From set-valued quantiles to risk measures: A set optimization approach to markets with frictions

Andreas H. Hamel, Daniel Kostner
Free University of Bozen-Bolzano

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Cheers to Yuri.

Theorem (Fenchel–Moreau, Hamel 2009)

$f\colon X \to \mathcal{G}(Z, C)$ proper closed convex, or identically $\emptyset$ or Z

if, and only if, $f = f^{**}$.

Consequence. (Kabanov 1999, superhedging in discrete time).

$$\Gamma := \left\{ v \in \mathbb{R}^d \colon V_T^{v,L} \succeq C \text{ for some } L \in \mathfrak{A} \right\}$$
$$= \bigcap_{Z \in \mathfrak{P}_0} \left\{ v \in \mathbb{R}^d \colon \widehat{Z}_0 v \geq E \widehat{Z}_T C \right\} =: D.$$

Proof. Γ is the value of a set-valued proper closed sublinear risk measure at $-C$, and D is the value of its biconjugate at $-C$. (Note: C here corresponds to x in the general theory. The additional dual variable z^* is $\widehat{Z}_0$.)

What is a quantile of a multivariate random variable?

For example, its median?

The question.

Let $X \colon \Omega \to \mathrm{I\!R}$ be a (univariate) random variable on a probability space $(\Omega, \mathcal{F}, P)$.

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where $F_X \colon \mathbb{R} \to [0,1]$, $q_X^-, q_X^+ \colon [0,1] \to \mathbb{R}$ (inverse!).

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Question. What about F_X, q_X^-, q_X^+ for $X \colon \Omega \to \mathbb{R}^d$, $d > 1$?

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Question. What if $\leq$ is replaced by $\leq_C$, $C \subseteq \mathbb{R}^d$ a closed convex cone?

Bad news.

- Even if $C = \mathbb{R}^d_+$, i.e., the case of joint distribution function $F_X^{jdf}(z) = P(X \leq_{\mathbb{R}^d_+} z)$, the formula

$$Q_X^-(p) = \inf \left\{ z \in \mathbb{R}^d \mid F_X^{jdf}(z) \geq p \right\}$$

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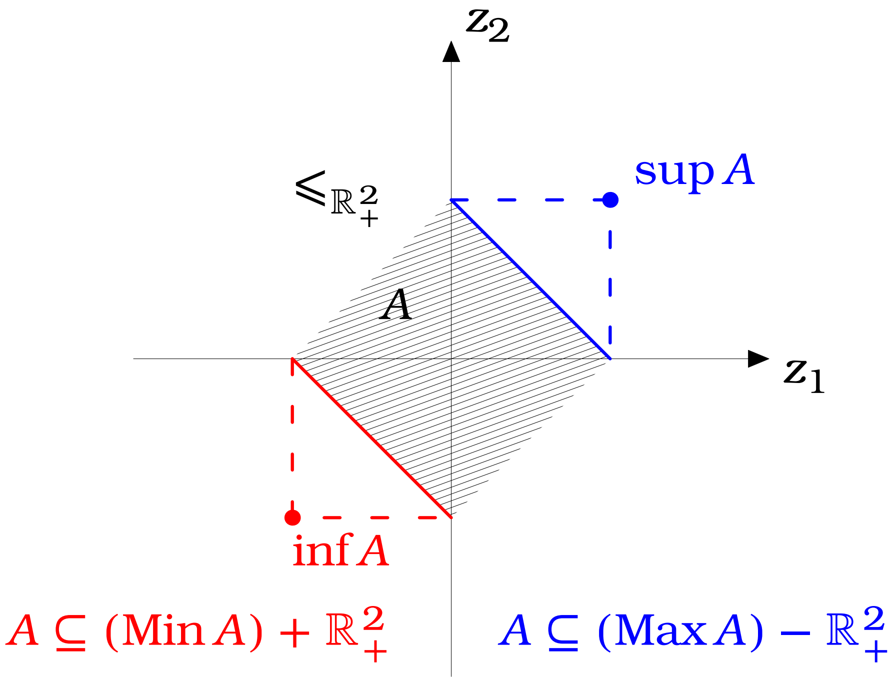
  doesn't work.
- Even worse, the joint distribution function is not very useful for statistical analysis: it 'does not mean much' (M. Hallin 2017). Instead, so-called *depth functions* are used.


z_2
$\sup A$
$\leqslant_{\mathbb{R}^2_+}$
A
z_1
$\inf A$
$A \subseteq (\operatorname{Min} A) + \mathbb{R}^2_+$
$A \subseteq (\operatorname{Max} A) - \mathbb{R}^2_+$

The question.

