have the representation*

$$Y_t^\star = Y_t + \varepsilon\dot u\left(t, X_t, \vartheta_0\right)\,\xi_t\left(\vartheta_0\right) + o\left(\varepsilon\right),$$
$$Z_t^\star = Z_t + \varepsilon^2\sigma\left(t, X_t\right)\dot u_x'\left(t, X_t, \vartheta_0\right)\,\xi_t\left(\vartheta_0\right) + o\left(\varepsilon^2\right),$$

where

$$\xi_t\left(\vartheta_0\right) = \mathbb{I}_t\left(\vartheta, x^t\right)^{-1}\int_0^t \frac{\dot S\left(\vartheta, x_s\right)}{\sigma\left(x_s\right)}\mathrm{d}W_s$$

The random process $\eta_{t,\varepsilon} = \varepsilon^{-1}\left(Y_t^\star - Y_t\right), \tau \leq t \leq T$ *for any* $\tau \in (0, T]$ *converges in distribution to the process* $\xi_t\left(\vartheta_0\right), \tau \leq t \leq T$.

Let us show that the proposed approximations are asymptotically efficient.

This means, that the means-quare errors

$$\mathbf{E}_\vartheta \left|Y_t - Y_t^\star\right|^2, \qquad \mathbf{E}_\vartheta \left|Z_t - Z_t^\star\right|^2,$$

of estimation Y_t and Z_t can not be improved. This will be done in two steps. First we establish a low bound on the risks of all estimators and then show that the proposed estimators attaint this bound.

Theorem 2 *For all estimators $\bar Y_t$ and $\bar Z_t$ and all $t \in [\tau_\varepsilon, T]$ we have the relations*

$$\varliminf_{\nu\to 0} \varliminf_{\varepsilon\to 0} \sup_{|\vartheta-\vartheta_0|\leq\nu} \varepsilon^{-2}\mathbf{E}_\vartheta \left|\bar Y_t - Y_t\right|^2 \geq \frac{\dot u^0\left(t, x_t\left(\vartheta_0\right), \vartheta_0\right)^2}{\mathbb{I}_t\left(\vartheta_0, x^t\left(\vartheta_0\right)\right)},$$

$$\varliminf_{\nu\to 0} \varliminf_{\varepsilon\to 0} \sup_{|\vartheta-\vartheta_0|\leq\nu} \varepsilon^{-4}\mathbf{E}_\vartheta \left|\bar Z_t - Z_t\right|^2$$

$$\geq \frac{\left(\dot u^0\right)'_x\left(t, x_t\left(\vartheta_0\right), \vartheta_0\right)^2 \sigma\left(t, x_t\left(\vartheta_0\right)\right)^2}{\mathbb{I}_t\left(\vartheta_0, x^t\left(\vartheta_0\right)\right)}$$

We call an approximation $Y_t^\star$ asymptotically efficient if for all $\vartheta_0 \in \Theta$ we have the equality

$$\lim_{\nu\to 0}\lim_{\varepsilon\to 0}\sup_{|\vartheta-\vartheta_0|\leq\nu}\varepsilon^{-2}\mathbf{E}_\vartheta\left|Y_t^\star - Y_t\right|^2 = \frac{\dot u^0\left(t, x_t\left(\vartheta_0\right), \vartheta_0\right)^2}{\mathbb{I}_t\left(\vartheta_0, x^t\left(\vartheta_0\right)\right)}$$

and the similar definition is valid in the case of the bound for Z_t.

Theorem 3 *The approximations*

$$Y_t^\star = u\left(t, X_t, \vartheta_{t,\varepsilon}^\star\right) \quad \text{and} \quad Z_t^\star = \varepsilon\sigma\left(t, X_t\right) u_x'\left(t, X_t, \vartheta_{t,\varepsilon}^\star\right)$$

are asymptotically efficient, i.e.,

$$\lim_{\nu\to 0}\lim_{\varepsilon\to 0}\sup_{|\vartheta-\vartheta_0|\leq\nu}\varepsilon^{-2}\mathbf{E}_\vartheta\left|Y_t^\star - Y_t\right|^2 = \frac{\dot u^0\left(t, x_t\left(\vartheta_0\right), \vartheta_0\right)^2}{\mathbb{I}\left(\vartheta_0, x^t\left(\vartheta_0\right)\right)},$$

$$\lim_{\nu\to 0}\lim_{\varepsilon\to 0}\sup_{|\vartheta-\vartheta_0|\leq\nu}\varepsilon^{-4}\mathbf{E}_\vartheta\left|Z_t^\star - Z_t\right|^2 = \frac{\sigma\left(t, x_t\left(\vartheta_0\right)\right)^2\left(\dot u^0\right)_x'\left(t, x_t, \vartheta_0\right)^2}{\mathbb{I}\left(\vartheta_0, x^t\left(\vartheta_0\right)\right)}$$

Small noise asymptotics. Case B.

We have two-dimensional linear diffusion process

$$\mathrm{d}X_t = -a\left(\vartheta, t\right) X_t \mathrm{d}t + \varepsilon b\left(\vartheta, t\right) \mathrm{d}V_t, \quad x_0 \neq 0,$$
$$\mathrm{d}R_t = A\left(\vartheta, t\right) X_t \mathrm{d}t + \varepsilon \sigma\left(t\right) \mathrm{d}W_t, \quad R_0 = 0, \quad 0 \leq t \leq T.$$

where $X^T = (X_t, 0 \leq t \leq T)$ is the Forward and the process $R^T = (R_t, 0 \leq t \leq T)$ is observed. Let us denote conditional expectation $\hat{X}_t = \mathbf{E}_\vartheta\left(X_t | R_s, 0 \leq s \leq t\right)$. We are given two functions $f\left(t, x, y, z\right)$ and $\Phi\left(x\right)$ and we have to construct the BSDE

$$\mathrm{d}Y_t = -f(t, \hat{X}_t, Y_t, Z_t)\mathrm{d}t + Z_t\, \mathrm{d}\bar{W}_t,$$

with the final value $Y_T = \Phi(\hat{X}_T)$.

Equations of optimal filtration:

$$\mathrm{d}\hat{X}_t = -a\left(\vartheta,t\right)\hat{X}_t \mathrm{d}t + c\left(\vartheta,t\right)\varepsilon\,\mathrm{d}\bar{W}_t, \qquad \hat{X}_0 = x_0,$$

$$\frac{\partial \gamma_t\left(\vartheta\right)}{\partial t} = -2a\left(\vartheta,t\right)\gamma_t\left(\vartheta\right) - \frac{\gamma_t\left(\vartheta\right)A\left(\vartheta,t\right)^2}{\sigma\left(t\right)^2} + b\left(\vartheta,t\right)^2, \qquad \gamma_0\left(\vartheta\right) = 0.$$

Here $c\left(\vartheta,t\right) = \gamma_t\left(\vartheta\right)A\left(\vartheta,t\right)\sigma\left(t\right)^{-1}$, $\hat{X}_t = \hat{X}_t\left(\vartheta\right)$ and

$$\mathrm{d}\bar{W}_t = \varepsilon^{-1}\sigma\left(t\right)^{-1}\left[\mathrm{d}R_t - A\left(\vartheta,t\right)\hat{X}_t\mathrm{d}t\right].$$

Itroduce $u\left(t,x,\vartheta\right)$ as solution

$$\frac{\partial u}{\partial t} - a\left(\vartheta,t\right)x\frac{\partial u}{\partial x} + \frac{c\left(\vartheta,t\right)^2\varepsilon^2}{2}\frac{\partial^2 u}{\partial x^2} = -f\left(t,x,y,c\left(\vartheta,t\right)\varepsilon\frac{\partial u}{\partial x}\right)$$

with the final value $u\left(T,x,\vartheta\right) = \Phi\left(x\right)$.

