Nonlinear pricing of American options in an incomplete market with default

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Innovative Research in Mathematical Finance

In honour of Yuri Kabanov's 65th birthday

Luminy, 4 September 2018

Important features of the market

- (Possible) default on the underlying risky asset
- The market is non-linear : the dynamics of the wealth process are non-linear (the driver is non-linear). The non-linearity of the driver can encode different lending and borrowing rates, repo rates, impact of a large investor ...
- The market is incomplete : not every contingent claim is replicable.

Goal

Study the pricing of an American option whose pay-off process is **not** necessarily **right-continuous**.

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- dual characterization in terms of a non-linear mixed problem of control and stopping
- characterization of in terms of the minimal supersolution a constrained reflected BSDE.

$\longrightarrow$ passes through establishing a non-linear optional decomposition.

Non-linear incomplete market with default

- Let $T > 0$ be a fixed terminal horizon.
- Let $(\Omega, \mathcal{G}, P)$ be a complete probability space.
- Let W be a one-dimensional Brownian motion.
- ϑ is a random variable which models a default time.
- We assume $P(\vartheta \geq t) > 0$ for all $t \geq 0$.
- Let N be the process defined by $N_t := \mathbf{1}_{\vartheta \leq t}$ for all $t \in [0, T]$,
- Let $\mathbb{G} = \{\mathcal{G}_t, t \geq 0\}$ be the (augmented) filtration generated by W and N.
- We assume that W is a $\mathbb{G}$-Brownian motion.

- Let (Λ_t) be the predictable compensator of the nondecreasing process $(N_t) = (\mathbf{1}_{\vartheta \leq t})$.
- We assume that $\Lambda_t = \int_0^t \lambda_s ds$, $t \geq 0$, where $\lambda_s \geq 0$ is the intensity process.
- To simplify the presentation, we assume that λ is bounded.
- Let M be the compensated martingale given by

$$M_t := N_t - \Lambda_t = N_t - \int_0^t \lambda_s ds\,.$$

We consider a market with :

- a risky asset $S = (S_t)_{0 \leq t \leq T}$

$$dS_t = S_{t^-}(\mu_t dt + \sigma_t dW_t + \beta_t dM_t) \text{ with } S_0 > 0.$$

The processes σ, μ, and β are predictable bounded with $\sigma_t > 0$ and $\beta_\vartheta > -1$.

- An investor, endowed with an initial wealth $x \in \mathbb{R}$.
- At each time t, the investor chooses the amount φ_t of wealth invested in the risky asset (where $\varphi \in \mathbb{H}^2$).
- The value of the associated portfolio (also called *wealth*) at time t is denoted by $V_t^{x,\varphi}$ (or simply V_t).
- The wealth process $V_t^{x,\varphi}$ (or simply V_t) satisfies the following dynamics :

$$-dV_t = f(t, V_t, \varphi_t\sigma_t)dt - \varphi_t\sigma_t dW_t - \varphi_t\beta_t dM_t,$$

with $V_0 = x$, where f is a *nonlinear* (non-convex) Lipschitz driver.

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The model includes many examples :

- f is a **linear** driver given by $f(t, y, \phi_t\sigma_t) = -r_t y - (\mu_t - r_t)\varphi_t = -r_t y - (\mu_t - r_t)\varphi_t$, where r_t is a risk-free interest rate.
- **different borrowing and lending interest rates** R_t and r_t such that $R_t \geq r_t$:
  $f(t, y, \varphi_t\sigma_t) = -r_t x - \varphi_t(\mu_t - r_t) + (R_t - r_t)(y - \varphi_t)^-$.
  (cf. Korn , El Karoui and Quenez (1997))
- **a repo market** on which the risky asset is traded
  $f(t, y, \varphi_t\sigma_t) = -r_t y - \varphi(\mu_t - r_t) - l_t\varphi_t^- + b_t\varphi_t^+$,
  where b_t (resp. l_t) the borrowing (resp. lending) repo rate.
  (cf. Brigo)
- **large seller**, ...