Remaining question.

What is a quantile of a multivariate random variable in the presence of an order relation for its values?

Quote.

Naturally, the quantiles of a multivariate random variable are also of interest, and the search for a multidimensional counterpart of the quantiles of a random variable has attracted considerable attention in the statistical literature.

The fundamental difficulty in reaching agreement on a suitable generalization of univariate quantiles is arguably the lack of a natural ordering in a multidimensional setting.

Belloni/Winkler 2011, The Annals of Statistics

Another quote.

Unlike the real line, the d-dimensional space $\mathbb{R}^d$, for $d \geq 2$, is not canonically ordered. As as a consequence, such fundamental and strongly order-related univariate concepts as quantile and distribution functions, and their empirical counterparts, involving ranks and signs, do not canonically extend to the multivariate context.

Hallin 2017, ECARES Working Paper

Observation.

If $X \colon \Omega \to \mathrm{I\!R}$ is univariate, the set

$$LQ_X(p) = \{r \in \mathrm{I\!R} \mid P(X \leq r) \geq p\}$$

is "directed upward," i.e.,

$$LQ_X(p) + \mathrm{I\!R}_+ = LQ_X(p) \qquad \text{and} \qquad q^-_X(p) = \inf LQ_X(p).$$

Observation.

On the other hand, the set

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Note. (lower quantiles are sufficient)

$$q^+_X(p) = -q^-_{-X}(1-p).$$

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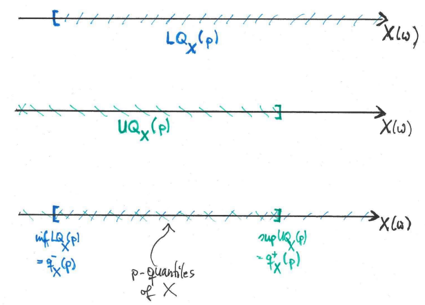
Recall. (V@R is a quantile)

$$V@R_\alpha(X) = q^-_{-X}(1-\alpha) \quad \text{for} \quad \alpha \in (0,1).$$

Observation.

$$LQ_X(p) \cap UQ_X(p) = [\inf LQ_X(p), \sup UQ_X(p)] = \left[q^-_X(p), q^+_X(p)\right].$$



Rest of the talk.

- Lower (real-valued) C-distribution functions
- Lower & upper (set-valued) C-quantiles
- Set-valued V@R & first order stochastic dominance
- Multivariate V@R for conical market models

Next.

- Lower (real-valued) C-distribution functions

Lower C-distribution functions.

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- closed homogeneous halfspace with normal $w \in \mathbb{R}^d \backslash \{0\}$:

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Bipolar theorem. $C = \bigcap_{w \in C^+} H^+(w)$.

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Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X \colon \Omega \to \mathbb{R}^d$ a random variable.

Definition

The w-distribution function $F_{X,w} \colon \mathbb{R}^d \to [0,1]$ of X is the composition of the cdf of $w^T X$ and $w \in \mathbb{R}^d \backslash \{0\}$, i.e.,

$$F_{X,w}(z) = F_{w^T X}(w^T z) = P\left(X \in z - H^+(w)\right).$$

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The lower C-distribution function $F_{X,C} \colon \mathbb{R}^d \to [0,1]$ of X is

$$F_{X,C}(z) = \inf_{w \in C^+ \backslash \{0\}} F_{X,w}(z).$$

Lower C-distribution function:

$$F_{X,C}(z) = \inf_{w \in C^+ \backslash \{0\}} F_{X,w}(z) = \inf_{w \in C^+ \backslash \{0\}} P\left(X \in z - H^+(w)\right).$$

Observation. (since $C \subseteq H^+(w)$ for all $w \in C^+ \backslash \{0\}$)

$$\forall z \in \mathbb{R}^d \colon F_{X,C}(z) \geq P(X \in z - C),$$

and this inequality is strict in general even for $C = \mathbb{R}^d_+$ (already for X bivariate normal and $z = 0 \in \mathbb{R}^2$).

