Quenched Mass Transport of Particles Towards a Target

Idris Kharroubi

LPSM
Sorbonne University
(Ex-Pierre et Marie Curie-Paris 6 University)

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Joint work with
B. Bouchard (Université Paris Dauphine)
and
B. Djehiche (KTH Stockholm)

Outline

1. Introduction
2. Quenched mean-field SDE
3. Stochastic target problem
4. The dynamic programming PDE
5. Conclusion

Stochastic target problems

• Consider a random dynamical system controlled by ν and starting at time t with the value x.

• Suppose that this system is described by a process $X^{t,x,\nu}$.

Look for the values x such that the system reaches a set K at a terminal time T by choosing an appropriate control ν.

Namely, the objective is to characterize the reachability sets

$$V(t) \;=\; \Big\{x \in \mathbb{R}^d : \quad X_T^{t,x,\nu} \in K \text{ for some admissible control } \nu\Big\}$$

for $t \in [0, T]$.

Brownian diffusions and HJB equations

• In the Brownian diffusion case, $v(t,.) = 1 - \mathbb{1}_V(t)$ is shown to be solution to an HJB PDE (Soner & Touzi 2001).

• The main motivation is the super-replication problem in finance:

find initial endowments such that there exists an investment strategy allowing the terminal wealth to be greater than a given pay-off (see e.g. El Karoui & Quenez 95).

Extension of the stochastic target problem

- In general the super-replication price is too high.

- Possible generalization: investment under terminal loss constraint:

$$V_\ell(t) = \Big\{ x \in \mathbb{R}^d : \quad \mathbb{E}[\ell(X_T^{t,x,\nu})] \geqslant 0 \text{ for some control } \nu \Big\}.$$

Motivation: relaxing the a.s. super-hedging constraint to get a lower price. In this case, we take $\ell(x) = \mathbb{1}_K(x) - p$ with $p \in [0,1]$.

Approach introduced in Fölmer and Leuckert 99. Then developed in Bouchard et al. 10.

Main idea of this last paper: use martingale representation to express the expectation constraint as an a.s. constraint on an extended process.

Our motivation

Study the stochastic target problem for non-linear controlled diffusions:

$$X_s^{t,\chi,\nu} = \chi + \int_t^s b_u(X_u^{t,\chi,\nu}, \mathbb{P}_{X_u^{t,\chi,\nu}}, \nu_u)du + \int_t^s \sigma_u(X_u^{t,\chi,\nu}, \mathbb{P}_{X_u^{t,\chi,\nu}}, \nu_u)dB_u,$$

where

- B is a standard Brownian motion,
- χ an independent random variable whose distribution can be interpreted as the initial repartition of a population.

Still with constraint $\mathbb{E}[\ell(X_T^{t,\chi,\nu})] \geqslant 0$.

Extended problem: conditional law

We consider a constraint on the condition law of $X_T^{t,\chi,\nu}$ given B:

$$V(t) = \Big\{ x \in \mathbb{R}^d : \ \mathbb{P}^B_{X_T^{t,\chi,\nu}} \in K \text{ for some control } \nu \Big\},$$

This includes the previous problem. Indeed, from Mart. Rep. Thm

$$\mathbb{E}[\ell(X_T^{t,\chi,\nu})] \ = \ \int \ell(x) d\mathbb{P}^B_{X_T^{t,\chi,\nu}}(x) - \int_0^T \alpha_s dB_s \ .$$

Hence the constraint $\mathbb{E}[\ell(X_T^{t,\chi,\nu})] \geqslant 0$ can be rewritten

$$L(\mathbb{P}^B_{\bar{X}_T^{t,\bar{\chi},\bar{\nu}}}) \ \geqslant \ 0$$

with $\bar{\nu} = (\nu, \alpha)$, $\bar{\chi} = (\chi, \eta)$, $\bar{X}^{t,\bar{\chi},\bar{\nu}} = (X^{t,\chi,\nu}, \eta + \int_t^{\cdot} \alpha_s dB_s)$ and

$$L(\mu) \ = \ \int (\ell(x) - y) \mu(dx, dy).$$

Extended problem

We can also extend the dynamics of $X_T^{t,\chi,\nu}$ as follows:

$$X_s^{t,\chi,\nu} = \chi + \int_t^s b_u(X_u^{t,\chi,\nu}, \mathbb{P}^B_{X_u^{t,\chi,\nu}}, \nu_u)du + \int_t^s \sigma_u(X_u^{t,\chi,\nu}, \mathbb{P}^B_{X_u^{t,\chi,\nu}}, \nu_u)dB_u.$$

Such general formulation is related to the probabilistic analysis of large scale particle systems.

