The joint distributions of terminal values of increasing processes and their compensators

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Definition

A probability measure μ on $(\mathbb{R}^2_+, \mathscr{B}(\mathbb{R}^2_+))$ belongs to $\mathbb{W}^*$ if there are a filtered probability space $\mathbb{B} = (\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geqslant 0}, \mathsf{P})$ and a locally integrable increasing process $X = (X_t)_{t\geqslant 0}$, $X_0 = 0$ defined on $\mathbb{B}$ such that

$$\mu = \mathrm{Law}(X_\infty, A_\infty),$$

where $A = (A_t)_{t\geqslant 0}$ is the compensator of X.

The main problem is to characterize the set $\mathbb{W}^*$.
If $\mu \in \mathbb{W}^+$, then we have necessarily

$$\int x\,\mu(dx, da) = \int a\,\mu(dx, da). \tag{1}$$

If the integrals in (1) are finite, we write $\mu \in \mathbb{W}$. This case is called integrable.

A necessary condition

Let X be a nonnegative local submartingale with the Doob–Meyer decomposition $X = M + A$, $X_0 = M_0 = A_0 = 0$, on some stochastic basis $(\Omega, \mathscr{F}, \mathbb{F}, \mathsf{P})$; M is a local martingale, A is a predictable integrable increasing process.

Proposition

Let X be a nonnegative local submartingale, $X_0 = 0$. Then, for any $\lambda > 0$, $(X - A + \lambda)\mathbb{1}_{\{A<\lambda\}}$ is a nonnegative supermartingale. In particular, P*-a.s.*

$$\{A_\infty < \infty\} \subseteq \{X \to\} \tag{2}$$

and, for any $\lambda \geqslant 0$,

$$\mathsf{E}X_\infty \mathbb{1}_{\{A_\infty \leqslant \lambda\}} \leqslant \mathsf{E}(A_\infty \wedge \lambda). \tag{3}$$

Proof.

Fix $\lambda > 0$ and put $H := \mathbb{1}_{\{A<\lambda\}} = \mathbb{1}_{[\![0,S[\![}$, where $S := \inf\{t \geqslant 0\colon A_t \geqslant \lambda\}$ is a predictable stopping time. It is clear that H is predictable and right-continuous. Then, for every semimartingale Y,

$$Y_0 + H \cdot Y = Y\mathbb{1}_{[\![0,S[\![} + Y_{S-}\mathbb{1}_{[\![S,\infty[\![}.$$

Applying this equality to $Y = X - A + \lambda$, we get

$$(X - A + \lambda)\mathbb{1}_{\{A<\lambda\}} = \lambda + H \cdot M - X_{S-}\mathbb{1}_{[\![S,\infty[\![} - (\lambda - A_{S-})\mathbb{1}_{[\![S,\infty[\![}.$$

Since all other processes in this formula are nonnegative, the local martingale $H \cdot M$ is bounded from below by $-\lambda$ and hence is a supermartingale. □

End of the proof.

Therefore, the nonnegative process $Z := (X - A + \lambda)\mathbb{1}_{\{A<\lambda\}}$ on the left, being the difference of a supermartingale and an increasing process, is also a supermartingale. Since a nonnegative supermartingale converges a.s., we obtain that X converges a.s. on the set $\{A_\infty < \lambda\}$, and (2) follows. Finally, it follows from the supermartingale property of Z that $\mathsf{E}Z_\infty \leqslant \mathsf{E}Z_0 = \lambda$, i.e. $\mathsf{E}X_\infty \mathbb{1}_{\{A_\infty<\lambda\}} \leqslant \mathsf{E}(A_\infty \wedge \lambda)$. Now, (3) follows because its right-hand side is continuous in λ. □

Equality in (3)