We propose the asymptotically optimal approximation as

$$\hat{Y}_t = u\left(t, \hat{X}_t\left(\vartheta_t^\star\right), \vartheta_t^\star\right), \qquad \hat{Z}_t = \varepsilon c\left(\vartheta_t^\star, t\right) \frac{\partial u\left(t, \hat{X}_t\left(\vartheta_t^\star\right), \vartheta_t^\star\right)}{\partial x},$$

where $\vartheta_t^\star$ is One-step MLE-process

$$\vartheta_t^\star = \bar{\vartheta}_{\tau_\varepsilon} + \varepsilon \frac{\Delta_t\left(\bar{\vartheta}_{\tau_\varepsilon}, X^t\right)}{\mathrm{I}_t\left(\bar{\vartheta}_{\tau_\varepsilon}\right)}, \quad \tau_\varepsilon < t \leq T$$

where

$$\Delta_t\left(\vartheta, X\right) = \int_{\tau_\varepsilon}^t \frac{\dot{A}\left(\vartheta, s\right)\hat{X}_s + A\left(\vartheta, s\right)\widehat{\dot{X}}_s}{\varepsilon\sigma\left(s\right)^2}\left[\mathrm{d}R_s - A\left(\vartheta, s\right)\hat{X}_s\left(\vartheta\right)\mathrm{d}s\right]$$

and

$$\mathrm{I}_t\left(\vartheta\right) = \int_\tau^t \frac{\left(\dot{A}\left(\vartheta, s\right)x_s\left(\vartheta\right) + A\left(\vartheta, s\right)\hat{\dot{x}}_s\left(\vartheta\right)\right)^2}{\sigma\left(s\right)^2}\mathrm{d}s$$

The random process $\widehat{\dot X}_t = \widehat{\dot X}_t\left(\bar\vartheta_{\tau_\varepsilon}\right)$ satisfies equation

$$\mathrm{d}\widehat{\dot X}_t = -\left[\dot a + \frac{\dot c A + c\dot A}{\sigma}\right]\hat X_t \mathrm{d}t - \left[a + \frac{cA}{\sigma}\right]\widehat{\dot X}_t \mathrm{d}t + \frac{\dot c}{\sigma}\mathrm{d}R_t.$$

Here

$$\dot a = \dot a\left(\bar\vartheta_{\tau_\varepsilon}, t\right), \quad \dot c = \dot c\left(\bar\vartheta_{\tau_\varepsilon}, t\right), \quad \dot A = \dot A\left(\bar\vartheta_{\tau_\varepsilon}, t\right),$$

and

$$\begin{aligned}\frac{\partial \dot\gamma_t\left(\bar\vartheta_{\tau_\varepsilon}\right)}{\partial t} &= -2\left[\dot a\left(\bar\vartheta_{\tau_\varepsilon}, t\right) + \frac{A\left(\bar\vartheta_{\tau_\varepsilon}, t\right)\dot A\left(\bar\vartheta_{\tau_\varepsilon}, t\right)}{\sigma\left(t\right)^2}\right]\gamma_t\left(\bar\vartheta_{\tau_\varepsilon}\right) \\ &\quad - \left[2a\left(\bar\vartheta_{\tau_\varepsilon}, t\right) + \frac{A\left(\bar\vartheta_{\tau_\varepsilon}, t\right)^2}{\sigma\left(t\right)^2}\right]\dot\gamma_t\left(\bar\vartheta_{\tau_\varepsilon}\right) + 2b\left(\bar\vartheta_{\tau_\varepsilon}, t\right)\dot b\left(\bar\vartheta_{\tau_\varepsilon}, t\right),\end{aligned}$$

with $\gamma_0\left(\bar\vartheta_{\tau_\varepsilon}\right) = 0$.

The error of estimation is

$$\varepsilon^{-1}\left(\hat{Y}_t - Y_t\right) = \left(u'_x \hat{\dot{x}}_t + \dot{u}_\vartheta\right)\varepsilon^{-1}\left(\vartheta_t^\star - \vartheta\right) + o\left(1\right)$$
$$\Longrightarrow \left(u'_x \hat{\dot{x}}_t + \dot{u}\right)\xi_t\left(\vartheta\right).$$

Here $\xi_t\left(\vartheta\right)$ is Gaussian process

$$\xi_t\left(\vartheta\right) = \mathrm{I}_t\left(\vartheta\right)^{-1}\int_0^t \frac{\dot{A}\left(\vartheta,s\right)x_s\left(\vartheta\right) + A\left(\vartheta,s\right)\hat{\dot{x}}_s\left(\vartheta\right)}{\sigma\left(s\right)}\,\mathrm{d}W_s$$

and

$$x_t\left(\vartheta\right) = x_0\exp\left\{-\int_0^t a\left(\vartheta,v\right)\mathrm{d}v\right\},$$
$$\hat{\dot{x}}_t\left(\vartheta\right) = -\int_0^t e^{-\int_s^t\left[a+\frac{Ac}{\sigma}\right]\mathrm{d}v}\left[\dot{a}\left(\vartheta,s\right) + \frac{c\left(\vartheta,s\right)\dot{A}\left(\vartheta,s\right)}{\sigma\left(s\right)}\right]x_s\mathrm{d}s$$

The similar result we have for the error $\varepsilon^{-2}\left(\hat{Z}_t - Z_t\right)$.

Unknown volatility (joint work with S. Gasparyan)

The forward equation is

$$\mathrm{d}X_t = S\left(t, X_t\right)\mathrm{d}t + \sigma\left(\vartheta, t, X_t\right)\,\mathrm{d}W_t, \quad X_0,\ 0 \leq t \leq T$$

where $\vartheta \in \Theta = (\alpha, \beta)$. We observe the solution of this equation in discrete times $t_i = i\frac{T}{n}$ and have to study the approximation $\hat{Y}_t = u(t, X_{t_k}, \hat{\vartheta}_{t_k}), k = 1, \ldots, n,$ where k satisfies the conditions $t_k \leq t \leq t_{k+1}$ and the estimator $\hat{\vartheta}_{t_k}$ is construct by the observations $X^k = (X_0, X_{t_1}, \ldots, X_{t_k})$. Our goal is to realize the same program as above: we study the one-step pseudo-MLE, which can be relatively easy in calculation and has some properties of optimality.

On parameter estimation in diffusion coefficient. First of all remind that ϑ can be calculated without error if we have continuous time observations. To illustrate it we give two examples.

Example. Suppose that $\sigma(\vartheta, t, x) = \sqrt{\vartheta} h(t, x), \vartheta \in (\alpha, \beta), \alpha > 0$, and the observed process is

$$\mathrm{d}X_t = S(t, X_t)\,\mathrm{d}t + \sqrt{\vartheta} h(t, X_t)\,\mathrm{d}W_t, \quad X_0, \quad 0 \leq t \leq T.$$

We suppose as well that $\int_0^t h(s, X_s)^2\,\mathrm{d}s > 0$.