Remark : This nonlinear market is **incomplete**.
Indeed, let $\eta \in L^2(G_T)$ be the terminal pay-off of a European option. Then, it might not be possible to find (x, φ) in $\mathbb{R} \times \mathbb{H}^2$ such that $V_T^{x,\varphi} = \eta$.

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In other words, the BSDE

$$-dV_t = f(t, V_t, Z_t)dt - Z_t dW_t - Z_t \sigma_t^{-1} \beta_t dM_t; \quad V_T = \eta,$$

might not be well-defined.
(Here, we have set as usual $Z_t := \varphi_t \sigma_t$.)

Seller's and buyer's superhedging price

- $\mathcal{T}$ is the set of $\mathbb{G}$-stopping times with values in $[0, T]$
- We consider an **American** option with maturity T and irregular pay-off process $\xi \in \mathbb{S}^2$
- $\mathbb{S}^2$ is the space of optional (not necessarily right-continuous) processes X such that $E[\text{ess sup}_{\tau \in \mathcal{T}} X_\tau^2] < +\infty$.

Example :

The pay-off is of the form $\xi_t = h(S_t)$, where h is a Borel function such that $(h(S_t))$ is in $\mathbb{S}^2$.

- American **digital call** option (with strike $K > 0$), where $h(x) := \mathbf{1}_{[K,+\infty)}(x)$.
- American **digital put** option, where $h(x) := \mathbf{1}_{(-\infty,K)}(x)$.

Definition (seller's superhedging price at time 0)

$$u_0 := \inf\{x \in \mathbb{R} : \exists \varphi \in \boldsymbol{H}^2 \text{ with } V_\tau^{x,\varphi} \geq \xi_\tau, \forall \tau \in \mathcal{T}\}.$$

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Definition (buyer's superhedging price at time 0)

$$\tilde{u}_0 := \sup\{z \in \mathbb{R} : \exists (\tau, \varphi) \in \mathcal{T} \times \boldsymbol{H}^2 \text{ with } V_\tau^{-z,\varphi} + \xi_\tau \geq 0 \text{ a.s. }\}.$$

References : Karatzas and Kou (1998), ...

No arbitrage considerations

We will see that :
The interval $[\tilde{u}_0, u_0]$ is a **no-arbitrage interval of prices** for the American option.

Definition (arbitrage opportunity for the seller)

Let $x \in \mathbb{R}$ be the initial price of the American option.
We say that $(y, \varphi) \in \mathbb{R} \times \mathbb{H}^2$ is an *arbitrage opportunity for the seller* of the American option with initial price x if

$$y < x \quad \text{and} \quad V_\tau^{y,\varphi} - \xi_\tau \geq 0 \quad \text{a.s.} \quad \text{for all } \tau \in \mathcal{T}.$$

Definition (arbitrage opportunity for the buyer)

Let $x \in \mathbb{R}$ be the initial price of the American option.
We say that (y, τ, φ) is an arbitrage opportunity for the buyer of the American option with initial price x, if

$$y > x \quad \text{and} \quad V_\tau^{-y,\varphi} + \xi_\tau \geq 0 \quad \text{a.s.}$$

Definition (arbitrage-free price)

A real number $x \in \mathbb{R}$ is called an *arbitrage-free price for the American option* if, neither the seller nor the buyer have arbitrage opportunity.

Proposition

The set of all arbitrage-free prices for the American option is given by $[\tilde{u}_0, u_0]$.

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Remark :

- It is possible that $u_0 < \tilde{u}_0$, and hence, that there does not exist an arbitrage-free price for the American option.

Dual characterization of the seller's price

Let $\mathcal{V}$ be the set of bounded predictable processes ν such that $\nu_t > -1$, for all $t \in [0, T]$, $\lambda_t dP \otimes dt$-a.e.

Driver f^ν

For $\nu \in \mathcal{V}$, we define

$$f^\nu(\omega, t, y, z, k) := f(\omega, t, y, z) + \nu_t(\omega)\lambda_t(\omega)\big(k - \beta_t(\omega)\sigma_t^{-1}(\omega)z\big).$$

The mapping f^ν is an example of a λ-*admissible* driver (cf. Dumitrescu, M.G., Quenez, Sulem (2018)).