Lower C-distribution functions.

Proposition

(a) Affine invariance: *if* $b \in \mathbb{R}^d$, $A \in \mathbb{R}^{d \times d}$ *invertible, then*

$$\forall z \in \mathbb{R}^d \colon F_{AX+b, A^T C}\left(Az + b\right) = F_{X,C}\left(z\right).$$

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(d) Right-continuity: *if* $\{z_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}^d$ *with* $\lim_{n \to \infty} z_n = \overline{z} \in \mathbb{R}^d$ *and* $z_{n+1} \leq_C z_n$ *for all* $n \in \{1, 2, \ldots\}$, *then*

$$\lim_{n \to \infty} F_{X,C}\left(z_n\right) = F_{X,C}\left(\overline{z}\right).$$

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(e) Behavior at infinity: *If* $L^+(z) = \{w \in C^+ \mid w^T z = 0\}$, *then*

$$\lim_{t\to\infty} F_{X,C}(tz) = \begin{cases} \min\{a,b\} & \textit{if} \quad z \in C \\ 0 & \textit{if} \quad z \notin C \end{cases}$$

where $a = \inf_{w \in L^+(z)\setminus\{0\}} F_{X,w}(0)$, $b = \lim_{t\to\infty} \inf_{w \in C^+\setminus L^+(z)} F_{X,w}(z)$ *and*

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$$\lim_{t\to-\infty} F_{X,C}(tz) = \begin{cases} 0 & \text{if} \quad z \notin -C \\ \min\{a,c\} & \text{if} \quad z \in -C \end{cases}$$

where $c = \lim_{t\to-\infty} \inf_{w\in C^+\setminus L^+(z)} F_{X,w}(z)$.

Summary.

- The lower C-distribution function $F_{X,C}$ almost behaves like a (univariate) cdf.
- It is, however, different from the joint distribution function even if $C = \mathrm{I\!R}^d_+$.
- For $d = 1$, $C = \mathrm{I\!R}_+$, one gets back the univariate cdf.

Next.

- Lower & upper C-quantiles

$$\mathcal{P}(\mathrm{I\!R}^d) = \left\{ D \mid D \subseteq \mathrm{I\!R}^d \right\} \quad \text{power set}$$

Definition

The lower w-quantile of X is $Q^-_{X,w} \colon [0,1) \to \mathcal{P}(\mathrm{I\!R}^d)$ defined by

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Straightforward results.

$$\forall p \in [0,1)\colon Q^-_{X,C}(p) = \bigcap_{w \in C^+\backslash\{0\}} Q^-_{X,w}(p).$$

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$$\forall z \in \mathbb{R}^d \colon F_{X,C}(z) = \sup \left\{ p \in [0,1) \mid z \in Q^-_{X,C}(p) \right\}$$

($F_{X,C}$ can be reconstructed from the quantile function.)

Lower & upper C-quantiles.

Proposition

(a) $Q^-_{X,C} \colon [0,1) \to \mathcal{P}(\mathrm{I\!R}^d)$ *has closed convex values and satisfies*

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(b) For all $b \in \mathrm{I\!R}^d$ *and all invertible* $A \in \mathrm{I\!R}^{d\times d}$,

$$Q^-_{X+b,C}(p) = Q^-_{X,C}(z) + b \quad \text{and} \quad Q^-_{AX,A^TC}(p) = AQ^-_{X,C}(p).$$

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(c) If $p_1 \geq p_2$ *then* $Q^-_{X,C}(p_1) \subseteq Q^-_{X,C}(p_2)$.
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Note. (a): $Q^-_{X,C}$ maps into the complete lattice ordered by $\supseteq$:

$$\mathcal{G}(\mathbb{R}^d, C) := \left\{D \subseteq \mathbb{R}^d \mid D = \operatorname{cl}\operatorname{co}(D + C)\right\}.$$

Tukey depth function.

$$HD_X(z) = \inf_{w \in \mathbb{R}^d \setminus \{0\}} P\left(X \in z - H^+(w)\right), \; z \in \mathbb{R}^d.$$

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Tukey depth region.

$$D_X(p) = \left\{z \in \mathbb{R}^d \mid HD_X(z) \geq p\right\}, \; p \in [0, 1/2].$$

Lower C-quantiles and Tukey depth.