Let us write the Itô formula for X_t^2:

$$X_t^2 = X_0^2 + 2\int_0^t X_s\,\mathrm{d}X_s + \vartheta \int_0^t h(s, X_s)^2\,\mathrm{d}s, \qquad 0 \leq t \leq T.$$

Hence, for all $t \in (0, T]$ we have with probability 1 the equality

$$\hat{\vartheta}_t = \frac{X_t^2 - X_0^2 - 2\int_0^t X_s\,\mathrm{d}X_s}{\int_0^t h(s, X_s)^2\,\mathrm{d}s} = \vartheta$$

The problem became more interesting if we consider the discrete time observations $X^n = (X_{t_1}, \ldots, X_{t_n})\,, t_j = j\frac{T}{n}$ and the problem of approximation in the *high frequency asymptotics* $(n \to \infty)$. Then in Example we obtain the estimator

$$\hat\vartheta_{t,k} = \frac{X_{t_k}^2 - X_0^2 - 2\sum_{j=1}^{k} X_{t_{k-1}}\left(X_{t_k} - X_{t_{k-1}}\right)}{\sum_{j=1}^{k} h\left(t_{j-1}, X_{t_{j-1}}\right)^2 \delta}, \qquad \delta = \frac{T}{n}.$$

It can be easily shown that if $n \to \infty$ then we have $\hat\vartheta_{t,n} \to \vartheta$ and we can use it in the approximation of Y_t as follows $\hat Y_{t,n} = u(t, X_t, \hat\vartheta_{t,n})$. We can describe the distribution of error $\sqrt{n}\left(\hat Y_{t,n} - Y_t\right)$, but the estimator is not asymptotically optimal. We consider a different estimator.

Let us introduce the equation

$$X_{t_{j+1}} = X_{t_j} + S\left(t_j, X_{t_j}\right)\delta + \sigma\left(t_j, X_{t_j}, \vartheta\right)\left[W_{t_{j+1}} - W_{t_j}\right].$$

Note that conditional $\left(X_{t_0}, \ldots, X_{t_j}\right)$ distribution

$$X_{t_{j+1}} - X_{t_j} - S\left(t_j, X_{t_j}\right)\delta \quad \sim \quad \mathcal{N}\left(0, \sigma\left(t_j, X_{t_j}, \vartheta\right)^2\delta\right),$$

therefore we can itroduce the log pseudo-likelihood ratio

$$\begin{aligned} L\left(\vartheta, X^k\right) = & -\frac{1}{2}\sum_{j=0}^{k-1}\ln\left[2\pi\sigma\left(t_j, X_{t_j}, \vartheta\right)^2\delta\right] \\ & - \frac{1}{2}\sum_{j=0}^{k-1}\frac{\left(X_{t_{j+1}} - X_{t_j} - S\left(t_j, X_{t_j}\right)\delta\right)^2}{\sigma\left(t_j, X_{t_j}, \vartheta\right)^2\delta} \end{aligned}$$

The corresponding contrast function is

$$U_k\left(\vartheta, X^k\right) = \sum_{j=0}^{k-1} \ln a\left(t_j, X_{t_j}, \vartheta\right) + \sum_{j=0}^{k-1} \frac{\left(X_{t_{j+1}} - X_{t_j} - S\left(t_j, X_{t_j}\right)\delta\right)^2}{a\left(t_j, X_{t_j}, \vartheta\right)\delta}$$

where $a\left(t, x, \vartheta\right) = \sigma\left(t, x, \vartheta\right)^2$. The estimator $\hat{\vartheta}_{t,n}$ is define by

$$U_k\left(\hat{\vartheta}_{t,n}, X^k\right) = \inf_{\vartheta\in\Theta} U_k\left(\vartheta, X^k\right)$$

It is known that this estimator is consistent, asymptotically conditionally normal

$$\sqrt{n}\left(\hat{\vartheta}_{t,n} - \vartheta_0\right) \Longrightarrow \mathcal{N}\left(0, \mathbb{I}_t\left(\vartheta_0\right)^{-1}\right),$$

$$\mathbb{I}_t\left(\vartheta_0\right) = 2\int_0^t \frac{\dot{\sigma}\left(s, X_s, \vartheta_0\right)^2}{\sigma\left(s, X_s, \vartheta_0\right)^2}\,\mathrm{d}s$$

and asymptotically efficient (Dohnal(1987), Genon-Catalot, Jacod (1993)).

Note that the approximation $\hat{Y}_t = u(t, X_{t_k}, \hat{\vartheta}_{t,n})$ is computationally difficult to realize. That is why we propose as above the one-step pseudo-MLE. Let us fix some (small) $\tau \in (0, T)$. The PMLE estimator $\hat{\vartheta}_{\tau,n}$ constructed by $X_{t_{0,n}}, X_{t_{1,n}}, \ldots, X_{t_{N,n}}$, where N is chosen from the condition $t_{N,n} \leq \tau < t_{N+1,n}$, is consistent and asymptotically conditionally normal.

Introduce the normalized pseudo score-function and the empirical Fisher information

$$\Delta_{k,n}(\vartheta) = \sum_{j=0}^{k-1} \frac{\left[\left(X_{t_{j+1}} - X_{t_j} - S_j\,\delta\right)^2 - a_j(\theta)\,\delta\right] \dot{a}_j(\vartheta)}{2a_j(\vartheta)^2\sqrt{\delta}},$$

$$\mathbb{I}_{k,n}(\vartheta) = \frac{1}{2}\sum_{j=0}^{k-1} \frac{\dot{a}_j(\vartheta)^2}{a_j(\vartheta)^2}\,\delta = 2\sum_{j=0}^{k-1} \frac{\dot{\sigma}\left(t_j, X_{t_j}, \vartheta\right)^2}{\sigma\left(t_j, X_{t_j}, \vartheta\right)^2}\,\delta.$$

We have the stable convergence

$$\Delta_{k,n}\left(\vartheta_0\right) \Longrightarrow \sqrt{2}\int_0^t \frac{\dot\sigma\left(s,X_s,\vartheta_0\right)}{\sigma\left(s,X_s,\vartheta_0\right)}\,\mathrm{d}w_s$$

and the convergence in probability

$$\mathbb{I}_{k,n}\left(\vartheta_0\right) \to \mathbb{I}_t\left(\vartheta_0\right).$$

The approximation of the random function Y_t we will do with the help of the following one-step PMLE

$$\vartheta^\star_{k,n} = \hat\vartheta_{\tau,n} + \sqrt{\delta}\,\frac{\Delta_{k,n}(\hat\vartheta_{\tau,n})}{\mathrm{I}_{k,n}(\hat\vartheta_{\tau,n})}$$

and show that this estimator is asymptotically efficient and easy calculated for all $t\in[\tau,T]$ (or $N<k\leq n$).

We have the lower bound (Dohnal 87)

$$\varliminf_{\gamma\to 0}\varliminf_{n\to\infty}\sup_{|\vartheta-\vartheta_0|<\gamma} nT^{-1}\mathbf{E}_\vartheta\left(\bar\vartheta_{t,n}-\vartheta\right)^2\geq \mathbf{E}_{\vartheta_0}\mathbb{I}_t\left(\vartheta_0\right)^{-1}.$$

The one-step PME is asymptotically efficient

$$\lim_{\gamma\to 0}\lim_{n\to\infty}\sup_{|\vartheta-\vartheta_0|<\gamma} nT^{-1}\mathbf{E}_\vartheta\left(\bar\vartheta_{t,n}-\vartheta\right)^2= \mathbf{E}_{\vartheta_0}\mathbb{I}_t\left(\vartheta_0\right)^{-1}.$$

Introduce the estimators $Y^\star_{t_k,n}=u\left(t,X_{t_k},\vartheta^\star_{k,n}\right)$ and $Z^\star_{t_k,n}=u'_x\left(t,X_{t_k},\vartheta^\star_{k,n}\right)\sigma\left(t,X_{t_k},\vartheta^\star_{k,n}\right)$ of the random functions Y_t and Z_t respectively.