In those systems, one is interested in the behavior of particles conditionally to the environment
('quenched' behaviors/properties).

Interpretation

One can also identify the initial condition χ as a law μ.

Then, our problem can be interpreted as a transport problem:

what is the collection of initial distributions μ of a population of particles, such that the terminal repartition $\mathbb{P}^B_{X_T^{t,\chi,\nu}}$, given the environment (modeled by B) satisfies the constraint ?

$$\mathcal{V}(t) \quad = \quad \Big\{\mu : \ \exists(\chi,\nu) \text{ s.t. } \mathbb{P}^B_\chi = \mu \ \text{ and } \ \mathbb{P}^B_{X_T^{t,\chi,\nu}} \in G\Big\}.$$

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Probabilistic setting

$T > 0$ fixed time horizon.

$\Omega^\circ = \{\omega^\circ \in \mathcal{C}([0, T], \mathbb{R}^d) \ : \ \omega^\circ_0 = 0\}$

$\mathbb{F}^\circ = (\mathcal{F}^\circ_t)_{t\leq T}$ filtration generated by the canonical process $B(\omega^\circ) := \omega^\circ$, $\omega^\circ \in \Omega^\circ$.

$\mathbb{P}^\circ$ Wiener measure on $(\Omega^\circ, \mathcal{F}^\circ_T)$.

$\bar{\mathbb{F}}^\circ = (\bar{\mathcal{F}}^\circ_t)_{t\leq T}$ the $\mathbb{P}^\circ$-completion of $\mathbb{F}^\circ$.

$\Omega' := [0, 1]^d$ endowed with σ-algebra $\mathcal{F}' := \mathcal{B}([0, 1]^d)$ Lebegues measure $\mathbb{P}'$.

Probability space

We then define the product filtered space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ by

- $\Omega := \Omega^\circ \times \Omega^{\text{I}}$,
- $\mathbb{P} = \mathbb{P}^\circ \otimes \mathbb{P}^{\text{I}}$,
- $\mathcal{F} = \mathcal{F}_T$ where $\mathbb{F} = (\mathcal{F}_t)_{t \leq T}$ is the completion of $(\mathcal{F}_t^\circ \otimes \mathcal{F}^{\text{I}})_{t \leq T}$.

We canonically extend the random variable ξ and the process B on Ω by setting $\xi(\omega) = \xi(\omega^{\text{I}})$ and $B(\omega) = B(\omega^\circ)$ for any $\omega = (\omega^\circ, \omega^{\text{I}}) \in \Omega$.

Wasserstein space

We define

$$\mathcal{P}_2 \quad := \quad \left\{ \mu \text{ probability measure on } (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \text{ s.t. } \int_{\mathbb{R}^d} |x|^2 \mu(dx) < +\infty \right\} .$$

This space is endowed with the 2-Wasserstein distance defined by

$$\mathcal{W}_2(\mu, \mu') := \inf \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 \pi(dy, dy) \ : \right.$$
$$\left. \text{s.t.} \quad \pi(\cdot \times \mathbb{R}^d) = \mu \text{ and } \pi(\mathbb{R}^d \times \cdot) = \mu' \right\}^{\frac{1}{2}} ,$$

for $\mu, \mu' \in \mathcal{P}_2$. $(\mathcal{P}_2, \mathcal{W}_2)$ is then Polish.

For later use, we also define the collection $\mathcal{P}_2^{\bar{\mathbb{F}}^\circ}$ of $\bar{\mathbb{F}}^\circ$-adapted continuous $\mathcal{P}_2$-valued processes.

Controlled diffusion

Let U be a closed subset of $\mathbb{R}^q$ for some $q \geqslant 1$ and $\mathcal{U}$ the set of U-valued $\mathbb{F}$-progressive processes.