It follows from the proof that, for a given $\lambda > 0$, equality in (3) holds if and only if

$$\mathsf{E}\mathbb{1}_{[\![0,S[\![} \cdot M_\infty = 0, \tag{4}$$

$$X_{S-}\mathbb{1}_{\{S<\infty\}} = 0 \quad \text{a.s.} \tag{5}$$

$$(\lambda - A_{S-})\mathbb{1}_{\{S<\infty\}} = 0 \quad \text{a.s.} \tag{6}$$

Let $A = (A_t)_{t\geqslant 0}$ be an adapted increasing process. For $s \geqslant 0$, define

$$C_s = \inf\{t \geqslant 0\colon A_t > s\}.$$

Then C_s is a stopping time for every s and trajectories $s \rightsquigarrow C_s$ are nondecreasing and right-continuous. The process $C = (C_s)$ is called the change of time generated by A.

Recall that, if $Y = (Y_t)_{t\geqslant 0}$ is a progressively measurable process, then $Y \circ C = \left(Y_{C_t}\right)_{t\geqslant 0}$ is the transform of Y via the change of time (C_t). This definition assumes implicitly that a random variable Y_∞ is defined on the set $\bigcup_t\{C_t = \infty\} = \{A_\infty < \infty\}$, and then $Y \circ C_t = Y_\infty$ for $t \geqslant A_\infty$. Since we have proved that X converges a.s. on this set, the transforms of X, M, and A are well defined..

Equality in (3) in terms of the time change

Proposition

Let X be a nonnegative local submartingale, $X_0 = 0$. The following assertions are equivalent:

(i) *for any* $\lambda \geqslant 0$,

$$\mathsf{E} X_\infty \mathbb{1}_{\{A_\infty \leqslant \lambda\}} = \mathsf{E}(A_\infty \wedge \lambda);$$

(ii)

$$A_{C_t} = A_\infty \wedge t, \quad X_{C_t} = X_\infty \mathbb{1}_{\{t \geqslant A_\infty\}}, \tag{7}$$

$\left(M_{C_t}\right)_{t\geqslant 0}$ *is a martingale relative to the filtration* $\left(\mathscr{F}_{C_t}\right)_{t\geqslant 0}$.

The proof is based on the analysis of relations (4)–(6).

If X is an increasing process, then necessary and sufficient conditions for equality in (2) have a simple form.

Proposition

An increasing process X satisfies assertions (i)–(ii) *in the first proposition if and only if X is locally integrable and, up to indistinguishability,*

$$X = \xi \mathbb{1}_{[\![T,\infty[\![},$$

where T is a totally inaccessible stopping time and ξ is a nonnegative $\mathscr{F}_T$-measurable random variable.

Predictability is important

Let N be a local martingale bounded from below by a uniformly integrable martingale, $N_0 = 0$. Put $\overline{N}_t = \sup_{0 \leqslant s \leqslant t} N_s$, then $\overline{N} = \left(\overline{N}_t\right)_{t \geqslant 0}$ is a locally integrable increasing process. Now put $X := \overline{N} - N$, $M := -N$, $A := \overline{N}$. Then the triple (X, M, A) satisfies all the assumptions of the first proposition except predictability of A. For $\lambda > 0$, define $T := \inf\{t \geqslant 0 \colon A_t > \lambda\}$. Then

$$0 \geqslant \mathsf{E} N_T \geqslant \lambda \mathsf{P}(\overline{N}_\infty > \lambda) + \int\limits_{\{\overline{N}_\infty \leqslant \lambda\}} N_\infty \, d\mathsf{P}$$

$$= \lambda \mathsf{P}(A_\infty > \lambda) + \int\limits_{\{A_\infty \leqslant \lambda\}} (A_\infty - X_\infty) \, d\mathsf{P},$$

i.e. it holds

$$\mathsf{E} X_\infty \mathbb{1}_{\{A_\infty \leqslant \lambda\}} \geqslant \mathsf{E}(A_\infty \wedge \lambda). \tag{8}$$

This inequality is opposite to inequality(3).

Equality in (8)

It follows from these arguments that we have equality in inequality (8) for all $\lambda \geqslant 0$ if and only if $\overline{N}$ is a.s. continuous. Sufficiency follows from the first proposition, and another explanation is that continuity of $\overline{N}$ implies that N_T is bounded from above for every λ.