Theorem 4 *Suppose that the conditions of regularity hold, then the estimators* $\left(Y^{\star}_{t,n}, t\in[\tau,T]\right)$ *and* $\left(Z^{\star}_{t,n}, t\in[\tau,T]\right)$ *are consistent*

$$Y^{\star}_{t_k,n}\longrightarrow Y_t,\qquad Z^{\star}_{t_k,n}\longrightarrow Z_t,$$

and asymptotically conditionally normal (stable convergence)

$$\begin{aligned}
\delta^{-1/2}\left(Y^{\star}_{t_k,n}-Y_{t_k}\right)&\Longrightarrow \dot u\left(t,X_t,\vartheta_0\right)\ \xi_t\left(\vartheta_0\right),\\
\delta^{-1/2}\left(Z^{\star}_{t_k,n}-Z_{t_k}\right)&\Longrightarrow \dot u'_x\left(t,X_t,\vartheta_0\right)\sigma\left(t,X_t,\vartheta_0\right)\ \ \xi_t\left(\vartheta_0\right)\\
&\qquad\qquad +u'_x\left(t,X_t,\vartheta_0\right)\dot\sigma\left(t,X_t,\vartheta_0\right)\ \ \xi_t\left(\vartheta_0\right),
\end{aligned}$$

The approximations $Y_t^\star$ and $Z_t^\star$ of the processes Y_t and Z_t are valid for the values $t \in [\tau, T]$. We take τ as a function of n, i.e., $\tau = \tau_n \to 0$. The rate of convergence of τ_n we take in such a way that the preliminary estimator $\hat{\vartheta}_{\tau_n}$ is still consistent and the one-step MLE $\vartheta_t^\star$ is asymptotically efficient.

Le us put $\tau_n = T/\ln n$. Then for $k = k_n \to \infty$ satisfying the condition $n^{-1}k_n \leq \tau_n < n^{-1}k_{n-1}$

Therefore, for the normalized the contrast-function we have the convergence

$$\tilde{U}_{k_n}\left(\vartheta, X^{k_n}\right) = \frac{U_{k_n}\left(\vartheta, X^{k_n}\right)}{\tau_n} \longrightarrow \ln a\left(0, x_0, \vartheta\right) + \frac{a\left(0, x_0, \vartheta_0\right)}{a\left(0, x_0, \vartheta\right)}.$$

Suppose that condition

$$\left|\frac{\dot\sigma\left(0,x_0,\vartheta\right)}{\sigma\left(0,x_0,\vartheta\right)}\right|\geq\kappa>0.$$

holds, then the estimator $\hat\vartheta_{\tau_n}$ defined with the help of this contrast function

$$\tilde U_{k_n}\left(\hat\vartheta_{\tau_n},X^{k_n}\right)=\inf_{\vartheta\in\Theta}\tilde U_{k_n}\left(\vartheta,X^{k_n}\right)$$

is consistent and asymptotically normal.

Introduce the one-step pseudo MLE

$$\vartheta^\star_{k,n}=\bar\vartheta_{\tau_n}+\sqrt{\delta}\,\frac{\Delta_{k,n}\left(\bar\vartheta_{\tau_n}\right)}{\mathrm{I}_{k,n}\left(\bar\vartheta_{\tau_n}\right)}$$

where $\bar\vartheta_{\tau_n}$ This estimator is asymptotically efficient and easy calculated for all $N<k\leq n$.

Theorem 5 *Suppose that the conditions of regularity hold then*

$$\hat{Y}_{t,n} = u\left(t, X_{t_k}, \vartheta^{\star}_{k,n}\right) \longrightarrow Y_t,$$

$$\hat{Z}_{t,n} = u'_x\left(t, X_{t_k}, \vartheta^{\star}_{k,n}\right) \sigma\left(t, X_{t_k}, \vartheta^{\star}_{k,n}\right) \longrightarrow Z_t,$$

and the errors of estimation are

$$\delta^{-1/2}\left(\hat{Y}_{t_k,n} - Y_{t_k}\right) \Longrightarrow \dot{u}\left(t, X_t, \vartheta_0\right)\ \xi_t\left(\vartheta_0\right),$$

$$\delta^{-1/2}\left(\hat{Z}_{t_k,n} - Z_{t_k}\right) \Longrightarrow \left[\dot{u}'_x\left(t, X_t, \vartheta_0\right) \sigma\left(t, X_t, \vartheta_0\right)\right.$$
$$\left. + u'_x\left(t, X_t, \vartheta_0\right) \dot{\sigma}\left(t, X_t, \vartheta_0\right)\right]\ \xi_t\left(\vartheta_0\right),$$

Observe that $Y_{t_k} - Y_t \sim O\left(\sqrt{\delta}\right)$. Below $\zeta \sim \mathcal{N}\left(0,1\right)$

$$\frac{\hat{Y}_{t_k,n} - Y_t}{\sqrt{\delta}} \Longrightarrow u'_x\left(t, X_t, \vartheta_0\right)\ \eta\, \sigma\left(t, X_t, \vartheta_0\right) \zeta + \dot{u}\left(t, X_t, \vartheta_0\right)\ \xi_t\left(\vartheta_0\right)$$

Example. The forward equation is

$$\mathrm{d}X_t = -X_t\mathrm{d}t + \sqrt{\theta + X_t^2}\,\mathrm{d}W_t, \quad X_0, \quad 0 \le t \le T.$$

Here $\vartheta \in \Theta = (\alpha, \beta)\,, \alpha > 0$ is unknown parameter. It is easy to see that in the case of continuous time observation the problem of parameter estimation is degenerated (singular), i.e., the unknown parameter ϑ can be estimated without error. Indeed, by Itô formula we can write

$$X_t^2 = X_0^2 + 2\int_0^t X_s\,\mathrm{d}X_s + \int_0^t \left[\vartheta + X_s^2\right]\mathrm{d}s.$$

Hence for all $t \in (0, T]$ we have the equality

$$\hat{\vartheta} = t^{-1}\left[X_t^2 - X_0^2 - 2\int_0^t X_s\,\mathrm{d}X_s - \int_0^t X_s^2\,\mathrm{d}s\right]$$

and $\hat{\vartheta} = \vartheta$

Our goal is to construct an asymptotically efficient estimator of the parameter ϑ. Note that the family of measures induced by the observations $X^k = (X_{t_0}, X_{t_1}, \ldots, X_{t_k})$ with t_k satisfying $t_k \leq t < t_{k+1}$ and fixed t are *locally asymptotically mixed normal* (LAMN) and for all estimators ϑ_k^* we have the lower bound on the risk

$$\varliminf_{\nu \to 0} \varliminf_{n \to \infty} \sup_{|\vartheta - \vartheta_0| < \nu} \mathbf{E}_\vartheta \ell \left(\sqrt{k} \left(\vartheta_k^* - \vartheta \right) \right) \geq \mathbf{E}_{\vartheta_0} \ell \left(\zeta_t \left(\vartheta_0 \right) \right).$$

The first consistent estimator we obtain as follows

$$\bar\vartheta_N = \frac{n}{TN} \left[X_{t_N}^2 - X_0^2 - 2 \sum_{j=1}^{N} X_{t_{j-1}} \left[X_{t_j} - X_{t_{j-1}} \right] - \sum_{j=1}^{N} X_{t_{j-1}}^2 \delta \right].$$