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Observation. Univariate quantile: $d = 1$, $C = C^+ = \mathbb{R}_+$.

Definition

For $w \in C^+\backslash\{0\}$, the function $Q^+_{X,w}\colon (0,1] \to \mathcal{P}(\mathbb{R}^d)$ defined by

$$Q^+_{X,w}(p) = \left\{z \in \mathbb{R}^d \mid P(w^T X < w^T z) \leq p\right\}$$

is called the *upper w-quantile function* of X. The function $Q^+_{X,C}\colon (0,1] \to \mathcal{P}(\mathbb{R}^d)$ defined by

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Remark. $Q^+_{X,C}\colon \mathbb{R}^d \to \mathcal{G}(\mathbb{R}^d, -C)$ (opposite to $Q^-_{X,C}$).

Surprising result. (no other concept known with this type of property)

$$Q^{+}_{X,C}(p) = Q^{-}_{X,-C}(1-p).$$

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$$Q^+_{X,C}(p) = Q^-_{X,-C}(1-p).$$

Compare this to

$$q^+_X(p) = -q^-_{-X}(1-p).$$

Lower & upper C-quantiles.

Summary.

- The lower C-quantile $Q^-_{X,C}$ is a set-valued inverse of $F_{X,C}$.
- It shares most properties with scalar quantiles.
- It also generalizes the Tukey (halfspace) depth function as well as the univariate quantile.

Lower & upper C-quantiles: how do they look like?

Example. (a four point uniform distribution)

- $\Omega = \left\{(-1,2)^T, (0,0)^T, (1,1)^T, (2,-1)^T\right\} \subseteq \mathbb{R}^2$,
- X uniformly distributed over Ω.

Lower and upper quantile sets for the seven cases

$p \in (0, \frac{1}{4})$, $p = \frac{1}{4}$, $p \in (\frac{1}{4}, \frac{1}{2})$, $p = \frac{1}{2}$, $p \in (\frac{1}{2}, \frac{3}{4})$, $p = \frac{3}{4}$, $p \in (\frac{3}{4}, 1)$.

Lower & upper C-quantiles: how do they look like?

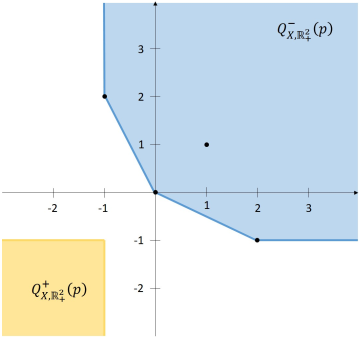


Figure : $p \in (0, \frac{1}{4})$

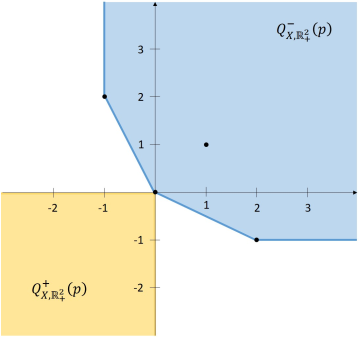


Figure : $p = \frac{1}{4}$

Lower & upper C-quantiles: how do they look like?

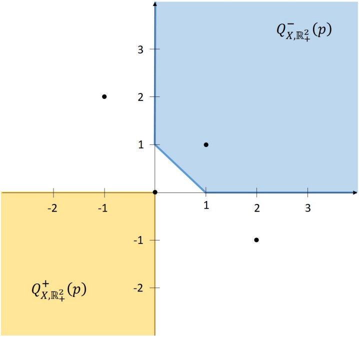


Figure : $p \in \left(\frac{1}{4}, \frac{1}{2}\right)$

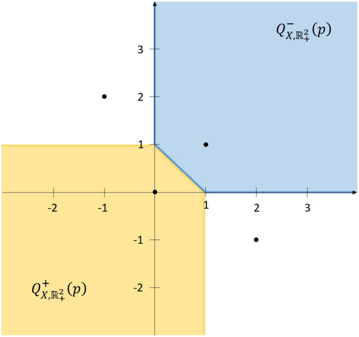


Figure : $p = \frac{1}{2}$

Lower & upper C-quantiles: how do they look like?