Rogers' Theorem

A probability measure μ on $\left(\mathbb{R}_+^2, \mathscr{B}(\mathbb{R}_+^2)\right)$ belongs to $\mathbb{W}_-$ if there are a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geqslant 0}, \mathsf{P})$ and a uniformly integrable martingale $N = (N_t)_{t\geqslant 0}$, $N_0 = 0$, on it such that

$$\mu = \mathrm{Law}(\overline{N}_\infty - N_\infty, \overline{N}_\infty),$$

where $\overline{N}_t = \sup_{0\leqslant s\leqslant t} N_s$.

Theorem (Rogers (1993))

A probability measure μ on $\mathbb{R}_+^2$ belongs to $\mathbb{W}_-$ if and only if

$$\int |x-a|\,\mu(dx,da) < \infty, \qquad \int (x-a)\,\mu(dx,da) = 0,$$

and

$$\int_{\{a\leqslant\lambda\}} x\,\mu(dx,da) \geqslant \int (a\wedge\lambda)\,\mu(dx,da) \qquad \text{for any } \lambda \geqslant 0.$$

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$$\int\limits_{\{a\leqslant\lambda\}} x\,\mu(dx,da) \geqslant \int (a\wedge\lambda)\,\mu(dx,da) \qquad \textit{for any } \lambda \geqslant 0.$$

If, for a local martingale N, there is equality in (8) for all $\lambda \geqslant 0$, then $\overline{N}$ is continuous, i.e. a local submartingale $X = \overline{N} - N$ satisfies $A = \overline{N}$ and, therefore, there is equality in (3) for all $\lambda \geqslant 0$.

Assume now that X is a local submartingale satisfying the assumptions of the first proposition and such that there is equality in (3) for all $\lambda \geqslant 0$. Put $N := A - X$. Then $A = \overline{N}$ and, therefore, there is equality in (8) for all $\lambda \geqslant 0$.

Conjecture

A probability measure μ on $\mathbb{R}_+^2$ belongs to $\mathbb{W}^*$ if and only if

$$\int x\,\mu(dx,da) = \int a\,\mu(dx,da) \tag{9}$$

and

$$\int_{\{a\leqslant\lambda\}} x\,\mu(dx,da) \leqslant \int (a\wedge\lambda)\,\mu(dx,da) \qquad \text{for any } \lambda \geqslant 0. \tag{10}$$

We know that the conjecture is true, in particular, if

1. There is equality in (10) for any $\lambda \geqslant 0$.
2. We have
$$\int (x-a)^+\,\mu(dx,da) \geqslant \int (a-x)^+\,\mu(dx,da).$$
3. $a = x + c$ μ-a.s. for some $c > 0$.

Proposition

Let V and W be random variables with values in $\mathbb{R}_+$ and $\overline{\mathbb{R}}_+$ respectively on a probability space $(\Omega, \mathscr{F}, \mathsf{P})$, moreover, $\{W = \infty\} \subseteq \{V = 0\}$ a.s. and

$$\mathsf{E} V \mathbb{1}_{\{W \leqslant \lambda\}} = \mathsf{E}(W \wedge \lambda) \qquad \text{for any } \lambda \geqslant 0. \tag{11}$$

Define $\mathscr{F}_t$ as the σ-field of subsets in $\mathscr{F}$ whose intersection with the set $\{W > t\}$ is either empty or coincides with $\{W > t\}$. Put

$$X_t := V 1_{\{t \geqslant W\}}, \qquad A_t := W \wedge t.$$

Then $X = (X_t)_{t \geqslant 0}$ is an $(\mathscr{F}_t)$-adapted locally integrable increasing process, $A = (A_t)_{t \geqslant 0}$ is its $(\mathscr{F}_t)$-compensator, and $(X_\infty, A_\infty) = (V, W)$ a.s.

Proof.