Given

- $\theta \in \bar{\mathcal{T}}^\circ$ (the set of $[0, T]$-valued $\bar{\mathbb{F}}^\circ$-stopping times),
- $\chi \in L^2(\Omega, \mathcal{F}_\theta, \mathbb{P}; \mathbb{R}^d)$,
- $\nu \in \mathcal{U}$,

we let $X^{\theta,\chi,\nu}$ denote the solution of

$$X \quad = \quad \mathbb{E}[\chi|\mathcal{F}_{\theta\wedge\cdot}] + \int_\theta^{\theta\vee\cdot} b_s\big(X_s, \mathbb{P}^B_{X_s}, \nu_s\big)\,ds + \int_\theta^{\theta\vee\cdot} a_s\big(X_s, \mathbb{P}^B_{X_s}, \nu_s\big)\,dB_s,$$

Existence and stability for SDEs

We suppose that b, a are continuous, bounded and Lipschitz: there exists a constant L such that

$$|b_t(x,\mu,\cdot) - b_t(x',\mu',\cdot)| + |a_t(x,\mu,\cdot) - a_t(x',\mu',\cdot)| \leqslant L\Big(|x - x'| + \mathcal{W}_2(\mu,\mu')\Big)$$

for all $t \in [0, T]$, $x, x' \in \mathbb{R}^d$ and $\mu, \mu' \in \mathcal{P}_2$.

Proposition

For all $\theta \in \bar{\mathcal{T}}^\circ$, $\nu \in \mathcal{U}$ and $\chi \in L^2(\mathcal{F}_\theta)$, the SDE admits a unique strong solution $X^{\theta,\chi,\nu}$, and it satisfies

$$\mathbb{E}\Big[\sup_{[0,T]} |X^{\theta,\chi,\nu}|^2\Big] \quad < \quad +\infty\,.$$

Stability: if $(t_n,\chi_n) \to (t,\chi)$ and $(\nu^n)_n \subset \mathcal{U}$ converges to ν $dt \otimes d\mathbb{P}$-a.e., then

$$\lim_{n\to\infty} \mathbb{E}[\mathcal{W}_2(\mathbb{P}^B_{X_T^{t_n,\chi_n,\nu^n}}, \mathbb{P}^B_{X_T^{t,\chi,\nu}})^2] \quad = \quad 0.$$

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First formulation

Look for the set of initial measures for the conditional law $\mathbb{P}^B_\chi$ such that the terminal conditional law of $X_T^{t,\chi,\nu}$ given B belongs to a fixed closed subset G of $\mathcal{P}_2$:

$$\mathcal{V}(t) \quad = \quad \Big\{\mu \in \mathcal{P}_2 : \ \exists(\chi,\nu) \in L^2(\mathcal{F}_t) \times \mathcal{U} \text{ s.t. } \mathbb{P}^B_\chi = \mu \ \text{ and } \ \mathbb{P}^B_{X_T^{t,\chi,\nu}} \in G\Big\}.$$

This formulation is not convenient for setting a DPP:

troubles come from $\exists \chi$ s.t. $\mathbb{P}^B_\chi = \mu$.

Strong formulation

Strong formulation allows to take any representing variable χ for the initial law μ.

Proposition

A measure $\mu \in \mathcal{P}_2$ belongs to $\mathcal{V}(t)$ if and only if for all $\chi \in L^2(\mathcal{F}_t)$ such that $\mathbb{P}^B_\chi = \mu$ there exists $\nu \in \mathcal{U}$ for which $\mathbb{P}^B_{X_T^{t,\chi,\nu}} \in G$:

$$\mathcal{V}(t) = \Big\{\mu \in \mathcal{P}_2 : \ \forall \chi \in L^2(\mathcal{F}_t) \text{ s.t. } \mathbb{P}^B_\chi = \mu \ \exists \nu \in \mathcal{U} \ \text{ and } \ \mathbb{P}^B_{X_T^{t,\chi,\nu}} \in G\Big\}.$$

Sketch of the proof

- Obviously

$\left\{\mu \in \mathcal{P}_2 : \ \forall \chi \in L^2(\mathcal{F}_t) \text{ s.t. } \mathbb{P}^B_\chi = \mu \ \exists \nu \in \mathcal{U} \ \text{ and } \ \mathbb{P}^B_{X_T^{t,\chi,\nu}} \in G\right\} =: \widetilde{\mathcal{V}}(t) \subset \mathcal{V}(t).$

- Turn to $\mathcal{V}(t) \subset \widetilde{\mathcal{V}}(t)$.

Let $\mu \in \mathcal{V}(t)$ and consider (χ, ν) such that $\mathbb{P}^B_\chi = \mu$ and $\mathbb{P}^B_{X_T^{t,\chi,\nu}} \in G$.

ν $\mathbb{F}$-progressive$\Rightarrow$ $\nu_s(\omega^\circ, \omega') = \mathrm{u}(s, B_{.\wedge s}(\omega^\circ), \xi(\omega'))$, $s \in [t, T]$, with u Borel.