The pseudo log-likelihood ratio function is

$$L\left(\vartheta, X^N\right) = -\frac{1}{2}\sum_{j=1}^{N} \ln\left(2\pi\left(\vartheta + X_{t_{j-1}}^2\right)\right) - \sum_{j=1}^{N} \frac{\left[X_{t_j} - X_{t_{j-1}} + X_{t_{j-1}}\delta\right]^2}{2\left(\vartheta + X_{t_{j-1}}^2\right)\delta}.$$

Denote the pseudo Fisher information as

$$\mathbb{I}_{t_k,n}\left(\vartheta\right) = \frac{1}{2}\sum_{j=1}^{k} \frac{\delta}{\left(\vartheta + X_{t_{j-1}}^2\right)^2} \longrightarrow \mathbb{I}_t\left(\vartheta_0\right) = \frac{1}{2}\int_0^t \frac{\mathrm{d}s}{\left(\vartheta + X_s^2\right)^2}.$$

The one-step MLE-process $\vartheta_{t_k,n}^{\star}, \tau \leq t_k \leq T$ is

$$\vartheta_{t_k,n}^{\star} = \bar\vartheta_N + \sqrt{\delta}\sum_{j=1}^{k} \frac{\left[X_{t_j} - X_{t_{j-1}} + X_{t_{j-1}}\delta\right]^2 - \left(\bar\vartheta_N + X_{t_{j-1}}^2\right)\delta}{2\mathbb{I}_{t_k,n}\left(\bar\vartheta_N\right)\left(\bar\vartheta_N + X_{t_{j-1}}^2\right)^2\sqrt{\delta}}.$$

Example. Black-Scholes model. The forward equation is

$$\mathrm{d}X_t = \alpha X_t \mathrm{d}t + \vartheta X_t \,\mathrm{d}W_t, \quad X_0 = x_0, \quad 0 \le t \le T$$

and the function $f(x, y, z) = \beta y + \gamma x z$. The corresponding partial differential equation is

$$\frac{\partial u}{\partial t} + (\alpha + \vartheta\gamma)\, x \frac{\partial u}{\partial x} + \frac{\vartheta^2 x^2}{2} \frac{\partial^2 u}{\partial x^2} + \beta u = 0, \qquad u(T, x, \vartheta) = \Phi(x).$$

The solution of this equation is the function

$$u(t, x, \vartheta) = \frac{e^{\beta(T-t)}}{\sqrt{2\pi\vartheta^2 (T-t)}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2\vartheta^2 (T-t)}} \Phi\left(e^{x + \left(\alpha + \vartheta\gamma - \frac{\vartheta^2}{2}\right)(T-t) - z}\right) \mathrm{d}z.$$

The estimator of Y_t is

$$\hat{Y}_{t_k} = \int \frac{e^{-\frac{z^2}{2\hat{\vartheta}^2_{t_k,n}(T-t_k)}+\beta(T-t_k)}}{\sqrt{2\pi\hat{\vartheta}^2_{t_k,n}\left(T-t_k\right)}} \Phi\left(e^{X_{t_k}+(\alpha+\hat{\vartheta}_{t_k,n}\gamma-\frac{\hat{\vartheta}^2_{t_k,n}}{2})(T-t_k)-z}\right) \mathrm{d}z,$$

where $k = \left[\frac{t}{T}n\right]$ and

$$\hat{\vartheta}_{t_k,n} = \left(\frac{1}{t}\sum_{j=0}^{k-1}\frac{(X_{t_{j+1}}-X_{t_j}-\alpha X_{t_j}\delta)^2}{X^2_{t_j}}\right)^{\frac{1}{2}}.$$

Approximation of $\hat{Z}_t$.

Note that $u\left(t,x,\vartheta\right)=e^{\beta(T-t)}\mathbf{E}_{\vartheta,x}\Phi\left(e^{m_t-\xi}\right)$, where

$$\xi\ \sim\ \mathcal{N}\left(-x,d_t^2\right),\quad m_t=(\alpha+\vartheta\gamma-\frac{\vartheta^2}{2})\left(T-t\right),\quad d_t^2=\vartheta^2\left(T-t\right)$$

Hence

$$u_x'(t,x,\theta)=-e^{\beta(T-t)}\mathbf{E}_\theta\left[\frac{(x+\xi)}{d_t^2}\Phi(e^{m_t-\xi})\right]$$

and therefore

$$\hat{Z}_{t_k}=-\frac{\hat{\theta}_{t_k,n}\,X_{t_k}}{d_{t_k}^3\sqrt{2\pi}}\int_{-\infty}^{\infty}(y+X_{t_k})\Phi(e^{m_{t_k}-y})e^{-\frac{\left(X_{t_k}+y\right)^2}{2d_{t_k}^2}+\beta(T-t_k)}\mathrm{d}y$$

Ergodic diffusion (joint work with A. Abakirova)

The observed diffusion process (forward) is

$$\mathrm{d}X_t = S\left(\vartheta, X_t\right)\mathrm{d}t + \sigma\left(X_t\right)\,\mathrm{d}W_t, \quad X_0,\ 0 \leq t \leq T$$

where $\vartheta \in \Theta = (\alpha, \beta)$. The process $X_t, t \geq 0$ has ergodic properties. We are given two functions $f\left(x, y\right)$, $\Phi\left(x\right)$ and we have to find a couple of stochastic processes $\left(\hat{Y}_t, \hat{Z}_t, 0 \leq t \leq T\right)$ which approximate well the solution of the BSDE

$$\mathrm{d}Y_t = -f\left(X_t, Y_t, Z_t\right)\,\mathrm{d}t + Z_t\,\mathrm{d}W_t, \qquad Y_0, \quad 0 \leq t \leq T$$

satisfying the condition $Y_T = \Phi\left(X_T\right)$. The functions $S\left(\cdot\right)$ and $\sigma\left(\cdot\right)$ are known and smooth. We have to minimize the errors

$$\mathbf{E}_\vartheta \left(\hat{Y}_t - Y_t\right)^2 \rightarrow \min, \qquad \mathbf{E}_\vartheta \left(\hat{Z}_t - Z_t\right)^2 \rightarrow \min .$$

as $T \rightarrow \infty$.

Solution: Introduce a family of functions
$\mathcal{U} = \{(u(t,x,\vartheta), t \in [0,T], x \in \mathbb{R}), \vartheta \in \Theta\}$ such that for all $\vartheta \in \Theta$ the function $u(t,x,\vartheta)$ satisfies the equation

$$\frac{\partial u}{\partial t} + S(\vartheta,x)\frac{\partial u}{\partial x} + \frac{\sigma(x)^2}{2}\frac{\partial^2 u}{\partial x^2} = -f\left(x,u,\sigma\left(x\right)u'_x\right)$$

and condition $u(T,x,\vartheta) = \Phi\left(x\right)$. If we put $Y_t = u\left(t,X_t,\vartheta\right)$, then by Itô's formula we obtain BSDE with $Z_t = \sigma\left(X_t\right)u'_x\left(t,X_t,\vartheta\right)$.

Let us change the variables $t = sT, s \in [0,1]$, and put
$v_\varepsilon\left(s,x,\vartheta\right) = u\left(sT,x,\vartheta\right)$, then

$$\varepsilon\frac{\partial v_\varepsilon}{\partial s} + S(\vartheta,x)\frac{\partial v_\varepsilon}{\partial x} + \frac{\sigma(x)^2}{2}\frac{\partial^2 v_\varepsilon}{\partial x^2} = -f\left(x,v_\varepsilon,\sigma\left(x\right)\left(v_\varepsilon\right)'_x\right),$$

where $v_\varepsilon\left(1,x,\vartheta\right) = \Phi\left(x\right)$ and $\varepsilon = T^{-1}$. The limit is $\varepsilon \to 0$.