For such a driver and the corresponding BSDE (with default), we have : existence and uniqueness of the solution ; representation result in the linear case ; comparison theorem, strict comparison theorem,

$\mathcal{E}^{f^\nu}$-expectation

Let $T' \in [0,T]$. Let $t \leq T'$.
We define :

$$\mathcal{E}^{f^\nu}_{t,T'} : L^2(\Omega, \mathcal{F}_{T'}, P) \longrightarrow L^2(\Omega, \mathcal{F}_t, P)$$
$$\eta \mapsto X^\nu_t,$$

where (X^ν, Z^ν, K^ν) is the unique solution of the BSDE

$$-dX^\nu_s = f^\nu(s, X^\nu_s, Z^\nu_s, K^\nu_s) - Z^\nu_s dW_s - K^\nu_s dM_s;$$
$$X^\nu_{T'} = \eta.$$

With this notation, $X^\nu_t = \mathcal{E}^{f^\nu}_{t,T'}(\eta)$.

Theorem (Dual characterization of the seller's price)

$$u_0 = \sup_{(\tau,\nu)\in\mathcal{T}\times\mathcal{V}} \mathcal{E}^{f^\nu}_{0,\tau}(\xi_\tau).$$

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1. The particular linear case (f is linear)
   $\longrightarrow$ gain some intuition about the duality result
2. Sketch of the proof
   $\longrightarrow$ a non-linear optional decomposition and a non-linear predictable decomposition

We consider the particular linear incomplete case, that is, the case where

$$f(t, y, z) = -r_t y - \theta_t z$$

Here, $\theta_t := \dfrac{\mu_t - r_t}{\sigma_t}$ is the risk premium at time t.
In this linear case the duality result for the seller's superhedging price u_0 for the American option with pay-off $\xi \in \mathbb{S}^2$ reduces to

(duality in the linear case)

$$u_0 = \sup_{(\tau,\nu)\in\mathcal{T}\times\mathcal{V}} \mathcal{E}^{f^\nu}_{0,\tau}(\xi_\tau) = \sup_{(\tau,\nu)\in\mathcal{T}\times\mathcal{V}} E_{R^\nu}(e^{-\int_0^\tau r_s ds}\xi_\tau),$$

where $\{R^\nu, \nu \in \mathcal{V}\}$ is the set of equivalent martingale measures.

Indeed, we notice the following :

Proposition (martingale measure)

The following assertions are equivalent :

(i) R is a martingale measure.

(ii) There exists $\nu \in \mathcal{V}$ such that $R = R^{\nu}$,
where R^{ν} is the probability measure with density process ζ^{ν} such that

$$d\zeta_t^{\nu} = \zeta_{t^-}^{\nu}[\alpha_t^{\nu} dW_t + \nu_t dM_t]; \zeta_0^{\nu} = 1.$$

Here, we have set $\alpha_t^{\nu} = (-\theta_t - \nu_t \lambda_t \beta_t \sigma_t^{-1})$.

Uses : $\mathbb{G}$-martingale representation (Kusuoka (1999)).

The general non-linear incomplete case : Main steps of the proof

We want to show :

Theorem (Dual characterization of the seller's price)

$$u_0 = \sup_{(\tau,\nu)\in\mathcal{T}\times\mathcal{V}} \mathcal{E}^{f^\nu}_{0,\tau}(\xi_\tau).$$

We consider the following non-linear problem of control and stopping :

$$Y(S) := ess \sup_{(\tau,\nu)\in\mathcal{T}_S\times\mathcal{V}} \mathcal{E}^{\nu}_{S,\tau}(\xi_\tau).$$

where

- $\mathcal{E}^\nu := \mathcal{E}^{f^\nu}$
- For S a stopping time in $\mathcal{T}_0$, we denote by $\mathcal{T}_S$ the set of stopping times τ such that $\tau \in [S, T]$ a.s.

Definition (strong $\mathscr{E}^{\nu}$-supermartingale)

Let $\nu \in \mathcal{V}$.
We say that a process $X \in \mathbb{S}^2$ is a strong $\mathscr{E}^{\nu}$-supermartingale if $\mathscr{E}^{\nu}_{S,\tau}(X_\tau) \leq X_S$, for all $S, \tau \in \mathcal{T}$ such that $S \leq \tau$ a.s.