Given $\bar{\chi} \in L^2(\mathcal{F}_t)$, we construct $\bar{\xi}$ such that

$$(\chi, \xi, B) \overset{(law)}{=} (\bar{\chi}, \bar{\xi}, B)$$

(Take ζ right-inverse of χ and set $\bar{\xi} = \zeta(\bar{\chi})$.)

Set $\bar{\nu} = \mathrm{u}(., ^{\cdot}B, \bar{\xi})$, then $(\chi, \nu, B) \overset{(law)}{=} (\bar{\chi}, \bar{\nu}, B)$ and $\mathbb{P}^B_{X_T^{t,\bar{\chi},\bar{\nu}}} = \mathbb{P}^B_{X_T^{t,\chi,\nu}} \in G$.

Dynamic programming principle

Theorem

Fix $t \in [0, T]$ *and* $\theta \in \bar{\mathcal{T}}^\circ$ *with values in* $[t, T]$. *Then,*

$$\mathcal{V}(t) = \Big\{ \mu \in \mathcal{P}_2 : \ \exists (\chi, \nu) \in L^2(\mathcal{F}_t) \times \mathcal{U} \text{ s.t. } \ \mathbb{P}^B_\chi = \mu \ \text{ and } \ \mathbb{P}^B_{X^{t,\chi,\nu}_\theta} \in \mathcal{V}(\theta) \Big\}.$$

Note that this DPP holds only for stopping times in $\bar{\mathcal{T}}^\circ$ *i.e.* stopping time w.r.t. the (completed) Brownian filtration.

Sketch of the proof (θ deterministic)

Denote by $\hat{\mathcal{V}}(t)$ the right hand side of the equality.

• $\mathcal{V}(t) \subset \hat{\mathcal{V}}(t)$.

Fix $\mu \in \mathcal{V}(t)$. Then, there exists (χ, ν) such that

$$\mathbb{P}^B_{\chi} = \mu \quad \text{and} \quad \mathbb{P}^B_{X_T^{t,\chi,\nu}} \in G \qquad \text{on} \qquad \widetilde{\Omega}^\circ \in \mathcal{F}^\circ \text{ with } \mathbb{P}^\circ(\widetilde{\Omega}^\circ) = 1.$$

From the flot property we have $\mathbb{P}^B_{X_T^{t,\chi,\nu}} = \mathbb{P}^B_{X_T^{\theta, X_\theta^{t,\chi,\nu},\nu}} \quad \in \quad G.$

Bp: $\mathbb{P}_{X_\theta^{t,\chi,\nu}} \neq \mathbb{P}^B_{X_\theta^{t,\chi,\nu}}$.

Solution: condition the value of $B_{\cdot\wedge\theta} = \widetilde{\omega}^0_{\cdot\wedge\theta}$ to get a r.v. independent of B with law $\mathbb{P}^B_{X_\theta^{t,\chi,\nu}}(\widetilde{\omega}^0)$.

Take the conditional initial condition and control $\chi^{\widetilde{\omega}^\circ} = X_\theta^{t,\chi,\nu}(\widetilde{\omega}^\circ)$ and $\nu^{\widetilde{\omega}^\circ}$

Since $\mathbb{P}_{X_T^{\theta,\chi^{\widetilde{\omega}^\circ},\nu^{\widetilde{\omega}^\circ}}} = \mathbb{P}^B_{X_T^{\theta,X_\theta^{t,\chi,\nu},\nu}}(\widetilde{\omega}^\circ) \in G$ we get $\mathbb{P}^B_{X_\theta^{t,\chi,\nu}} \in \mathcal{V}(\theta)$.

Sketch of the proof (θ deterministic)

• $\hat{\mathcal{V}}(t) \subset \mathcal{V}(t)$

Fix $\mu \in \hat{\mathcal{V}}(t)$ and (χ, ν) such that $\mathbb{P}^B_\chi = \mu$ and $\mathbb{P}^B_{X_\theta^{t,\chi,\nu}} \in \mathcal{V}(\theta)$.
By measurable selection results, there exists a measurable map ϑ such that

$$\mathbb{P}^B_{X_T^{\theta,\chi',\vartheta(\chi')}} \quad \in \quad G \quad \mathbb{P}^\circ - a.s. \text{ for } \mathfrak{P} - a.e. \ \chi'$$

where $\mathfrak{P}$ is the probability measure induced by $\omega^\circ \mapsto X_\theta^{t,\chi,\nu}(\omega^\circ, .)$.
Define the process $\bar{\nu} \in \mathcal{U}$ by

$$\bar{\nu}(\omega) \quad = \quad \nu(\omega)\mathbb{1}_{[0,\theta)} + \vartheta(\theta, X_\theta^{t,\chi,\nu}(\omega^\circ, \cdot))(\omega)\mathbb{1}_{[\theta,T]} \, .$$

We get $\mathbb{P}^B_{X_T^{t,\chi,\bar{\vartheta}}} \in G$ and $\mu \in \mathcal{V}(t)$.