We have a family of solutions $v_\varepsilon(s,y,\vartheta), 0\leq s\leq 1$. Fix some (small) $\delta>0$ and define the estimators

$$\hat Y_{sT}=v_\varepsilon\left(s,X_{sT},\vartheta^\star_{sT}\right),\qquad \hat Z_{sT}=\sigma\left(X_{sT}\right)\left(v_\varepsilon\right)'_x\left(s,X_{sT},\vartheta^\star_{sT}\right)$$

where $\vartheta^\star_{sT}, s\in[\delta,1]$ is one-step MLE, which is constructed as follows. Suppose that we have an estimator $\bar\vartheta_{\delta T}$ constructed by the observations $X^{\delta T}=(X_t,0\leq t\leq\delta T)$, which is consistent and asymptotically normal

$$\sqrt{\delta T}\left(\bar\vartheta_{\delta T}-\vartheta\right)\Longrightarrow\mathcal N\left(0,D_\delta^2\right).$$

Then we calculate the one-step MLE

$$\vartheta^\star_{sT}=\vartheta^*_{\delta T}+\frac{\Delta_{sT}\left(\vartheta^*_{\delta T},X^{sT}_{\delta T}\right)+\Delta_\delta\left(\vartheta^*_{\delta T},X^{\delta T}\right)}{\sqrt{s\,T}\,\mathrm{I}\left(\vartheta^*_{\delta T}\right)},\qquad \delta\leq s\leq 1,$$

where

$$\Delta_{sT}\left(\vartheta, X_{\delta T}^{sT}\right) = \frac{1}{\sqrt{sT}} \int_{\delta T}^{sT} \frac{\dot S\left(\vartheta, X_t\right)}{\sigma\left(X_t\right)^2}\ \left[{\rm d}X_t - S\left(\vartheta, X_t\right)\,{\rm d}t\right], \qquad s \in [\delta, 1],$$

$$\Delta_{\delta}\left(\vartheta, X^{\delta T}\right) = \frac{A\left(\vartheta, X_\delta\right)}{\sqrt{sT}} - \frac{1}{2\sqrt{sT}} \int_0^{\delta} B_x'\left(\vartheta, X_t\right)\sigma\left(X_t\right)^2 {\rm d}t$$

$$- \int_0^{\delta} \frac{\dot S\left(\vartheta, X_t\right) S\left(\vartheta, X_t\right)}{\sqrt{sT}\sigma\left(X_t\right)^2}{\rm d}t,$$

$$B\left(\vartheta, x\right) = \frac{\dot S\left(\vartheta, x\right)}{\sigma\left(x\right)^2}, \qquad A\left(\vartheta, x\right) = \int_{x_0}^{x} B\left(\vartheta, z\right)\,{\rm d}z.$$

Note that under regularity conditions (K. 2004)

$$\sqrt{sT}\left(\vartheta_{sT}^{\star} - \vartheta\right) \Longrightarrow \mathcal{N}\left(0, \mathbb{I}\left(\vartheta\right)^{-1}\right)$$

$$\sqrt{sT}\left(\hat Y_{sT} - Y_{sT}\right) \sim \dot v_\varepsilon\left(s, X_{sT}, \vartheta\right)\sqrt{sT}\left(\vartheta_{sT}^{\star} - \vartheta\right),$$

$$\sqrt{sT}\left(\hat Z_{sT} - Z_{sT}\right) \sim \sigma\left(X_{sT}\right)\left(\dot v_\varepsilon\right)_x'\left(s, X_{sT}, \vartheta\right)\sqrt{sT}\left(\vartheta_{sT}^{\star} - \vartheta\right)$$

Two-step MLE. Khasminskii and K. [**?**] recently considered the problem of parameter estimation by the observations of diffusion process and showed that Mullti-step procedure can provide asymptotically efficiennt estimation even if the preliminary estimators have bad rate of convergence.

Let us take the *first* estimator $\tilde\vartheta_{\tau_\delta}$ constructed by the observations $X^{T^\delta}=\left(X_t,,0\leq t\leq T^\delta\right)$ with $\delta\in(\frac{1}{3},\frac{1}{2}]$. We suppose that this estimator is consistent, asymptotically normal and the moments converge too:

$$\tilde v_{\tau_\delta}=T^{\frac{\delta}{2}}\left(\tilde\vartheta_{\tau_\delta}-\vartheta_0\right)\Longrightarrow\mathcal{N}\left(0,\mathbb{M}\left(\vartheta_0\right)\right),\qquad\sup_{\vartheta_0\in\mathbb{K}}\mathbf{E}_{\vartheta_0}\left|\tilde v_{\tau_\delta}\right|^p\leq C,$$

for any $p>0$. Here $\mathbb{M}\left(\vartheta_0\right)$ is some matrix and $C>0$ does not depend on T. It can be the MLE, MDE, BE or the EMM (see [6]).

Introduce the *second* preliminary estimator, which is estimator-process

$$\bar{\vartheta}_\tau = \tilde{\vartheta}_{\tau_\delta} + (\tau T)^{-1/2}\, \mathbb{I}\left(\tilde{\vartheta}_{\tau_\delta}\right)^{-1} \Delta_{\tau T}\left(\tilde{\vartheta}_{\tau_\delta}, X_{T^\delta}^{\tau T}\right), \quad \tau \in [\tau_\delta, 1]$$

where $\tau_\delta = T^{-1+\delta}$. Note that $T^\gamma \left(\bar{\vartheta}_\tau - \vartheta_0\right) \to 0$ for $\gamma \in (1-\delta, 2\delta)$

$$\Delta_{\tau T}\left(\vartheta, X_{T^\delta}^{\tau T}\right) = \frac{1}{\sqrt{\tau T}} \int_{T^\delta}^{\tau T} \frac{\dot{S}(\vartheta, X_t)}{\sigma(X_t)^2} \left[\mathrm{d}X_t - S(\vartheta, X_t)\,\mathrm{d}t\right].$$

The *Two-step MLE-process* we define as follows

$$\vartheta_\tau^{\star\star} = \bar{\vartheta}_\tau + \frac{\mathbb{I}\left(\bar{\vartheta}_\tau\right)^{-1}}{\sqrt{\tau T}} \hat{\Delta}_{\tau T}\left(\tilde{\vartheta}_{\tau_\delta}, \bar{\vartheta}_\tau, X_{T^\delta}^{\tau T}\right), \quad \tau_\delta \le \tau \le 1,$$

where

$$\hat{\Delta}_{\tau T}\left(\vartheta_1, \vartheta_2, X_{T^\delta}^{\tau T}\right) = \frac{1}{\sqrt{\tau T}} \int_{T^\delta}^{\tau T} \frac{\dot{S}(\vartheta_1, X_t)}{\sigma(X_t)^2} \left[\mathrm{d}X_t - S(\vartheta_2, X_t)\,\mathrm{d}t\right].$$

Note that $\hat{\Delta}_{\tau T}\left(\vartheta, \vartheta, X_{T^\delta}^{\tau T}\right) = \Delta_{\tau T}\left(\vartheta, X_{T^\delta}^{\tau T}\right)$.

