Efficient estimation of present value distributions for long-dated contracts

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The problem

Present value

$$X_0 = \int_0^{T_0} D_u f(Z_u)\mathrm{d}u + D_{T_0} g(Z_{T_0}).$$

- **Payment** flow $f(Z)$ and lump sum $g(Z)$ depend on factors Z.
- **Discounting** D depends on Z, potential extra randomness.
- **Termination** T_0 occurs with rate depending on factors Z.

Aim

Calculate the law of X_0 given Z_0.

- With $Z_0 \sim p$ known, equivalent to obtain the *joint* law π of (Z_0, X_0).
- Very few cases have known "closed form" answers.
- PDE or MC methods either impossible or extremely inefficient.

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Factor

Dynamics

Z ergodic diffusion, living in $(\underline{e}, \overline{e})$, $-\infty \leq \underline{e} < \overline{e} \leq \infty$, such that:

$$\mathrm{d}Z_t = m(Z_t)\mathrm{d}t + \sigma(Z_t)\mathrm{d}W_t, \quad t \in \mathbb{R}.$$

Let p be the invariant (stationary) probability.

Ergodicity of Z

This happens exactly when, with $z_0 \in (\underline{e}, \overline{e})$ and

$$\Psi(z) := \exp\left(-2\int_{z_0}^{z} \frac{m(s)}{\sigma^2(s)}\mathrm{d}s\right),$$

$$\int_{\underline{e}}^{z_0} \Psi(z)\mathrm{d}z = \infty = \int_{z_0}^{\overline{e}} \Psi(z)\mathrm{d}z, \qquad \text{and} \qquad \int_{\underline{e}}^{\overline{e}} \frac{\Psi(z)}{\sigma^2(z)}\mathrm{d}z < \infty,$$

In this case, $p \propto \Psi/\sigma^2$.

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Discounting and Termination

Discounting

With $D_0 = 1$,

$$-\frac{\mathrm{d}D_t}{D_t} = r(Z_t)\mathrm{d}t + \widetilde{\theta}(Z_t)\mathrm{d}W_t + \eta(Z_t)\mathrm{d}B_t$$
$$= a(Z_t)\mathrm{d}t + \theta(Z_t)\mathrm{d}Z_t + \eta(Z_t)\mathrm{d}B_t, \quad t \in \mathbb{R}.$$

Termination

N: Cox process (of contract terminations).

- Given $Z \equiv (Z_t;\ t \in \mathbb{R})$, N is inhomogeneous Poisson with rate $\lambda(Z)$.
- Assume that $p[\lambda > 0] > 0$. (Contracts terminate in finite time.)
- Define the time-of-next-contract-termination

$$T_t := \inf\{u \geq t \mid \Delta N_u = 1\}, \quad t \in \mathbb{R}.$$

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The law of (Z_0, X_0) via ergodicity: a first idea

Extend X_0 to a process

$$X_t = \int_t^{T_t} \frac{D_u}{D_t} f(Z_u) \mathrm{d}u + \frac{D_{T_t}}{D_t} g(Z_{T_t}), \quad t \in \mathbb{R}.$$

- The joint process (Z, X) is ergodic, with invariant joint law π.

Dynamics of (Z, X)?

- Idea: Write dynamics for (Z, X) in its filtration. Then, simulate (Z, X) starting from *any* $(Z_0, X_0) = (z, x)$; by the ergodic theorem, the empirical laws converge to the actual one.
- Barrier: X is "forward-looking"; idea does not seem implementable.

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Flip the idea backwards

Time reversal

Define the processes

$$\zeta_t = Z_{-t}, \quad \chi_t = X_{-t}, \quad t \in \mathbb{R}.$$

The process (ζ, χ) is ergodic and Markov, same invariant law π as (Z, X). Now, χ depends on the *past* of ζ. Define also

$$\nu_t = -N_{-t}, \quad \omega_t = -W_{-t}, \quad \beta_t = -B_{-t}, \quad t \in \mathbb{R}.$$

Dynamics of (ζ, χ)?

- Dynamics of ζ from Haussmann-Pardoux '86. (See result later.)
- Given ζ, dynamics for χ follow. (See next slides.)