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The value function

Let $v : [0,T] \times \mathcal{P}_2 \to \mathbb{R}$ be the indicator function of the complement of the reachability set $\mathcal{V}$:

$$v(t,\mu) \;=\; 1 - \mathbb{1}_{\mathcal{V}(t)}(\mu)\,, \quad (t,\mu) \in [0,T] \times \mathcal{P}_2.$$

Aim: provide a characterization of v as a (viscosity) solution of a fully non-linear second order parabolic PDE.

Lift on $\mathcal{P}_2$

Aim: define derivatives for functions defined on $\mathcal{P}_2$.

Pb: $\mathcal{P}_2$ is not a vector space.

Possible approach: Lifting (by Lions)

Fix $\widetilde{\Omega}'$ a polish space, $\widetilde{\mathcal{F}}'$ its Borel σ-algebra and $\widetilde{\mathbb{P}}'$ atomless probability.
In particular $\mathcal{P}_2 = \{\widetilde{\mathbb{P}}'_\xi \ : \ \xi \in L_2(\widetilde{\Omega}', \widetilde{\mathcal{F}}', \widetilde{\mathbb{P}}'; \mathbb{R}^d)\}$.

For a function $w: \ \mathcal{P}_2 \to \mathbb{R}$, we define its lift as $W: \ L^2(\widetilde{\Omega}', \widetilde{\mathcal{F}}', \widetilde{\mathbb{P}}'; \mathbb{R}^d) \to \mathbb{R}$ such that

$$W(X) \quad = \quad w(\widetilde{\mathbb{P}}_X)\,, \qquad \text{for all } X \in L^2(\widetilde{\Omega}', \widetilde{\mathcal{F}}', \widetilde{\mathbb{P}}'; \mathbb{R}^d)\,.$$

Allows to consider functions defined on the Hilbert space $L^2(\widetilde{\Omega}', \widetilde{\mathcal{F}}', \widetilde{\mathbb{P}}'; \mathbb{R}^d)$.

Derivatives on $\mathcal{P}_2$ (first order)

We then say that w is Fréchet differentiable (resp. $\mathcal{C}^1$) on $\mathcal{P}_2$ if its lift W is (resp. continuously) Fréchet differentiable on $L_2(\widetilde{\Omega}', \widetilde{\mathcal{F}}', \widetilde{\mathbb{P}}'; \mathbb{R}^d)$.

Then $DW(X) \in L^2(\widetilde{\Omega}', \widetilde{\mathcal{F}}', \widetilde{\mathbb{P}}'; \mathbb{R}^d)$ admits the representation

$$DW(X) \quad = \quad \partial_\mu w(\widetilde{\mathbb{P}}_X)(X)$$

with $\partial_\mu w(\widetilde{\mathbb{P}}_X) : \ \mathbb{R}^d \to \mathbb{R}^d$ measurable map, called the derivative of w at $\widetilde{\mathbb{P}}_X$.

We have $\partial_\mu w(\mu) \in L^2(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \mu; \mathbb{R}^d)$ for $\mu \in \mathcal{P}_2$.

Derivatives on $\mathcal{P}_2$ (second order)

w is said to be fully $\mathcal{C}^2$ if it is $\mathcal{C}^1$ on $\mathcal{P}_2$ and

- the map $(\mu, x) \mapsto \partial_\mu w(\mu)(x)$ is continuous at any $(\mu, x) \in \mathcal{P}_2 \times \mathbb{R}^d$,
- for any $\mu \in \mathcal{P}_2$, the map $x \mapsto \partial_\mu w(\mu)(x)$ is continuously differentiable and the map $(\mu, x) \mapsto \partial_x \partial_\mu w(\mu)(x)$ is continuous at any $(\mu, x) \in \mathcal{P}_2 \times \mathbb{R}^d$,
- for any $x \in \mathbb{R}^d$, the map $\mu \mapsto \partial_\mu w(\mu)(x)$ is differentiable in the lifted sense and its derivative, regarded as the map $(\mu, x, x') \mapsto \partial^2_\mu w(\mu)(x, x')$, is continuous at any $(\mu, x) \in \mathcal{P}_2 \times \mathbb{R}^d \times \mathbb{R}^d$.