Then we use this estimator to construct One-step $(\vartheta_t^\star, \tau \leq t \leq T)$ and Two-step MLE-processes like (ergodic case)

$$\vartheta_t^\star = \bar\vartheta_\tau + T^{-1}\mathbb{I}\left(\bar\vartheta_\tau\right)^{-1/2}\int_\tau^t \frac{\dot S\left(\bar\vartheta_\tau, X_s\right)}{\sigma\left(X_s\right)^2}\left[\mathrm{d}X_s - S\left(\bar\vartheta_\tau, X_s\right)\mathrm{d}s\right].$$

This estimator process is easy to calculate, uniformely on $\tau \leq t \leq T$ consistent, asymptotically normal and asymptotically efficient.

The contribution of this talk: we can choose $\tau = \tau_T$ smaller than before.

Theorem 6 *Suppose that the conditions of regularity hold. Then the Two-step MLE-process* $\vartheta_\tau^{\star\star}, \tau_\delta \leq \tau \leq 1$ *is consistent, asymptotically normal*

$$\sqrt{T}\left(\vartheta_\tau^{\star\star} - \vartheta_0\right) \Longrightarrow \mathcal{N}\left(0, \tau^{-1}\mathbb{I}\left(\vartheta_0\right)^{-1}\right),$$

and asymptotically efficient. The random process

$$\eta_{\tau,T}\left(\vartheta_0\right) = \tau\sqrt{T}\mathbb{I}\left(\vartheta_0\right)^{-1/2}\left(\vartheta_\tau^{\star\star} - \vartheta_0\right), \qquad \tau_* \leq \tau \leq 1$$

for any $\tau_* \in (0,1)$ *converges in distribution to the Wiener process* $W\left(\tau\right), \tau_* \leq \tau \leq 1$.

Example. *Ergodic diffusion.* Fix a learning interval $[0,\tau]$, where $\tau=\tau_T\to\infty, \tau_T=o\left(T\right)$ and obtain the preliminary estimator $\bar\vartheta_\tau$. Then we use this estimator to construct One-step $(\vartheta_t^\star, \tau\leq t\leq T)$ and Two-step $(\vartheta_t^{\star,\star}, \tau\leq t\leq T)$ MLE-processes. Say,

$$\vartheta_t^\star=\bar\vartheta_\tau+T^{-1}\mathbb{I}\left(\bar\vartheta_\tau\right)^{-1/2}\int_\tau^t\frac{\dot S\left(\bar\vartheta_\tau,X_s\right)}{\sigma\left(X_s\right)^2}\left[{\rm d}X_s-S\left(\bar\vartheta_\tau,X_s\right){\rm d}s\right].$$

This estimator-process is easy to calculate, it is uniformely on $\tau\leq t\leq T$ consistent, asymptotically normal and asymptotically efficient.

The main contribution of this talk: we can choose $\tau=\tau_T$ smaller than before.

Example. *Time series.* (K. and Motrunich) Introduce the time series

$$X_j = X_{j-1} + 3\frac{\vartheta - X_{j-1}}{1 + \left(X_{j-1} - \vartheta\right)^2} + \varepsilon_j, \quad j = 1, \ldots, n,$$

where $\left(\varepsilon_j\right)_{j\geq 1}$ are i.i.d. standard Gaussian random variables and X_0 is given. The unknown parameter $\vartheta \in \Theta = (-1, 1)$.

Case $N = n^\delta, \frac{1}{2} < \delta \leq 1$. Note that the unknown parameter is the shift parameter and that the invariant density function is symmetric with respect to ϑ. Hence we can take the EMM

$$\bar\vartheta_N = \frac{1}{N}\sum_{j=1}^{N} X_j \longrightarrow \vartheta, \qquad N = \left[n^{3/4}\right].$$

Of course, the limit variance of the EMM $\bar\vartheta_N$ is greater than that of the MLE, but this estimator is much more easier to calculate.

The score-function process is

$$\Delta_k(\vartheta, X^k) = \frac{1}{\sqrt{k}} \sum_{j=1}^{k} \dot{\ell}\left(\vartheta, X_{j-1}, X_j\right), \qquad N+1 \leq k \leq n.$$

where

$$\dot{\ell}\left(\vartheta, x, x'\right) = 3\left(x' - x - 3\frac{\vartheta - x}{1 + \left(\vartheta - x\right)^2}\right) \frac{1 - (\vartheta - x)^2}{\left(1 + (\vartheta - x)^2\right)^2}.$$

Therefore we can calculate the one-step MLE-process as follows

$$\vartheta^\star_{k,n} = \bar{\vartheta}_N$$
$$+ \frac{3}{\mathbb{I}_k k} \sum_{j=1}^{k} \left(X_j - X_{j-1} - 3\frac{\bar{\vartheta}_N - X_{j-1}}{1 + \left(\bar{\vartheta}_N - X_{j-1}\right)^2}\right) \frac{1 - (\bar{\vartheta}_N - X_{j-1})^2}{\left(1 + (\bar{\vartheta}_N - X_{j-1})^2\right)^2}$$

Here $\mathbb{I}_k$ is the empirical Fisher information.

Case $N = n^\delta, \frac{1}{4} < \delta \leq \frac{1}{2}$. The choice of the learning period of observations $N = \left[n^\delta\right]$ with $\delta \in (1/2, 1)$ allows us to construct an estimator process for the values $k > N$ only. It can be interesting to see if it is possible to take more short learning interval. Our goal is to show that the learning period can be $N = \left[n^\delta\right]$ with $\delta \in (1/4, 1/2]$.

Suppose that $N = \left[n^\delta\right]$ with $\delta \in (1/4, 1/2)$. The asymptotically efficient estimator we construct in three steps. By the first N observations as before we obtain the preliminary estimator $\bar\vartheta_{N,1}$ which is asymptotically normal with the rate $\sqrt{N}$, i.e.,

$$n^{\frac{\delta}{2}}\left(\bar\vartheta_{N,1} - \vartheta\right) \Longrightarrow \mathcal{N}\left(0, \mathbb{B}\left(\vartheta\right)\right).$$

This can be the same estimator as in the preceding case. It can be, for example, the EMM, BE or MLE.

The two-step MLE-process $\vartheta_n^{\star\star} = \left(\vartheta_{k,n}^{\star\star}, k = N+1, \ldots, n\right)$ we construct as follows. Introduce the second preliminary estimator-process

$$\bar\vartheta_{k,2} = \bar\vartheta_{N,1} + \frac{1}{\sqrt{k}}\mathbb{I}\left(\bar\vartheta_{N,1}\right)^{-1}\Delta_k(\bar\vartheta_{N,1}, X^k),$$