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Theorem (Aggregation and Snell-type characterization)

- There exists an r.u.s.c. process $(Y_t) \in \mathbb{S}^2$ which aggregates the value family $(Y(S))$ of the problem of control and stopping.
- The process (Y_t) is **a strong $\mathscr{E}^{\nu}$-supermartingale for all $\nu \in \mathcal{V}$** and $Y_t \geq \xi_t$, for all $t \in [0, T]$, a.s.
- Moreover, the process (Y_t) is **the smallest** process in $\mathbb{S}^2$ satisfying these properties.

Remark : If the pay-off process ξ is right-continuous, then so is Y.

Strong $\mathscr{E}^{\nu}$-supermartingales for all $\nu \in \mathcal{V}$: optional and predictable decomposition

Theorem (Non-linear optional decomposition)

Let $(Y_t) \in \mathbb{S}^2$ be a $\mathcal{E}^{\nu}$-strong supermartingale for each $\nu \in \mathcal{V}$. Then, there exists a unique $Z \in \mathbb{H}^2$, a unique $C \in \mathbb{C}^2$ and a unique nondecreasing optional RCLL process h, with $h_0 = 0$ and $E[h_T^2] < \infty$ such that

$$-dY_t = f(t, Y_t, Z_t)dt - Z_t\sigma_t^{-1}(\sigma_t dW_t + \beta_t dM_t) + dC_{t^-} + dh_t.$$

$\mathbb{C}^2$ is the set of adapted non-decreasing RCLL purely discontinuous processes (X_t) such that $X_{0_-} = 0$ and $E[X_T^2] < +\infty$.

Remark

In the case where f is linear and X is right-continuous, the above $\mathscr{E}$-optional decomposition is reduced to the usual optional decomposition of an RCLL process, which is a (right-continuous) supermartingale under all martingale probability measures (up to a discounting and a change of probability measure procedure).

References : El Karoui and Quenez (1995), Kramkov (1996), Föllmer and Kabanov (1998).

Theorem (Predictable decomposition)

Let $(X_t) \in \mathbb{S}^2$ be a strong $\mathscr{E}^{\nu}$-supermartingale for all $\nu \in \mathcal{V}$. There exists a unique process $(Z, K, A, C) \in \mathbb{H}^2 \times \mathbb{H}^2_{\lambda} \times \mathcal{A}^2 \times \mathbb{C}^2$ such that

$$-dX_t = f(t, X_t, Z_t)dt - Z_t dW_t - K_t dM_t + dA_t + dC_{t-}$$

$$A_{\cdot} + \int_0^{\cdot} (K_s - \beta_s \sigma_s^{-1} Z_s)\lambda_s ds \in \mathcal{A}^2 \quad \text{and}$$

$$(K_t - \beta_t \sigma_t^{-1} Z_t)\lambda_t \leq 0, \; t \in [0, T], \; dP \otimes dt - \text{a.e.}$$

The set $\mathcal{A}^2$ is the set of

Characterization of the value process as the minimal supersolution of a constrained reflected BSDE

Definition

Let $\xi \in \mathbb{S}^2$. A process $Y' \in \mathbb{S}^2$ is said to be a *supersolution* of the *constrained reflected BSDE* with driver f and obstacle ξ if there exists a process $(Z', K', A', C') \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda \times \mathcal{A}^2 \times \mathbb{C}^2$ such that

$$
\begin{aligned}
& -dY'_t = f(t, Y'_t, Z'_t)dt + dA'_t + dC'_{t-} - Z'_t dW_t - K'_t dM_t; \\
& Y'_T = \xi_T \text{ a.s.} \quad \text{and} \quad Y'_t \geq \xi_t \text{ for all } t \in [0,T] \text{ a.s.}; \\
& (Y'_\tau - \xi_\tau)(C'_\tau - C'_{\tau-}) = 0 \text{ a.s. for all } \tau \in \mathcal{T}; \\
& A'_\cdot + \int_0^\cdot (K'_s - \beta_s \sigma_s^{-1} Z'_s)\lambda_s ds \in \mathcal{A}^2 \quad \text{and} \\
& (K'_t - \beta_t \sigma_t^{-1} Z'_t)\lambda_t \leq 0,\ t \in [0,T],\ dP \otimes dt\text{-a.e.};
\end{aligned}
$$

Proposition (Characterization of the seller's price process)

The seller's price process (Y_t) is a supersolution of the *constrained reflected BSDE* from the previous definition.
Moreover, it is the minimal one, that is, if (Y'_t) is another supersolution, then $Y'_t \geq Y_t$ for all $t \in [0, T]$ a.s.