Chain rule

Proposition

Let $w \in \mathcal{C}_b^{1,2}([0,T] \times \mathcal{P}_2)$. Given $(t, \chi, \nu) \in [0,T] \times L^2(\mathcal{F}_t) \times \mathcal{U}$, set $X = X^{t,\chi,\nu}$. Then,

$$
\begin{aligned}
w(s, \mathbb{P}^B_{X_s}) = w(t, \mathbb{P}^B_{\chi}) &+ \int_t^s \mathbb{E}_B \Big[\partial_t w(r, \mathbb{P}^B_{X_r}) + \partial_\mu w(r, \mathbb{P}^B_{X_r})(X_r) b_r(X_r, \mathbb{P}^B_{X_r}, \nu_r) \Big] dr \\
&+ \frac{1}{2} \int_t^s \mathbb{E}_B \Big[Tr \Big(\partial_x \partial_\mu w(r, \mathbb{P}^B_{X_r})(X_r) a_r a_r^\top (X_r, \mathbb{P}^B_{X_r}, \nu_r) \Big) \Big] dr \\
&+ \frac{1}{2} \int_t^s \mathbb{E}_B \Big[\widetilde{\mathbb{E}}_B \Big[Tr \Big(\partial_\mu^2 w(r, \mathbb{P}^B_{X_r})(X_r, \widetilde{X}_r) a_r \widetilde{a}_r^\top \Big) \Big] \Big] dr \\
&+ \int_t^s \mathbb{E}_B \Big[\partial_\mu w(r, \mathbb{P}^B_{X_r})(X_r) a_r(X_r, \mathbb{P}^B_{X_r}, \nu_r)) \Big] dB_r
\end{aligned}
$$

for all $s \in [t, T]$, where $(\widetilde{X}, \widetilde{a})$ is a copy of (X, a) on $\widetilde{\Omega} = \Omega^\circ \times \Omega'$.

Sketch of the proof

Same idea as in Sznitman 1989: replace the laws by empirical means and make things converge by LLN. (Approach used by Chassagneux et al 2015)

Fix $(\chi, \nu) = (x(\xi), u(\cdot, B_{\wedge\cdot}, \xi))$ and define $(\chi^{\ell}, \nu^{\ell}) := (x(\xi^{\ell}), u(\cdot,^{\cdot} B, \xi^{\ell}))$, for $\ell \geq 1$ where $(\xi^{\ell})_{\ell \geqslant 1}$ IID sequence with $\xi^1 = \xi$.

Then $X^{t,\chi^{\ell},\nu^{\ell}}$, $\ell \geqslant 1$ is IID given B.

Apply classical Itô formula to $(X_r^{t,\chi^{\ell},\nu^{\ell}})_{1 \leqslant \ell \leqslant N} \mapsto w(r, \frac{1}{N}\sum_{\ell=1}^{N} \delta_{X_r^{t,\chi^{\ell},\nu^{\ell}}})$.

From LLN conditionally to B we have $\mathcal{W}_2(\bar{\mu}_r^N, \mathbb{P}^B_{X_r^1}) \to 0$ a.s. as $N \to \infty$ for all $r \in [t, s]$.

The approximated Itô's formula and this conditional LLN give the result.

Chain rule on L^2

Corollary

Let $W : [0,T] \times L^2(\widetilde{\Omega}', \widetilde{\mathcal{F}}', \widetilde{\mathbb{P}}'; \mathbb{R}^d) \to \mathbb{R}$ *be the lift of a* $\mathcal{C}_b^{1,2}$ *function. Set* X *a copy of* $X^{t,\chi,\nu}$ *on* $\widetilde{\Omega}$*. Then,*

$$
\begin{aligned}
W(s, X_s) = W(t, \chi) \\
&+ \int_t^s \widetilde{\mathbb{E}}_B \left[\partial_t W(r, X_r) + DW(r, X_r) b_r(X_r, \widetilde{\mathbb{P}}^B_{X_r}, \nu_r) \right] dr \\
&+ \frac{1}{2} \int_t^s \widetilde{\mathbb{E}}_B \left[D^2 W(r, X_r)(X_r)(a_r Z)(a_r Z)^\top (X_r, \widetilde{\mathbb{P}}^B_{X_r}, \nu_r) \right] dr \\
&+ \int_t^s \widetilde{\mathbb{E}}_B \left[DW(r, X_r) a_r(X_r, \widetilde{\mathbb{P}}^B_{X_r}, \nu_r)) \right] dB_r
\end{aligned}
$$

for all $s \in [0,T]$*, with* $Z \sim N(0, I_d)$ *and* $Z \perp\!\!\!\perp \chi, B$ *.*

Lack of regularity for lift function

Unfortunately, lift of $\mathcal{C}^2$ functions in the sens of measures are not necessarily C^2 in the sens L^2 (Buckdahn & al.).