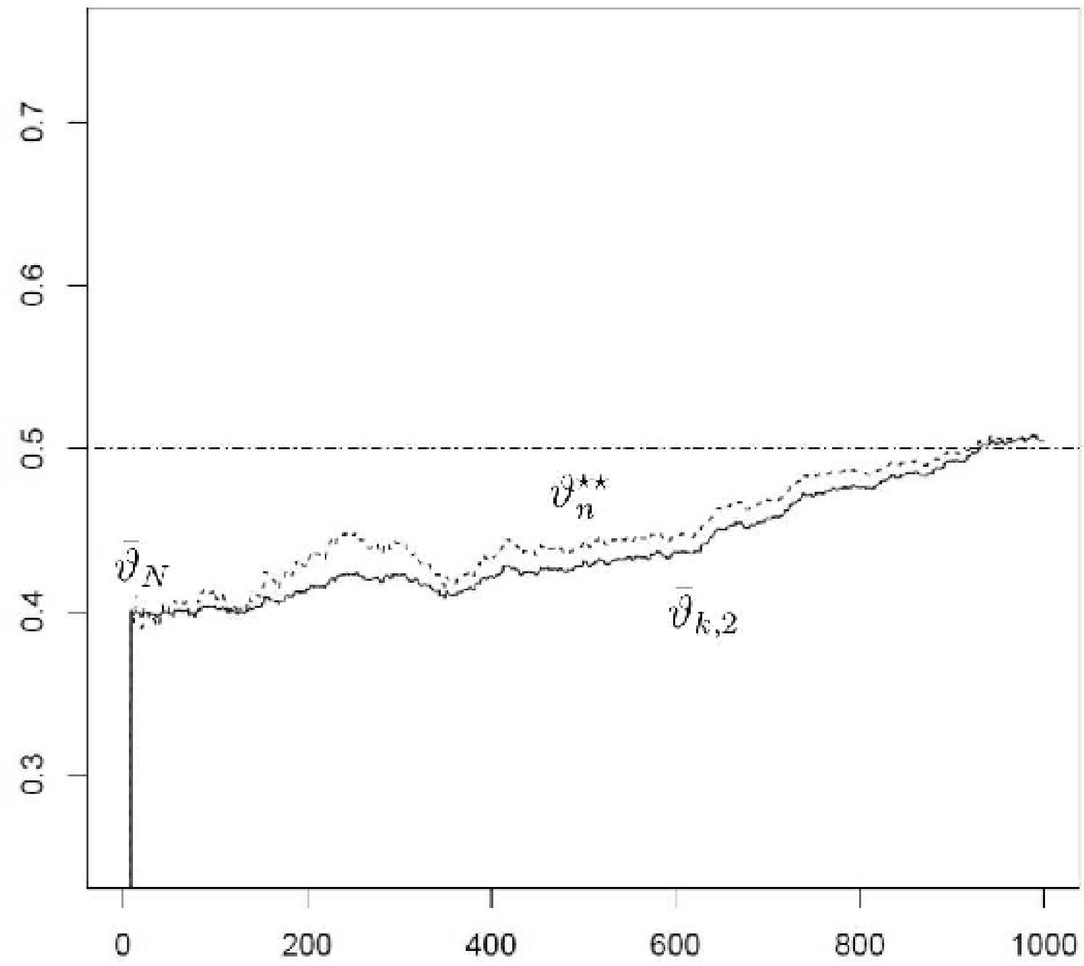
and two-step MLE-process

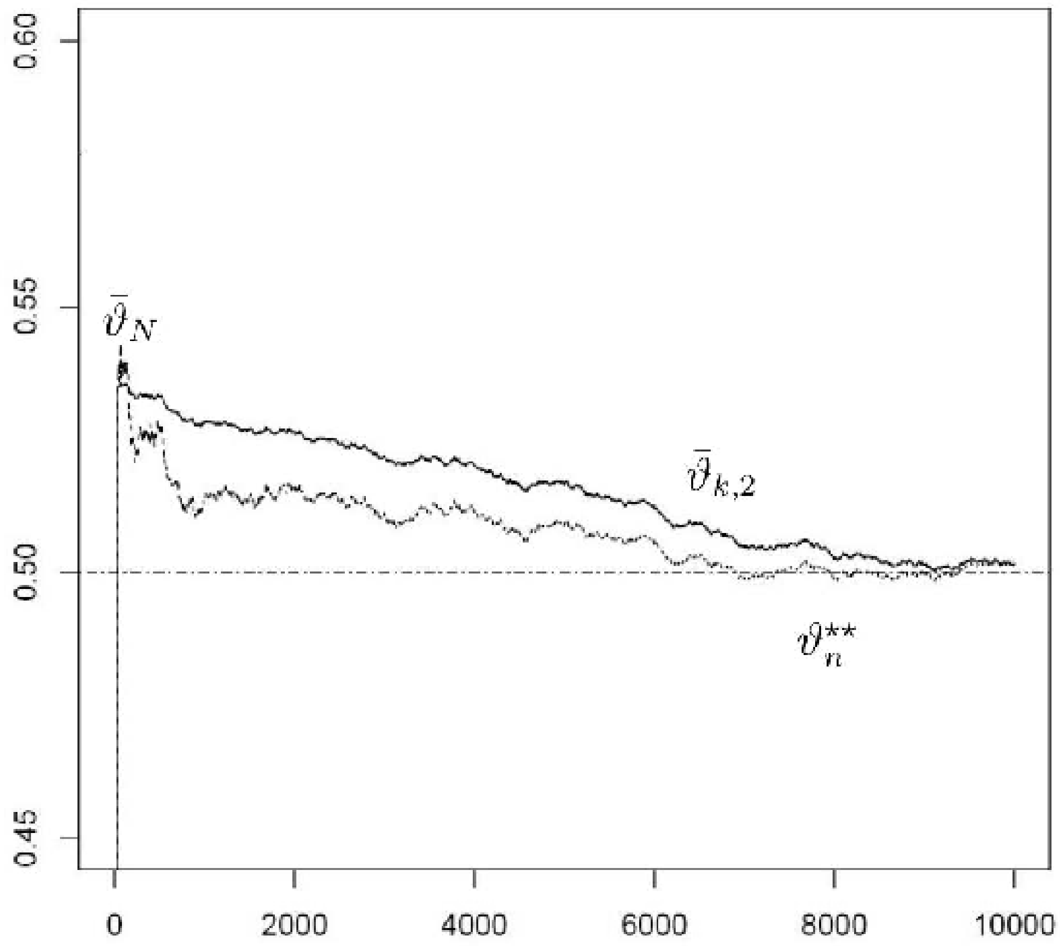
$$\vartheta_k^{\star\star} = \bar\vartheta_{k,2} + \frac{1}{\sqrt{k}}\mathbb{I}\left(\bar\vartheta_{k,2}\right)^{-1}\Delta_k(\bar\vartheta_{k,2}, X^k).$$

In the next theorem we realize this program.

Theorem 7 *Suppose that the conditions of regularity are fulfilled, then the estimator $\vartheta_n^{\star}$ is asymptotically normal*

$$\sqrt{k}(\vartheta_{k,n}^{\star\star} - \vartheta) \Longrightarrow \mathcal{N}\left(0, \mathbb{I}\left(\vartheta\right)^{-1}\right).$$


0.7
0.6
0.5
0.4
0.3
0
200
400
600
800
1000
$\bar{\vartheta}_N$
$\vartheta_n^{\star\star}$
$\bar{\vartheta}_{k,2}$



0.60
0.55
0.50
0.45
0
2000
4000
6000
8000
10000
$\bar{\vartheta}_N$
$\bar{\vartheta}_{k,2}$
$\vartheta_n^{\star\star}$

It is shown that the one-step MLE-process admits the recurrent representation

$$\vartheta^\star_{k+1,n} = \frac{k\,\vartheta^\star_{k,n}}{k+1} + \frac{\bar\vartheta_N}{k+1} + \frac{1}{k+1}\mathbb{I}\left(\bar\vartheta_N\right)^{-1}\dot\ell\left(\bar\vartheta_N, X_k, X_{k+1}\right).$$

It allows us to calculate $\vartheta^\star_{k+1,n}$ using the values $\bar\vartheta_N, \vartheta^\star_{k,n}$ and observations X_k, X_{k+1} only.

The similar structure can be obtained for the two-step MLE-process too. Note that this is not a particular case of the well-known algorithms of stochastic approximation

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