Trivially, X and A are adapted and increasing. Moreover, A is continuous and hence predictable. In our terms equality (11) can be rewritten as $\mathsf{E}X_t = \mathsf{E}A_t$ for any $t \geqslant 0$, in particular, X_t is integrable. Let $s < t$. Then $X_s = X_t$ and $A_s = A_t$ on the set $\{W \leqslant s\}$. Therefore,

$$\int_{\{W>s\}} \big((X_t - A_t) - (X_s - A_s)\big)\, d\mathsf{P} = \int_{\Omega} \big((X_t - A_t) - (X_s - A_s)\big)\, d\mathsf{P} = 0.$$

It follows from the definition of $\mathscr{F}_s$ that $\mathsf{E}\big((X_t - A_t) - (X_s - A_s) \big| \mathscr{F}_s\big) = 0$, i.e. $X - A$ is a martingale. □

Remark

If $\mathsf{E}|V - W| < \infty$ *in the previous proposition, then* $X - A$ *is a uniformly integrable martingale if and only if any one of the following two equivalent conditions holds:*
(1) $\mathsf{E}(V - W) = 0$;
(2) $\lim_{\lambda\to\infty} \lambda \mathsf{P}(W > \lambda) = 0$.
Indeed, necessity of the first condition is obvious, and it implies $\mathsf{E}\big((X_\infty - A_\infty) - (X_t - A_t)\big|\mathscr{F}_t\big) = 0$ *for any* $t \geqslant 0$ *similarly to the previous proof. Equivalence of the first and the second conditions follows easily from* (11) *if one takes into account that* $W < \infty$ *a.s. under the current assumptions.*

Second key proposition

The proof in Case 2 is based on the following coupling-type proposition.

Proposition

Let a probability measure μ *on* $\left(\mathbb{R}^2_+, \mathscr{B}(\mathbb{R}^2_+)\right)$ *satisfy*

$$\int (x-a)^+\,\mu(dx,da) \geqslant \int (a-x)^+\,\mu(dx,da), \tag{12}$$

$$\int_{\{a\leqslant\lambda\}} x\,\mu(dx,da) \leqslant \int (a\wedge\lambda)\,\mu(dx,da) \qquad \text{for any } \lambda \geqslant 0. \tag{13}$$

Then there exists a probability space $(\Omega, \mathscr{F}, \mathrm{P})$ *and random variables* X, Y, Z *on it such that* $\mathrm{Law}(X,Y) = \mu$, $0 \leqslant Z \leqslant X \wedge Y$, *and*

$$\int_{\{Y-Z\leqslant\lambda\}} (X - Y + \lambda)\,d\mathrm{P} = \lambda \qquad \text{for any } \lambda \geqslant 0.$$

Two remarks

Remark

(14) *and* (16) *imply*

$$\int x\,\mu(dx, da) = \int a\,\mu(dx, da).$$

Remark

The statement of the proposition guarantees that the random variables $V := X - Z$ *and* $W := Y - Z$ *satisfy the assumptions of the proposition from Case 1.*

The constructions in Cases 1 and 2 have important property: one can always construct an increasing process X with a continuous compensator A. Since Case 2 includes the integrable case, we have

Corollary

(i) *If* $\mu \in \mathbb{W}^*$. *there are a filtered probability space* $\mathbb{B} = (\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \geqslant 0}, \mathsf{P})$ *and a locally integrable increasing process* $X = (X_t)_{t \geqslant 0}$, $X_0 = 0$ *defined on* $\mathbb{B}$ *such that*

$$\mu = \mathsf{Law}(X_\infty, A_\infty),$$

where the compensator A *of* X *is continuous.*
(ii) *If* X *is a nonnegative local submartingale with the Doob–Meyer decomposition* $X = M + A$, $X_0 = M_0 = A_0 = 0$, T *is a stopping time, then* $\mathsf{Law}(X_T, A_T) \in \mathbb{W}^*$.