Pb: need to have C^2-function in L^2 for the viscosity properties (test functions).

Solution (Pham& Wei): consider test functions that are not measure invariant $\Rightarrow$ need to extend the previous chain rule to general C^2 functions on L^2.

Holds true by (Carmona & Delarue).

The PDE

We aim at proving that $V : [0,T] \times L^2(\widetilde{\Omega}', \widetilde{\mathcal{F}}', \widetilde{\mathbb{P}}'; \mathbb{R}^d) \to \mathbb{R}$ lift of v is solution on $[0,T) \times L^2(\widetilde{\Omega}', \widetilde{\mathcal{F}}', \widetilde{\mathbb{P}}'; \mathbb{R}^d)$ of

$$-\partial_t W(t,\chi) + \mathcal{H}\big(t,\chi, DW(t,\chi), D^2 W(t,\chi)\big) = 0 .$$

where $\mathcal{H} = \lim_{\varepsilon \to 0+} \mathcal{H}_\varepsilon$ with

$$\begin{aligned}
\mathcal{L}_t^u(\chi, P, Q) &:= \widetilde{\mathbb{E}}_B\Big[b_t^\top(\chi, \widetilde{\mathbb{P}}_\chi, u)P + \frac{1}{2} Q\big(a_t(\chi, \widetilde{\mathbb{P}}_\chi, u)Z\big) a_t(\chi, \widetilde{\mathbb{P}}_\chi, u)Z\Big] \\
\mathcal{H}_\varepsilon(t,\chi,P,Q) &:= \sup_{u \in \mathcal{N}_\varepsilon(t,\chi,P)} \Big\{ -\mathcal{L}_t^u(\chi,P,Q)\Big\} \\
\mathcal{N}_\varepsilon(t,\chi,P) &:= \Big\{ u \in L^0(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}; U) \ : \ |\widetilde{\mathbb{E}}_B[a_t(\chi, \widetilde{\mathbb{P}}_\chi, u)P]| \le \varepsilon \Big\},
\end{aligned}$$

Continuity assumption

We need the following assumption. It ensures the existence of a regular feedback control 'close' to the kernel $\mathcal{N}_0$.

(Continuity Assumption)
Let $\mathcal{O}$ be an open subset of $[0, T] \times [L^2(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}; \mathbb{R}^d)]^2$ such that $\mathcal{N}_0 \neq \emptyset$ on $\mathcal{O}$. Then, for every $\varepsilon > 0$, $(t_0, \chi_0, P_0) \in \mathcal{O}$ and $u_0 \in \mathcal{N}_0(t_0, \chi_0, P_0)$, there exists

- $\mathcal{O}'$ open neighborhood of (t_0, χ_0, P_0)
- $\hat{u} : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \Omega' \to U$ measurable

such that

(i) $\mathbb{E}_B[|\hat{u}_{t_0}(\chi_0, P_0, \xi) - u_0|] \leqslant \varepsilon$,

(ii) there exists $C > 0$ for which

$$\mathbb{E}[|\hat{u}_t(\chi, P, \xi) - \hat{u}_t(\chi', P', \xi)|^2] \leq C\mathbb{E}[|\chi - \chi'|^2 + |P - P'|^2]$$

for all $(t, \chi, P), (t, \chi', P') \in \mathcal{O}'$,

(iii) $\hat{u}_t(\chi, P, \xi) \in \mathcal{N}_0(t, \chi, P)$ $\mathbb{P}^\circ - a.e.$, for all $(t, \chi, P) \in \mathcal{O}'$,

Viscosity property

We also suppose that there exists a constant C and a function $m: \mathbb{R}_+ \to \mathbb{R}$ such that $m(t) \xrightarrow[t\to 0]{} 0$ and

$$|b_t(x,\mu,u) - b_{t'}(x,\mu,u')| + |a_t(x,\mu,u) - a_{t'}(x,\mu,u')| \leqslant m(t-t') + C|u-u'|.$$

for all $t, t' \in [0,T]$, $x \in \mathbb{R}^d$, $\mu \in \mathcal{P}_2$ and $u, u' \in U$.