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Dynamics for χ

Recall that

$$D_t X_t = \int_t^{T_t} D_u f(Z_u) \mathrm{d}u + D_{T_t} g(Z_{T_t}), \quad t \in \mathbb{R}.$$

Using "$-$" and "$+$" to denote sampling at the left- and right end-point respectively and "δ" for differences, we obtain (excluding high order terms)

$$X_- = f(Z_-)\delta t + \frac{D_+}{D_-} X_+ (1 - \delta N) + g(Z_+)\delta N \quad \Rightarrow$$

$$-\delta X = f(Z_-)\delta t + X_+ \frac{\delta D}{D_-} + (g(Z_+) - X_+)\, \delta N.$$

Since $f(Z_-)\delta t = f(Z_+)\delta t$, it follows that

$$\delta\chi = f(\zeta_-)\delta t + \chi_- \frac{\delta D}{D_-} + (g(\zeta_-) - \chi_-)\, \delta\nu.$$

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Time-reversed dynamics for $\delta D / D_-$

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\begin{aligned}
-\frac{\delta D}{D_-} &= a(Z_-)\delta t + \theta(Z_-)\delta Z + \eta(Z_-)\delta B \\
&= a(Z_+)\delta t - \theta(Z_+)(-\delta Z) - \eta(Z_+)(-\delta B) \\
&\quad - \delta a(Z)\delta t + \delta\theta(Z)(-\delta Z) + \delta\eta(Z)(-\delta B).
\end{aligned}
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But, $\delta a(Z)\delta t = 0 = \delta\eta(Z)(-\delta B)$, and

$$\delta\theta(Z)(-\delta Z) = -\theta'(Z_+)\sigma^2(Z_+)\delta t = -\theta'(\zeta_-)\sigma^2(\zeta_-)\delta t.$$

Putting everything together, with

$$\alpha := \theta'\sigma^2 - a,$$

$$\frac{\delta D}{D_-} = \alpha(\zeta_-)\delta t + \theta(\zeta_-)\delta\zeta + \eta(\zeta_-)\delta\beta.$$

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The main result

Theorem

- With $\zeta_0 \sim p$, $\chi_t^x = x$ (for $x \in \mathbb{R}$), let (ζ, χ^x) satisfy

$$\begin{aligned} \mathrm{d}\zeta_t &= m(\zeta_t)\mathrm{d}t + \sigma(\zeta_t)\mathrm{d}\omega_t, \\ \mathrm{d}\chi_t^x &= f(\zeta_t)\mathrm{d}t + \chi_{t-}^x\left(\alpha(\zeta_t)\mathrm{d}t + \theta(\zeta_t)\mathrm{d}\zeta_t + \eta(\zeta_t)\mathrm{d}\beta_t\right) \\ &\quad + \left(g(\zeta_t) - \chi_{t-}^x\right)\mathrm{d}\nu_t, \qquad t \in \mathbb{R}_+, \end{aligned}$$

where (ω, β) are independent Brownian motions, and ν a Cox process with rate $\lambda(\zeta)$. Define the *occupation measure* $\widehat{\pi}_t^x$ via

$$\widehat{\pi}_t^x[A] = \frac{1}{t}\int_0^t \mathbf{1}_A(\zeta_s, \chi_s^x)\mathrm{d}s, \qquad t > 0.$$

- Then, it almost surely holds that

$$\lim_{t \to \infty} \widehat{\pi}_t^x = \pi \text{ (weakly)}, \quad \forall x \in \mathbb{R}.$$

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Extensions

Multi-dimensional diffusive factor models

- Difficult to *check* for ergodicity (tests involving Lyapunov functions, adjoint equations).
- Even more difficult to *calculate* invariate measure p (gradient conditions, special cases like multi-dimensional OU models).
- Dynamics for ζ involve p.

Continuous Markov chain factor models

- Allow for different payoff during sojourns, transition and termination.
- More tractable: piecewise deterministic χ between transitions of ζ.
- Results in density estimation: convergence of laws in total variation.

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The End

Yuri, thanks.
Looking forward to your 80th!