On a multi-asset version on the Kusuoka limit theorem of option superreplication under transaction costs

Yuri Kabanov

Lomonosov Moscow State University and Laboratoire de Mathématiques, Université de Franche-Comté and Federal Research Center Informatics and Control, Moscow

Based on a joint work with Julien Grépat

Luminy, CIRM, September 6, 2018

Binomial and multinomial approximations of stock prices

Let S^i be scalar geometric Brownian motions describing the prices of $d-1$ risky securities, i.e.

$$dS^i_t = \sigma^i S^i_t dW^i_t, \quad S^i_0 = 1, \quad i = 1, \dots, d-1,$$

where W^i are independent Wiener processes. It is well-known that the financial market model based on such a description is complete. However, the standard binomial approximation leads to a complete market only in the case of one risky securities. A good alternative to save completeness of approximating models is multinomial approximation spanned by a sequence of independent random variables taking values in the vertices of a regular simplex.

Let $\mathcal{V} := \{v_1, ..., v_d\}$ be the set of vertices of a regular simplex $\Sigma \subset \mathbb{R}^{d-1}$, i.e. d affine independent vectors such that $|v_i|^2 = d-1$, $\sum_{j\le d} v_j = 0$, and $v_i v_j = -1$ for $i \neq j$. Also,

$$(1/d)\sum_{j\le d} v_j \otimes v_j = (1/d)\sum_{j\le d} v_j v_j' = I_{d-1}.$$

Let σ be an invertible symmetric $(d-1)\times(d-1)$-matrix. Define the discrete-time process $\eta^n = (\eta^n_l)_{l\le n}$ with values in $\mathbb{R}^{d-1}$ by putting $\eta^n_0 = \mathbf{1} := (1, ..., 1)$,

$$\ln \eta^n_l = \sum_{k=1}^{l} \ln\left(\mathbf{1} + n^{-1/2}\sigma\xi^n_k\right). \tag{1}$$

Put $S^n_t := \eta^n_{[nt]}$, $t \in [0,1]$. Then S^n is a piecewise-constant process in $\mathbb{R}^{d-1}$ jumping at $t_k = t^n_k := k/n$ modeling the price evolution of $d-1$ risky assets. The price process of all traded assets, including the numéraser is d-dimensional, $\tilde S^n := (1, S^n)$. One can check that $\mathcal{L}(S^n|P^n)$ converge to the law of process S with $S^i = \mathcal{E}((\sigma W)^i)$.

Details

So, $S^n_t := \eta^n_{[nt]}$, $t \in [0,1]$, $\ln \eta^n_l := \sum_{k=1}^l \ln\big(\mathbf{1} + n^{-1/2}\sigma\xi^n_k\big)$.
The identities $\sum v_j = 0$ and $(1/d)\sum v_j v_j' = I_{d-1}$ implies that $E\xi^n_k = 0$, $E\xi^n_k(\xi^n_k)' = I_{d-1}$, $E(\sigma\xi^n_k)(\sigma\xi^n_k)' = \sigma\sigma'$. Then

$$E\ln\big(\mathbf{1} + n^{-1/2}\sigma\xi^n_k\big) = -(1/(2n))\mathrm{diag}\,\sigma\sigma' + o(1/n),$$

$$E\ln\big(\mathbf{1} + n^{-1/2}\sigma\xi^n_k\big)\big(\ln\big(\mathbf{1} + n^{-1/2}\sigma\xi^n_k\big)\big)' = (1/n)\sigma\sigma' + o(1/n).$$

The supermartingale $\ln S^n$ has the Doob decomposition $\ln S^n = L^n + E^n \ln S^n$ where $L^n := \ln S^n - E^n \ln S^n$ is a martingale, $\langle L^n\rangle_t = \sigma\sigma' t + o(1)$ and $E^n \ln S^n_t = -(1/2)\mathrm{diag}\,\sigma\sigma' t + o(1)$ By the CLT for martingales the laws $\mathcal{L}(\ln S^n|P^n)$ converge in $\mathbb{D}_{d-1}$ to the law of the Gaussian process $X_t := \sigma w_t - (1/2)\mathrm{diag}\,\sigma\sigma' t$ where w is the standard Wiener process in $\mathbb{R}^{d-1}$. Hence, $\mathcal{L}(S^n|P^n)$ converge to the law of process S with components $S^i = \mathcal{E}((\sigma w)^i)$.