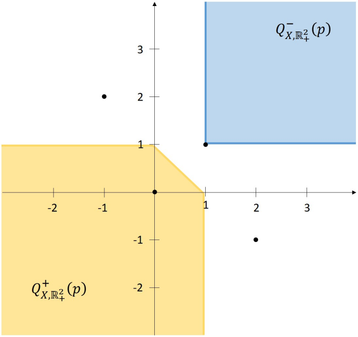


Figure : $p \in \left(\frac{1}{2}, \frac{3}{4}\right)$

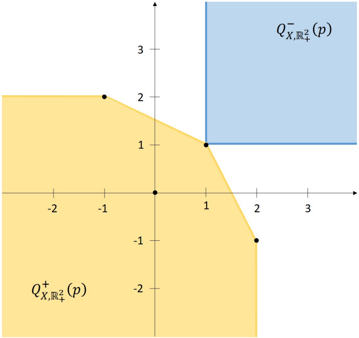


Figure : $p = \frac{3}{4}$

Lower & upper C-quantiles: how do they look like?

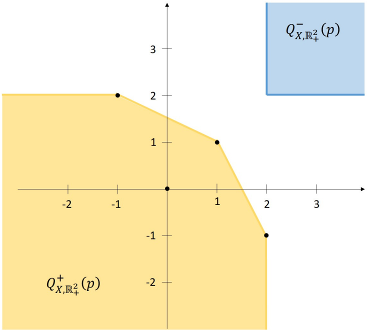


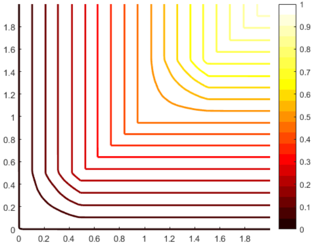
Figure : $p \in (\frac{3}{4}, 1)$

Remark. In general,

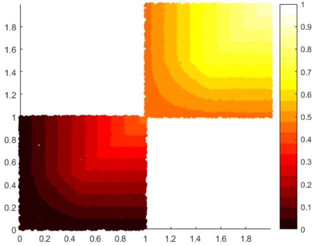
$$Q^{-}_{X,C}(p) \bigcap Q^{+}_{X,C}(p) = \emptyset.$$

Compare this to

$$LQ_X(p) \cap UQ_X(p) = [\inf LQ_X(p), \sup UQ_X(p)] = \left[q^{-}_X(p), q^{+}_X(p)\right].$$

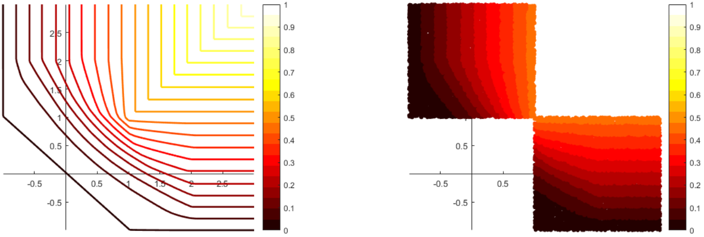


Contours of $Q^-_{X,\mathbb{R}^2_+}(p)$



Sample points in $Q^-_{X,\mathbb{R}^2_+}(p)$

Figure : Lower C-quantiles in 5% increments for $C = \mathbb{R}^2_+$ and $X \sim \text{U}niform\big((0,1) \times (0,1) \cup (1,2) \times (1,2)\big)$.



Contours of $Q^-_{X,\mathrm{I\!R}^2_+}(p)$ Sample points in $Q^-_{X,\mathrm{I\!R}^2_+}(p)$

Figure : Lower C-quantiles in 5% increments for $C = \mathrm{I\!R}^2_+$ and $X \sim \text{Uniform}\big((-1,1)\times(1,3)\cup(1,3)\times(-1,1)\big)$.

The joint distribution lower quantile.

$$Q_X^{jdf}(p) := \left\{ z \in \mathbb{R}^2 \mid F_X^{jdf}(z) \geq p \right\}$$

Example. (same as before)

- $Q_X^{jdf}(p)$ is different from lower C-quantile in general.
- $Q_X^{jdf}(p)$ is non-convex in general.
- $Q^-_{X,C}(p)$ is also different from the convex hull of $Q_X^{jdf}(p)$.