Now it can be shown that $\mathbb{W}^*$ is the set of all joint laws of (X_T, A_T), T is a stopping time, or (X_∞, A_∞), where A is a predictable increasing process in the Doob–Meyer decomposition of X, $X_0 = 0$, where X runs over any of the following classes:

- all nonnegative local submartingales;
- all locally integrable increasing processes;
- all quasi-left continuous locally integrable increasing processes;
- all processes of the form $|M|^p$, $p \geqslant 1$ is fixed, where M is a local martingale with $M_0 = 0$ and $M \in \mathcal{H}^p_{\mathrm{loc}}$;
- the quadratic variations $[M, M]$ of locally square-integrable martingales M with $M_0 = 0$ (in this case the compensator of $[M, M]$ is the quadratic characteristic $\langle M, M \rangle$).

Theorem

Let μ be a probability measure on $\{y = x + 1,\ x \geqslant 0\}$ and $\int x\,\mu(dx, dy) = \infty$. Then there exists a stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geqslant 0}, \mathsf{P})$, and a locally integrable increasing process $X = (X_t)_{t\geqslant 0}$, $X_0 = 0$ defined on it, such that $\mathrm{Law}(X_\infty, A_\infty) = \mu$, where $A = (A_t)_{t\geqslant 0}$, $A_0 = 0$, is the compensator of X.

To prove Theorem 2, it is sufficient to construct, on the same stochastic basis, a nonnegative martingale $M = (M_n)_{n=0,1,2,\dots}$, $M_0 = 1$, and a r.v. Y with distribution $\mathsf{P}(Y \in B) = \mu(B \times \mathbb{R}_+)$ satisfying the inequality $S := \sum_{k=1}^{\infty} M_k \leqslant Y$. If we define an increasing process $X_t := \sum_{1 \leqslant k \leqslant t} M_k$, then its compensator has the form $A_t = 1 + \sum_{1 \leqslant k \leqslant t-1} M_k$, $t \geqslant 1$. Thus, $A_\infty = 1 + \sum_{k=1}^{\infty} M_k = 1 + X_\infty = S + 1$. It is easy to check that the increasing process, $X_{t/(1-t)} 1_{\{t<1\}} + Y 1_{\{t \geqslant 1\}}$, is as required. In turn, for the desired construction, it is sufficient to prove a weaker statement (see the last lemma below).

Lemma 1

Lemma

Let $p > 0$, $c > 0$, *and a nonnegative r.v.* Z *with* $\mathsf{E}Z = \infty$ *be given. Then there exist* $d \geqslant c$ *and a r.v.* L *taking values in* $\{0\} \cup [c, d]$ *such that* $\mathsf{P}(L > 0) \leqslant p$, $\mathsf{E}L = 1$, *and*

$$\inf_{x \leqslant d} \big[\mathsf{P}(Z > x) - \mathsf{P}(L > x)\big] > 0. \tag{14}$$

Proof. Denote $G(x) := \frac{1}{2}\mathrm{P}(Z > x)$. Since $\mathrm{E}Z = \infty$, we have $\int_0^\infty [\delta \wedge G(x)]\,dx = \infty$ for every $\delta := p \wedge c^{-1} \wedge G(c)$. Then there exists $d \geqslant c$ such that $\int_0^d [\delta \wedge G(x)]\,dx = 1$. A r.v. $L \geqslant 0$ with $\mathrm{P}(L > x) = [\delta \wedge G(x)]1_{\{x<d\}}$, $x \geqslant 0$, (which clearly exists) satisfies all the requirements; in particular,
$\mathrm{P}(Z > x) - \mathrm{P}(L > x) \geqslant G(x) \geqslant G(d) > 0$ for $x \leqslant d$. This proves the lemma.

Lemma 2

Lemma

Let a r.v. L_i*,* $i = 1, 2$*, take values in* $\{0\} \cup [c_{i-1}, c_i]$*, where* $0 < c_0 \leqslant c_1 \leqslant c_2$*, and* $\mathrm{E}L_1 = \mathrm{E}L_2$*. Then* L_2 *is greater than* L_1 *in convex order, i.e.* $\mathrm{E}f(L_1) \leqslant \mathrm{E}f(L_2)$ *for any convex function* f*.*

Proof. The statement is evident for a convex function f satisfying $f(0) = f(c_1) = 0$. The general case follows from this by subtracting a linear function from f.