BSDE with default

(Definition) λ-*admissible* driver

A function $g : [0,T] \times \Omega \times \mathbb{R}^3 \to \mathbb{R}; (\omega, t, y, z, k) \mapsto g(\omega, t, y, z, k)$ is said to be a λ-*admissible* driver if

- (measurability) g is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^3)-$ measurable
- (integrability) $g(.,0,0,0) \in \mathbb{H}^2$.
- there exists $C \geq 0$ such that $dP \otimes dt$-a.e. , for each (y_1, z_1, k_1), (y_2, z_2, k_2),

$$|g(t, y_1, z_1, k_1) - g(t, y_2, z_2, k_2)| \leq C(|y_1 - y_2| + |z_1 - z_2| + \sqrt{\lambda_t}|k_1 - k_2|).$$

Definition (BSDE with λ-admissible driver)

Let g be a λ-*admissible* driver. Let $\eta \in L^2(\mathcal{G}_T)$.

- A process (Y, Z, K) in $\mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\lambda$ is said to be a solution of the BSDE with default jump associated with (T, g, η) if it satisfies :

$$-dY_t = g(t, Y_t, Z_t, K_t)dt - Z_t dW_t - K_t dM_t; \quad Y_T = \eta.$$

comment upon the comparison with Poisson

BSDE theory in the case of a λ-admissible driver
(cf. ,)

- existence and uniqueness of the solution for (non-linear) BSDE with default jump.
- **representation** result in the case of a **linear** λ-admissible driver, that is, the case where $g(t, y, z, k) = a_t y + b_t z + c_t \lambda_t k + d_t$.
- comparison theorem for (non-linear) BSDEs with default under the additional assumption
  (A) There exists a predictable process (γ_t) with
  $$(\gamma_t \sqrt{\lambda_t}) \text{ bounded } \text{ and } \quad \gamma_t \geq -1, \quad dt \otimes dP - \text{ a.s.}$$
  such that
  $$g(t, y, z, k_1) - g(t, y, z, k_2) \geq \gamma_t (k_1 - k_2)\lambda_t, \; t \in [0, T], \; dt \otimes dP - \text{ a.e.}$$
- strict comparison theorem if the inequality is strict in assumption (A), that is if $\gamma_t > -1$.

- **Definition :** A process (X_t) in $\mathcal{S}^2$ is called a **strong supermartingale** if $X_S \geq E[X_\tau \mid \mathcal{F}_S]$ a.s., for all $S, \tau \in \mathcal{T}_0$ such that $S \leq \tau$ a.s.

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- A strong supermartingale (X_t) in $\mathcal{S}^2$ has the following **Mertens decomposition**

$$X_t = X_0 + M_t - A_t - C_{t-}, \ 0 \leq t \leq T \text{ a.s.},$$

where

  - M is a square-integrable martingale ($M_t = \int_0^t Z_s dW_s$, with $Z \in \mathbb{H}^2$)
  - A is a non-decreasing right-continuous predictable process such that $A_0 = 0$ and $E(A_T^2) < \infty$
  - C is a non-decreasing right-continuous optional process purely discontinuous with $C_{0-} = 0$ and $E[C_T^2] < \infty$.

A pair of processes (A, C) satisfying the above properties will be called a **Mertens process**.

We use also

- Gal'chouk-Lenglart formula for optional semimartingales (a generalization of Itô's formula)

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- Gal'chouk-Lenglart formula for optional semimartingales (a generalization of Itô's formula)
- some results from our previous work (the case of one barrier)

G.M., Imkeller P., Offen E., Ouknine Y. and M.-C. Quenez : Reflected BSDEs when the obstacle is not right-continuous and optimal stopping, (2017), Annals of Applied Probability.