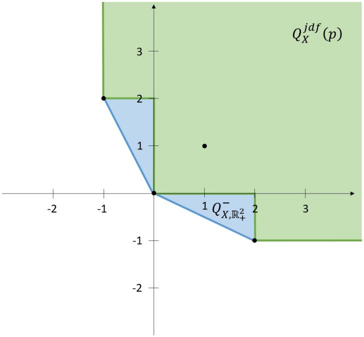


Figure : $p = \frac{1}{4}$

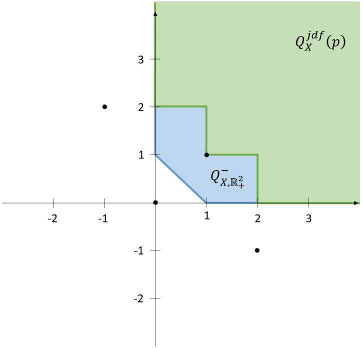


Figure : $p = \frac{1}{2}$

Next.

- Set-valued V@R & first order stochastic dominance

Definition

The Value-at-Risk of $X \colon \Omega \to \mathbb{R}^d$ at level $\alpha \in (0,1)$ is

$$V@R_{\alpha,C}(X) = Q^-_{-X,C}\,(1-\alpha).$$

Value-at-Risk.

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$$V@R_{\alpha,C}(X) = Q^-_{-X,C}(1-\alpha).$$

Recall. (univariate V@R)

$$V@R_\alpha(X) = q^-_{-X}(1-\alpha) \quad \text{for} \quad \alpha \in (0,1).$$

Basic formula.

$$V@R_{\alpha,C}(X) = \left\{ z \in \mathbb{R}^d \mid \sup_{w \in C^+ \setminus \{0\}} P\left(X + z \in -\operatorname{int} H^+(w)\right) \leq \alpha \right\}$$

$$= \bigcap_{w \in C^+ \setminus \{0\}} \left\{ z \in \mathbb{R}^d \mid P\left(X + z \in -\operatorname{int} H^+(w)\right) \leq \alpha \right\}$$

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Interpretation. $V@R_{\alpha,C}(X)$ includes all deterministic portfolios z (= deposits) which keep the probability of bankruptcy $\leq \alpha$ when added to the random position X – for all "price" vectors $w \in C^+$ since

$$X + z \in -\operatorname{int} H^+(w) \quad \Leftrightarrow \quad w^T(X + z) < 0.$$

Proposition

(1) The function $X \mapsto V@R_{\alpha,C}(X)$ *maps into* $\mathcal{G}(\mathbb{R}^d, C)$ *and is positively homogeneous. In particular,* $V@R_{\alpha,C}(0) = C$.

Value-at-Risk.

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(2) $X \mapsto V@R_{\alpha,C}(X)$ *is* $\mathbb{R}^d$*-translative, i.e.,*

$$\forall y \in \mathbb{R}^d \colon V@R_{\alpha,C}(X + y1\!\!1) = V@R_{\alpha,C}(X) - y.$$

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(3) $X \mapsto V@R_{\alpha,C}(X)$ *is monotone non-increasing w.r.t.* $\leq_C$, *i.e.,* $X \leq_C Y$ *implies* $V@R_{\alpha,C}(X) \subseteq V@R_{\alpha,C}(Y)$.

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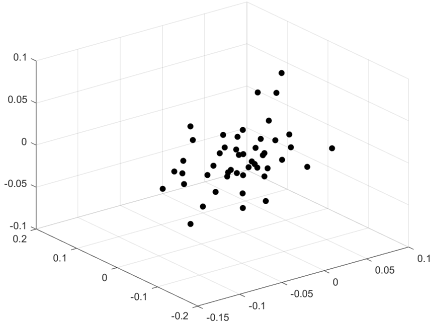
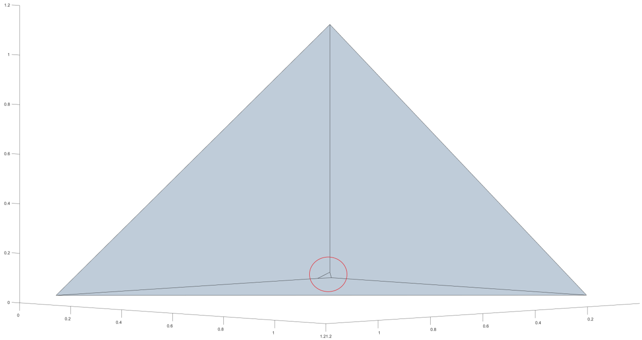
Note. This qualifies $V@R$ as a set-valued risk measure in the sense of Hamel/Heyde 2010 SIFIN.