Theorem

The function V is a viscosity supersolution of the HJB equation. If in addition the **Continuity Assumption** *holds, then V is also a viscosity subsolution of the HJB equation.*

Idea of the proof (regular case)

Fix (t,χ,ν) and set $X = X^{t,\chi,\nu}$. From Itô's formula we have

$$V(t+h,X_s) = V(t,\chi) + \int_t^{t+h} \widetilde{\mathbb{E}}_B\left[\partial_t V(r,X_r) + \mathcal{L}_r^u(X_r, DV(X_r), D^2V(X_r))\right] dr + \int_t^{t+h} \widetilde{\mathbb{E}}_B\left[DV(r,X_r)a_r(X_r, \widetilde{\mathbb{P}}^B_{X_r}, \nu_r)\right] dB_r$$

Idea of the proof (regular case)

- Suppose that $v(t, \mathbb{P}^B_\chi) = 1$.

Fix $u_0 \in \mathcal{N}_0(t, \chi, DV(t, \chi))$
Take ν_s feedback valued in $\mathcal{N}_0(s, X_s, DV(s, X_s))$ close to u_0.
Existence of X is ensured by **Continuity assumption**.
Chain rule gives

$$\int_t^{t+h} \widetilde{\mathbb{E}}_B \Big[\partial_t V(r, \widetilde{\mathbb{P}}^B_{X_r}) + \mathcal{L}^\nu_r(X_r, DV(r, X_r), D^2V(r, X_r))\Big]\, dr \quad \leqslant \quad 0$$

Then we get by dividing by h and sending $h \to 0$

$$\partial_t V(t, \mathbb{P}_\chi) + \mathcal{H}(\chi, DV(t,\chi), D^2V(t,\chi)) \quad \leqslant \quad 0\,.$$

Idea of the proof (regular case)

- Suppose that $v(t, \mathbb{P}^B_\chi) = 0$.

From the DDP, there is some control ν such that $v(t+h, \mathbb{P}^B_{X_s}) = 0$ for $h \in (0, T-t)$. Then we have

$$\int_t^{t+h} \mathbb{E}_B \left[\partial_t V(r, \mathbb{P}^B_{X_r}) + \mathcal{L}^\nu_r(X_r, DV(r, X_r), D^2 V(r, X_r)) \right] dr + \int_t^{t+h} \mathbb{E}_B \left[DV(r, X_r) a_r(X_r, \widetilde{\mathbb{P}}^B_{X_r}, \nu_r) \right] dB_r = 0 .$$

Identification of the MG and BV parts gives

$$\widetilde{\mathbb{E}}_B \left[DV(r, X_r) a_r(X_r, \widetilde{\mathbb{P}}^B_{X_r}, \nu_r) \right] = 0$$

$$\int_t^{t+h} \widetilde{\mathbb{E}}_B \left[\partial_t V(r, \widetilde{\mathbb{P}}^B_{X_r}) + \mathcal{L}^\nu_r(X_r, DV(r, X_r), D^2 V(r, X_r)) \right] dr = 0 .$$

Then we get by dividing by h and sending $h \to 0$

$$\partial_t V(t, \mathbb{P}_\chi) + \mathcal{H}(\chi, DV(t, \chi), D^2 V(t, \chi)) \geqslant 0 .$$

Parabolic boundary condition

Define the function g by

$$g(\chi) \;=\; 1-\mathbb{1}_G(\widetilde{\mathbb{P}}_\chi)\,, \quad \chi\in L^2(\widetilde{\Omega}',\widetilde{\mathcal{F}}',\widetilde{\mathbb{P}}';\mathbb{R}^d)$$

and g_* and g^* its lower and upper semi-continuous envelopes.

Theorem

Under **(H1)***, the function V satisfies*

$$V^*(T,.)\;=g^* \quad \textit{and} \quad V_*(T,.)\;=g_*$$

on $L^2(\Omega',\mathcal{F}',\mathbb{P}';\mathbb{R}^d)$.

Outline

1 Introduction

2 Quenched mean-field SDE

3 Stochastic target problem

4 The dynamic programming PDE

5 Conclusion

Conclusion and perspectives

- Define a new stochastic target problem with potential financial and physical applications.
- Get a dynamic programming principle and PDE properties

Extensions and open problems

- Uniqueness (comparison at least) for the PDE
- Target for $\mathbb{P}_{X_T}$ (unconditional law)
- Numerical methods.

Thank You!