G.M., Imkeller P., Ouknine Y. and M.-C. Quenez : Optimal stopping with f-expectations : the irregular case, (2017), submitted.

This talk :
G.M., Imkeller P., Ouknine Y. and M.-C. Quenez : Doubly Reflected BSDEs and $\mathscr{E}^f$-Dynkin games : beyond the right-continuous case, (2017), submitted.

Definition

Let $(\xi_t)_{t\in[0,T]}$ and $(\zeta_t)_{t\in[0,T]}$ be two (irregular) processes in $\mathcal{S}^2$ such that

- $\xi_t \leq \zeta_t$ for all t a.s.
- $\xi_T = \zeta_T$ a.s.

A pair of such processes will be called an **admissible** pair.

DRBSDE with irregular barriers

Some literature : Cvitanić and Karatzas ('96), Hamadène and Lepeltier ('00), Lepeltier and Xu ('07), Crépey and Matoussi ('08), ...

Definition

A process (Y, Z, A, C, A', C') is said to be a solution to the DRBSDE with parameters (g, ξ, ζ), where (ξ, ζ) is an admissible pair and g is a Lipschitz driver if

$$(Y, Z, A, C, A', C') \in \mathcal{S}^2 \times \boldsymbol{H}^2 \times (\mathcal{S}^2)^4$$

$$Y_t = \xi_T + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s + A_T - A_t + C_{T-} - C_{t-}$$

$$-(A'_T - A'_t) - (C'_{T-} - C'_{t-}) \text{ for all } t \in [0, T] \text{ a.s.}$$

$\zeta_t \geq Y_t \geq \xi_t$ for all $t \in [0, T]$ a.s.

(A, C) is a Mertens process, (A', C') is a Mertens process

(A, C) and (A', C') satisfy conditions of minimality ...

$dA_t \perp dA'_t$ and $dC_t \perp dC'_t$ (mutual singularity)

...Minimality conditions (Skorokhod conditions)

$$\int_0^T \mathbf{1}_{\{Y_{t-} > \xi_{t-}\}} dA_t = 0 \text{ a.s.}$$

$$(Y_t - \xi_t)(C_t - C_{t-}) = 0 \text{ for all } t \text{ a.s.}$$

$$\int_0^T \mathbf{1}_{\{Y_{t-} < \zeta_{t-}\}} dA'_t = 0 \text{ a.s.}$$

$$(Y_t - \zeta_t)(C'_t - C'_{t-}) = 0 \text{ for all } t \text{ a.s.}$$

Remark : We have assumed that ξ_{t-} and ζ_{t-} exist. This assumption can be relaxed.

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$$(Y, Z, A, C, A', C') \in \mathcal{S}^2 \times \boldsymbol{H}^2 \times (\mathcal{S}^2)^4$$

$$Y_t = \xi_T + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s + A_T - A_t + C_{T-} - C_{t-}$$

$$-(A'_T - A'_t) - (C'_{T-} - C'_{t-}) \text{ for all } t \in [0, T] \text{ a.s.}$$

$\zeta_t \geq Y_t \geq \xi_t$ for all $t \in [0, T]$ a.s.

(A, C) is a Mertens process, (A', C') is a Mertens process

(A, C) and (A', C') satisfy conditions of minimality ...

$dA_t \perp dA'_t$ and $dC_t \perp dC'_t$ (mutual singularity)

We introduce the following condition

Mokobodzki's condition

There exist two nonnegative strong supermartingales H and H' in $\mathcal{S}^2$ such that

$$\xi_t \leq H_t - H'_t \leq \zeta_t, \quad 0 \leq t \leq T \quad \text{a.s.}$$

Theorem (existence)

The following assertions are equivalent :

- The DRBSDE(g, ξ, ζ) has a solution.
- The pair (ξ, ζ) satisfies Mokobodzki's condition.

Theorem (uniqueness)

If (ξ, ζ) satisfies Mokobodzki's condition, the solution of DRBSDE(g, ξ, ζ) is unique.