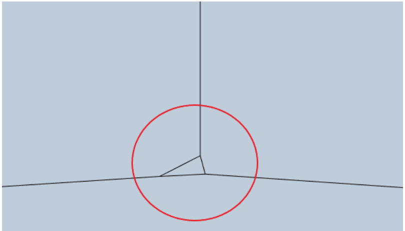
Value-at-Risk.



$$\tilde{X} = (R_{DAX} \mid R_{HSI} \mid R_{S\&P})$$



$$V@R_{10\%,\mathbb{R}^3_+}(\tilde{X})$$



Vertices of $V@R$

Figure : Value-at-Risk with $C = \mathrm{I\!R}^3_+$ for the monthly return of three major stock indexes.

Scalarization formula.

$$V@R_{\alpha,C}(X) = \bigcap_{w \in C^+ \setminus \{0\}} \left\{ z \in \mathbb{R}^d \mid w^T z \geq V@R_\alpha^{sca}(w^T X) \right\}$$

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Special case. $C = \mathrm{I\!R}^d_+$

$$V@R_{\alpha,\mathrm{I\!R}^d_+}(X) = \bigcap_{w \in \mathrm{I\!R}^d_+ \setminus \{0\}} \left\{ z \in \mathrm{I\!R}^d \mid w^T z \geq V@R_\alpha^{sca}(w^T X) \right\}$$

$$\subseteq \bigcap_{i \in \{1,\ldots,d\}} \left\{ z \in \mathrm{I\!R}^d \mid z_i \geq V@R_\alpha^{sca}(X_i) \right\},$$

i.e., $V@R_{\alpha,\mathrm{I\!R}^d_+}$ is “more conservative” than the component-wise Value-at-Risk.

Value-at-Risk.

Embrechts/Puccetti 06. F_X^{jdf} jdf and

$$V@R_\alpha^{jdf}(X) = \partial \left\{ z \in \mathbb{R}^d \mid F_X^{jdf}(z) \geq \alpha \right\}.$$

Value-at-Risk.

Embrechts/Puccetti 06. F_X^{jdf} jdf and

$$V@R_\alpha^{jdf}(X) = \partial \left\{ z \in \mathbb{R}^d \mid F_X^{jdf}(z) \geq \alpha \right\}.$$

Associated to order generated by $\mathbb{R}^d_+$; even if it is replaced by C and ∂ is dropped,

$$V@R_{\alpha,C}(X) = Q^-_{-X,C}(1-\alpha) = \left\{ z \in \mathbb{R}^d \mid F_{-X,C}(z) \geq 1-\alpha \right\}$$
$$\supseteq \left\{ z \in \mathbb{R}^d \mid P(-X \in z - C) \geq 1-\alpha \right\}$$

since

$$\forall z \in \mathbb{R}^d \colon F_{-X,C}(z) \geq P(-X \in z - C).$$

Note. $V@R_{\alpha,C}$ is "less conservative" than the E/P version.

Stochastic dominance.

Definition

The random variable $Y \in L^0_d(\Omega, \mathcal{F}, P)$ is said to C-stochastically dominate $X \in L^0_d(\Omega, \mathcal{F}, P)$, written as $Y \succeq^C_{FSD} X$, iff

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$$\forall z \in \mathbb{R}^d \colon F_{Y,C}(z) \leq F_{X,C}(z) \,.$$

Proposition

For $X, Y \in L^0_d(\Omega, \mathcal{F}, P)$, $Y \succeq^C_{FSD} X$ is true if, and only if,

$$\forall p \in (0,1) \colon Q^-_{Y,C}(p) \subseteq Q^-_{X,C}(p) \,,$$

and if, and only if,

$$\forall \alpha \in (0,1) \colon V@R_{\alpha,C}(X) \subseteq V@R_{\alpha,C}(Y).$$

Next.