Geometric formulation of models transaction costs, 1

- Piecewise constant price process $S^n = (S^n_t)_{t\in[0,1]}$ with jumps at $t_k = k/n$, $k = 1,\dots,n$, $S_0 = \mathbf{1}$, $S^0 \equiv 1$.
- Solvency cone in terms of the numéraire $K^n \in \mathbb{R}^d$ with $K^{n*} = \mathbb{R}_+(\mathbf{1} + n^{-1/2}U_n)$ where the sequence of convex compact sets U_n lays in a bounded subset of the linear subspace $\{x \in \mathbb{R}^d : \; x^0 = 0\}$ (identified with $\mathbb{R}^{d-1}$) and converges to a convex compact U. E.g., $U_n = [-\kappa_0, \kappa_0]$.
- Solvency cone in terms of physical units $\hat{K}^n_t := K^n/S^n_t$ with the dual $\hat{K}^{n*}_t := D^n_t K^{n*}$, where $D^n_t := \mathrm{diag}\, S^n_t$.
- Price dynamics

$$\widehat{V}_t = x + \sum_{t_k \le t} \widehat{\Delta B}_{t_k}, \qquad \widehat{\Delta B}_{t_k} \in L^0(-\widehat{K}^n_{t_k}, \mathcal{F}^n_{t_k}).$$

  We denote $\widehat{\mathcal{A}}^n_x(1)$ the set of terminal values of such processes.
- Consistent price systems are martingales $Z \in \mathcal{M}_0(\widehat{K}^{n*} \setminus \{0\})$, i.e $Z_t \in L^1(\widehat{K}^{*n}_t)$, Z is constant on $[t_{k-1}, t_k[$, $Z^0_0 = 1$.

Geometric formulation of models transaction costs, 2

- Contingent claim is an arbitrary $\mathbb{R}^d$-valued random variable ζ^n (a function of the price, i.e. $\zeta^n = \widehat{F}(S^n)$).
- Hedging set is the set of initial values allowing portfolio processes with the terminal values dominating the contingent claim $\widehat{F}(S^n)$, i.e.

$$\Gamma^n = \{x \in \mathbb{R}^d : \ \widehat{F}(S^n) \in \widehat{\mathcal{A}}^n_x(1)\}.$$

- The hedging theorem gives the following description :

$$\Gamma^n = \{x \in \mathbb{R}^d : \ Z_0 x \geq EZ_1 \widehat{F}(S^n) \ \ \forall Z \in \mathcal{M}_0(\widehat{K}^{n*})\}.$$

- **Problem** : given a model for S^n find the limit Γ^∞ of Γ^n.
- Closed limit convergence $\Gamma^n \to \Gamma^\infty$:
  1. For any $x \in \Gamma^\infty$, there are $x^n \in \Gamma^n$ such that $x^n \to x$.
  2. For any convergent subsequence of a sequence of vectors $x^n \in \Gamma^n$, its limit x belongs to Γ^∞.

Grépat theorem (simplified form)

- $d = 2$. The price process $\tilde{S}^n_t := (1, \eta_{[nt]})$ where

$$\eta^n_k = \prod_{j \le k} \left(1 + n^{-1/2}\sigma\xi^n_j\right),$$

  ξ^n_k, $k = 1, \ldots, n$, are independent and take values -1 and 1 with equal probabilities.
- $U_n = (1 + n^{-1/2}\lambda)^{-1} - 1, n^{-1/2}\lambda)$.
- $\widehat{F}$ is a bounded continuous function on the Skorohod space.
- Let W be a Wiener process and let g be a predictable process,

$$\sigma(\sigma - 2\lambda)^+ \le g^2 \le \sigma(\sigma + 2\lambda).$$

  Put $\Gamma^\infty := \{x \in \mathbb{R}^2 \colon \ \mathbf{1}x \ge EX_1\widehat{F}(X) \ \ \forall X = (1, \mathcal{E}(g \cdot W))\}$.