Links with non-linear Dynkin games and non-linear extended Dynkin games

g-expectation and g-conditional expectation

Let g be a Lipschitz driver
Let $t \in [0, T]$. Let $s \leq t$.
The g-conditional expectation at time s is defined by

$$\mathcal{E}^g_{s,t} : L^2(\Omega, \mathcal{F}_t, P) \longrightarrow L^2(\Omega, \mathcal{F}_s, P)$$
$$\eta \mapsto X_s,$$

where the pair (X, π) is the unique solution (in $\mathcal{S}^2 \times \boldsymbol{H}^2$) of the BSDE

$$X_s = \eta + \int_s^t g(u, X_u, \pi_u) du - \int_s^t \pi_u dW_u, \; s \in [0, t].$$

- particular case : if $g \equiv 0$, then $\mathcal{E}^g_{s,t}(\cdot) = \mathbb{E}(\cdot | \mathcal{F}_s)$.

We consider the following non-linear Dynkin game (over stopping times)

- Two players A and B
- Each of the players chooses a strategy in $\mathcal{T}_0$.
- If agent A chooses a strategy $\tau \in \mathcal{T}_0$ and agent B chooses a strategy $\sigma \in \mathcal{T}_0$, the pay-off at time $\tau \wedge \sigma$ (when the game ends) is $I(\tau, \sigma)$, where
$$I(\tau, \sigma) := \xi_\tau \mathbf{1}_{\tau \leq \sigma} + \zeta_\sigma \mathbf{1}_{\sigma < \tau}.$$
- The pay-off is assessed by a (non-linear) g-expectation. At time 0, player A receives $\mathcal{E}^g_{0,\tau\wedge\sigma}[I(\tau, \sigma)]$ and player B receives $-\mathcal{E}^g_{0,\tau\wedge\sigma}[I(\tau, \sigma)]$

- The *upper value* $\overline{V}(0)$ and *the lower value* $\underline{V}(0)$ at time 0 are defined by

$$\overline{V}(0) := \inf_{\sigma \in \mathcal{T}_0} \sup_{\tau \in \mathcal{T}_0} \mathcal{E}^g_{0,\tau\wedge\sigma}[I(\tau,\sigma)]$$

$$\underline{V}(0) := \sup_{\tau \in \mathcal{T}_0} \inf_{\sigma \in \mathcal{T}_0} \mathcal{E}^g_{0,\tau\wedge\sigma}[I(\tau,\sigma)].$$

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$$\underline{V}(0) := \sup_{\tau\in\mathcal{T}_0} \inf_{\sigma\in\mathcal{T}_0} \mathcal{E}^g_{0,\tau\wedge\sigma}[I(\tau,\sigma)].$$

- More generally, the *upper value* $\overline{V}(S)$ and *the lower value* $\underline{V}(S)$ at time S (where $S\in\mathcal{T}_0$) are defined by

$$\overline{V}(S) := \text{ess inf}_{\sigma\in\mathcal{T}_S} \text{ess sup}_{\tau\in\mathcal{T}_S} \mathcal{E}^g_{S,\tau\wedge\sigma}[I(\tau,\sigma)]$$

$$\underline{V}(S) := \text{ess sup}_{\tau\in\mathcal{T}_S} \text{ess inf}_{\sigma\in\mathcal{T}_S} \mathcal{E}^g_{S,\tau\wedge\sigma}[I(\tau,\sigma)].$$

As usual, $\underline{V}(S) \leq \overline{V}(S)$ a.s.

Theorem (Existence and characterization of the value)

Let (ξ, ζ) be an admissible pair of processes satisfying Mokobodzki condition and such that ξ and $-\zeta$ are **right u.s.c.** Then, for all $S \in \mathcal{T}_0$, we have

$$Y_S = \overline{V}(S) = \underline{V}(S) \quad \text{a.s.}$$

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- In the **linear** case, that is, $g = 0$, the equality $\overline{V}(S) = \underline{V}(S)$ has been proven by Alario-Nazaret, Lepeltier and Marchal (1982).
- When ξ and $-\zeta$ are **right-continuous**, we recover well-known results
  - for the case $g = 0$: Cvitanić and Karatzas (1996), Hamadène and Lepeltier (2000), Lepeltier and Xu (2007), ...
  - for the case g non-linear : Dumitrescu, Quenez and Sulem (2016).