- Multivariate V@R for conical market models

V@R for the market.

Obvious idea. Make the cone C random.

Motivation (from finance). Conical market models.

Motivation (from statistics). I hardly have a clue, just ideas. Ilya, would it make sense to analyse data with an order relation which changes with the data point?

The model.

- $K(\omega) \subseteq \mathrm{I\!R}^d$ closed convex cones (measurable) generating on L^0_d the order

$$X^1 \leq_K X^2 \quad \Leftrightarrow \quad X^2 - X^1 \in K \; (P-\text{a.s.})$$

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- Potential quantiles: $z \in \mathbb{R}^m$ ordered via closed convex cone $K_0 \subseteq \mathbb{R}^m$ with $1 \leq m \leq d$:

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- A linear operator $T \colon \mathbb{R}^m \to L^0_d(\Omega, \mathcal{F}, P)$ which generates a another cone $TK_0 \subseteq L^0_d$.

Example. $Z^1, \dots, Z^m \in L^0_d$ linearly independent and $Tz = \sum z_i Z^i$. In particular, $Z^i = b^i 1\!\!1$ with $b^i \in \mathbb{R}^d$ (riskless portfolios used as deposits).

Example. $Z^1, \ldots, Z^m \in L^0_d$ linearly independent and $Tz = \sum z_i Z^i$. In particular, $Z^i = b^i 1\!\!1$ with $b^i \in \mathbb{R}^d$ (riskless portfolios used as deposits).

Pointwise cones.

- $C(\omega) = (TK_0)(\omega) + K(\omega)$,
- $C^+(\omega) = [(TK_0)(\omega) + K(\omega)]^+ = (TK_0)(\omega)^+ \cap K(\omega)^+$.

This can be seen as a one-period market model with m eligible assets.

Definition

The composition of the function $F_{X,Y}\colon L^0_d \to [0,1]$ with T which is defined by

$$F_{X,Y}(Tz) = \Pr[Y^\top X \leq Y^\top (Tz)]$$

is called the Y-distribution function of X with $Y \in L^0_d$.

V@R for the market.

Definition

The composition of the function $F_{X,Y} \colon L_d^0 \to [0,1]$ with T which is defined by

$$F_{X,Y}(Tz) = \Pr[Y^\top X \leq Y^\top (Tz)]$$

is called the Y-distribution function of X with $Y \in L_d^0$.
The composition of the function $F_{X,C} \colon L_d^0 \to [0,1]$ with T defined by

$$F_{X,C}(Tz) = \inf\{F_{X,Y}(Tz) \mid Y \in C^+\}$$

is called the C-distribution function of X (with respect to T).

Definition

The lower Y-quantile of X is $Q^-_{X,Y} \colon [0,1) \to \mathcal{P}(\mathrm{I\!R}^m)$ defined by

$$Q^-_{X,Y}(p) = \{z \in \mathrm{I\!R}^m \mid F_{X,Y}(Tz) \geq p\}.$$

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$$Q^-_{X,C}(p) = \{z \in \mathrm{I\!R}^m \mid F_{X,C}(Tz) \geq p\}.$$

Note. Many (not all) properties carry over from the constant cone case.

V@R for the market.

Definition

The Value-at-Risk of $X\colon \Omega \to \mathbb{R}^m$ at level $\alpha \in (0,1)$ is

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V@R for the market.

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Perspective. stochastic dominance, AV@R, Kusuoka representations ... data analysis etc.

Coda.

Conclusion. There seems to be a canonical way to introduce quantiles in the multivariate case in the presence of an in general non-complete order relation for the values of random variables.

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Test question. Where is the infimum in

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Coda.

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Reference. Hamel/Kostner: *Cone distribution functions and quantiles for multivariate random variables*
JMVA 167 (2018) 97-113 and follow-ups

THANK YOU FOR FOLLOWING ...

... INTO THE DEPTH OF STATISTICS.

SET OPTIMIZATION for APPLICATIONS

4th International Conference on Set Optimization

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at Friedrich Schiller University Jena, Germany

February 11-15, 2019