Lemma

Let Y be a nonnegative r.v. with $\mathsf{E}Y = \infty$. Then there exists a nonnegative martingale $M = (M_n)_{n=0,1,2,\dots}$, $M_0 = 1$, such that

$$\mathsf{P}(S > x) \leqslant \mathsf{P}(Y > x) \quad \textit{for any } x \geqslant 0, \textit{ where} \quad S := \sum_{n=1}^{\infty} M_n. \tag{15}$$

Proof. We proceed by induction. Suppose that for some $n \in \{0, 1, 2, \dots\}$ we have constructed numbers $1 = c_0 \leqslant c_1 \leqslant \dots \leqslant c_n$ and r.v.'s $M_1, \dots, M_n$ with the following properties: M_k, $k = 1, \dots, n$, takes values in $\{0\} \cup [c_{k-1}, c_k]$, the sequence $1, M_1, \dots, M_n$ is a martingale, and

$$\varepsilon_n := \inf_{x \leqslant C_n} \big[\mathsf{P}(Y > x) - \mathsf{P}(S_n > x)\big] > 0, \qquad (16)$$

where $S_n := \sum_{k=1}^n M_k$ and $C_n := \sum_{k=1}^n c_k$. We need to construct a number c_{n+1}, and a r.v. M_{n+1}, such that the aforementioned properties hold when n is replaced by $n + 1$.

To achieve this, we apply Lemma 1 with $(Y - C_n)^+$ as Z, a positive $p < \mathsf{P}(M_n > 0) \wedge \varepsilon_n$, and $c = c_n$. Then we obtain a r.v. L with mean 1, with $\mathsf{P}(L > 0) \leqslant p$, which takes values in $\{0\} \cup [c_n, c_{n+1}]$, where $c_{n+1} := d$, and which satisfies (14). Denote by q_n the value of the expression on the left in (14). By Lemma 2, L is greater than M_n in convex order. It is well known that, under this condition, we can construct (on the same space or on its extension) a r.v. M_{n+1} such that $\mathrm{Law}(M_{n+1}) = \mathrm{Law}(L)$ and $\mathsf{E}(M_{n+1}|M_1, \dots, M_n) = M_n$. The martingale property implies that $\{M_k = 0\} \subseteq \{M_{k+1} = 0\}$ a.s., $k = 1, \dots, n$. Therefore, $\{M_k > 0\} \subseteq \{S_k \in [1 + C_{k-1}, C_k]\}$ a.s. Since the intervals $[1 + C_{k-1}, C_k]$ do not intersect, the last inclusion is equality a.s.

To check (16) with n replaced by $n+1$, we consider 3 cases for the location of x. If $x < 1 + C_{n-1}$, then $\{S_n \leqslant x\} \subseteq \{M_n = 0\} \subseteq \{S_{n+1} = S_n\}$ a.s. And hence $\mathsf{P}(S_n > x) = \mathsf{P}(S_{n+1} > x)$. For $x \in [1 + C_{n-1}, C_n]$, we have that $\mathsf{P}(S_{n+1} > x) \leqslant \mathsf{P}(S_n > x) + \mathsf{P}(M_{n+1} > 0)$ and hence that $\mathsf{P}(Y > x) - \mathsf{P}(S_{n+1} > x) \geqslant \varepsilon_n - \mathsf{P}(M_{n+1} > 0) > 0$. Finally, since $S_n \leqslant C_n$, we have that $\mathsf{P}(S_{n+1} > x) \leqslant \mathsf{P}(L > x - C_n) \leqslant \mathsf{P}(Y > x) - q_n$ for $x \in (C_n, C_{n+1}]$. The induction step has been proved. Since $\mathsf{P}(S > x) = \lim_n \mathsf{P}(S_n > x)$, this proves the lemma.

Thank you for the attention!