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- Application in **financial mathematics** : superhedging price of a game option

The case where ξ and $-\zeta$ are completely irregular : Non-linear **extended** Dynkin game

- Two players A and B
- Agent A chooses a strategy $\rho = (\tau, H)$ where $\tau \in \mathcal{T}_0$ and $H \in \mathcal{F}_\tau$.
  Agent B chooses a strategy $\delta = (\sigma, G)$, where $\sigma \in \mathcal{T}_0$ and $G \in \mathcal{F}_\sigma$.
- The game ends at time $\tau \wedge \sigma$
- The pay-off at time $\tau \wedge \sigma$ is

  $$I(\rho, \delta) := \xi^u_\rho \mathbf{1}_{\tau \leq \sigma} + \zeta^l_\delta \mathbf{1}_{\sigma < \tau},$$

  where

  $$\xi^u_\rho := \xi_\tau \mathbf{1}_H + \overline{\xi}_\tau \mathbf{1}_{H^c} \text{ and } \zeta^l_\delta := \zeta_\sigma \mathbf{1}_G + \underline{\zeta}_\sigma \mathbf{1}_{G^c},$$

  with $\overline{\xi}_t := \limsup_{s \downarrow t, s > t} \xi_s$ (right upper- semicontinuous envelope) and $\underline{\zeta}_t := \liminf_{s \downarrow t, s > t} \zeta_s$ (right lower- semicontinuous envelope)
- The pay-off is assessed by a (non-linear) g-expectation.

- Upper and Lower Value at time 0

$$\overline{V}(0) := \inf_{\delta=(\sigma,G)} \sup_{\rho=(\tau,H)} \mathcal{E}^{g}_{0,\tau\wedge\sigma}[I(\rho,\delta)]$$

$$\underline{V}(0) := \sup_{\rho=(\tau,H)} \inf_{\delta=(\sigma,G)} \mathcal{E}^{g}_{0,\tau\wedge\sigma}[I(\rho,\delta)].$$

- Upper and Lower Value at time S (where $S \in \mathcal{T}_0$) ...

Theorem (Existence and characterization of the value)

Let (ξ, ζ) be an admissible pair of completely irregular processes satisfying Mokobodzki condition. Then, for all $S \in \mathcal{T}_0$, we have

$$Y_S = \overline{V}(S) = \underline{V}(S) \quad \text{a.s.}$$

Theorem (Existence and characterization of the value)

Let (ξ, ζ) be an admissible pair of completely irregular processes satisfying Mokobodzki condition. Then, for all $S \in \mathcal{T}_0$, we have

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Theorem (Existence and characterization of the value)

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- This theorem is useful in showing **a comparison theorem** and ***a priori* estimates with universal constants** (i.e. depending only on the terminal time T and the common Lipschitz constant K) for DRBSDEs with completely irregular barriers.

Theorem (Gal'chouk-Lenglart)

Let X be a 1-dimensional optional semimartingale with decomposition $X = X_0 + M + A + B$, where M and A are as in the right-continuous case, and B is an adapted left-continuous process of finite variation which is purely discontinuous and such that $B_0 = 0$. Let $F \in C^2(\mathbb{R})$. Then, a.s, for all $t \geq 0$,

$$F(X_t) = F(X_0) + \int_{]0,t]} F'(X_{s-})d(A+M)_s + \frac{1}{2}\int_{]0,t]} F''(X_{s-})d\langle M^c\rangle_s + \sum_{0<s\leq t}\left[F(X_s) - F(X_{s-}) - F'(X_{s-})\Delta X_s\right] + \int_{[0,t[} F'(X_s)dB_{s+} + \sum_{0\leq s<t}\left[F(X_{s+}) - F(X_s) - F'(X_s)\Delta_+ X_s\right].$$

Notation : $\Delta X_s = X_s - X_{s-}$ and $\Delta_+ X_s = X_{s+} - X_s$.