Theorem (Grépat, 2013)

$\Gamma^n \to \Gamma^\infty$.

Multidimensional result

$$\mathbb{B} := \{\beta = (1/d)\sum_{j\le d} u_j \otimes v_j : \ u_j \in U\},$$

$$\mathfrak{M} := \{\sigma\sigma' + \sigma\beta + \beta'\sigma', \ \beta \in \mathbb{B}\},$$

$$\mathfrak{M}_+ := \mathfrak{M} \cap \mathbb{S}_+.$$

A : $\beta + \sigma$ *is invertible* $\forall\, \beta \in \mathbb{B}$ *and* $(\beta+\sigma)^{-1}\beta\Sigma \subseteq \operatorname{int}\Sigma$.

Theorem

If **A** *is fulfilled, then* $\Gamma^n \to \Gamma^\infty$ *where*
$\Gamma^\infty := \{x \in \mathbb{R}^d : \ \mathbf{1}x \ge E\tilde{X}_1\widehat{F}(X)\ \forall\, X = \mathcal{E}(H\cdot W), \ HH' \in \mathfrak{M}_+\}$.

A particular case : Bank–Dolinsky–Perkkiö theorem

$U_n := [-\kappa_-, \kappa_+] \subset \mathbb{R}^{d-1}$.

- Let $\mathbf{G}$ be the set of $(d-1) \times (d-1)$-matrices of the matrix
$$\sigma\sigma' + \sigma\beta + \beta'\sigma'$$
where β is a matrix which column $\beta_k = \sum_{j \le d} w_{jk} v_j$ for some $w_{jk} \in [0, (1/d)(\kappa_- + \kappa_+)]$ (the set of such β is denoted by $\mathbf{B}$).
- Let W be $(d-1)$-valued Wiener process defined on some stochastic basis and let $\mathcal{H}$ be the set of predictable $(d-1)$-valued processes such that $HH' \in \mathbf{G}$.

Theorem

Suppose that for any $\beta \in \mathbf{B}$ the matrix $\sigma' + \beta$ is invertible and $v_i'\beta(\sigma' + \beta)^{-1}v_j > -1$ for all i, j. Then

$$\lim \bar{x}^n = \sup_{H \in \mathcal{H}} E\bar{F}(1, \mathcal{E}(H^2 \cdot W^2), \ldots, \mathcal{E}(H^d \cdot W^d)).$$

- Bank P., Dolinsky Y., Perkkiö A-P. The scaling limit of superreplication prices with small transaction costs in the multivariate case. Finance and Stochastics, **21**, 2, 487-508 (2017)
- Grépat J. : On the Limit Behavior of Option Hedging Sets under Transaction Costs. Recent Advances in Financial Engineering, 75–91 (2013)
- He H. : Convergence from discrete to continuous time contingent claim prices. Rev. Fin. Stud., **3**, 523-546 (1990)
- Kabanov Yu., Safarian M. : On Leland's strategy of option pricing with transaction costs. Finance and Stochastics, **1**, 3, 239-250 (1997)
- Kusuoka, S. : Limit theorem on option replication cost with transaction costs. Ann. Appl. Probab., **5**, 198–221 (1995)

- Kabanov Yu., Safarian M. : Markets with Transaction Costs. Springer, Berlin (2009)
- Kusuoka, S. : Limit theorem on option replication cost with transaction costs. Ann. Appl. Probab., **5**, 198–221 (1995)
- Leland H. E. : Option pricing and replication with transaction costs, J. Finance, **40**, 1283–1301 (1985)
- Lépinette E. : Modified Leland's strategy for constant transaction costs rate. Mathematical Finance **22**, 4 (2012)