Hedging under small transaction costs

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Dedicated to the 70th anniversary of Yuri Kabanov

Yuri in Japan since 2008

2008 Aug Tokyo
2008 Sep Kyoto
2009 Feb-Mar Osaka
2009 Aug Kyoto
2009 Sep Kyoto
2010 Aug Tokyo
2012 Feb Sapporo
2013 Feb Sapporo
2014 Feb Sapporo
2014 Mar Kyoto
2016 Feb Sapporo
2016 Apr-Jun Tokyo (Apr Osaka, Jun Hakodate)
2017 Feb Sapporo

Summary and Plan

This talk is

- Based on Cai and Fukasawa (F&S, 2016)
- An extended framework (reformulation)

Plan

- Revisit to the Leland-Lott strategy
- A class of regular and singular controls
- Homogenization
- A deterministic problem
- Asymptotically optimal strategy
- Fixed transaction costs
- An open problem

Leland-Lott strategy : enlarged volatility

Consider the Black-Scholes : $\mathrm{d}S_t = \sigma S_t \mathrm{d}B_t$.
European pricing PDE with enlarged volatility :

$$p(s, T) = f(s), \quad \frac{\partial p}{\partial t} + \frac{1}{2}\left(1 + \frac{2}{\alpha}\right)\sigma^2 s^2 \frac{\partial^2 p}{\partial s^2} = 0.$$

Itô's formula gives

$$f(S_T) = p(S_0, 0) + \int_0^T X_t \mathrm{d}S_t - \frac{1}{\alpha}\int_0^T \Gamma_t \mathrm{d}\langle S \rangle_t$$

with

$$X_t = \frac{\partial p}{\partial s}(S_t, t), \quad \Gamma_t = \frac{\partial^2 p}{\partial s^2}(S_t, t).$$

Use the third term to pay transaction costs ($\Gamma \geq 0$ if f is convex).

Leland-Lott strategy : regular discretization

A discrete hedging strategy :

$$X^h_t = X_{ih}, \quad t \in (ih, (i+1)h], \quad i = 0, 1, 2, \dots$$

P&L under proportional transaction costs :

$$\int_0^T X^h_t \mathrm{d}S_t - \kappa \sum_{0<t\leq T} S_t |\Delta X^h_t|$$

The trick : by choosing $h = \frac{2}{\pi}\frac{\kappa^2\alpha^2}{\sigma^2}$, as $\kappa \to 0$,

$$\int_0^T X^h_t \mathrm{d}S_t \to \int_0^T X_t \mathrm{d}S_t, \quad \kappa \sum_{0<t\leq T} S_t |\Delta X^h_t| \to \frac{1}{\alpha}\int_0^T \Gamma_t \mathrm{d}\langle S \rangle_t.$$

Central limit theorem

The hedging error of $O(\kappa)$ is

$$\frac{1}{\kappa}\left(\int_0^T X_t \mathrm{d}S_t - \int_0^T X_t^h \mathrm{d}S_t\right)$$

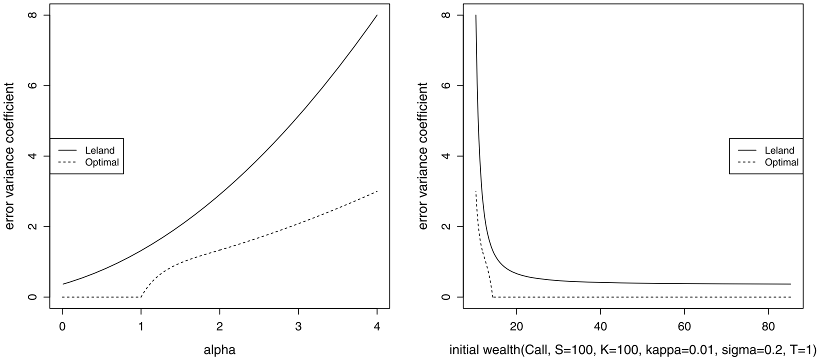
minus

$$\frac{1}{\kappa}\left(\frac{1}{\alpha}\int_0^T \Gamma_t \mathrm{d}\langle S\rangle_t - \kappa \sum_{0<t\leq T} S_t|\Delta X_t^h|\right).$$

Denis and Kabanov (F&S, 2010): it converges in $D[0,T]$ to a time-changed BM W_Q, where

$$Q = \eta_L(\alpha)\int_0^{\cdot} |\Gamma_t S_t|^2 \mathrm{d}\langle S\rangle_t, \quad \eta_L(\alpha) = \frac{1}{\pi}\alpha^2 + \frac{2}{\pi}\alpha + 1 - \frac{2}{\pi}.$$

$\eta_L(\alpha)$


error variance coefficient
0
2
4
6
8
0
1
2
3
4
alpha
Leland
Optimal
error variance coefficient
0
2
4
6
8
20
40
60
80
initial wealth(Call, S=100, K=100, kappa=0.01, sigma=0.2, T=1)
Leland
Optimal

References

Leland (1985), Henrotte (1991), Lott (1993), Toft (1996)
Grannan and Swindle (1996), Ahn et al (1998)
Barles and Soner (1998)

Kabanov and Safarian (1997):

On Leland's strategy of option pricing with transaction costs.

Gamys and Kabanov (2009), Denis and Kabanov (2010):

Mean square error for the Leland-Lott hedging strategy

Lepinette, Sekine and Yano, Nguyen and Pergamenshchikov
Fukasawa, Cai and Fukasawa

Abstraction

Let S be a continuous local martingale and $X = \Gamma \cdot S + \varphi \cdot \langle S \rangle$. The question is how to approximate

$$X \cdot S - \frac{1}{\alpha}\langle X, S \rangle$$

by

$$\hat{X} \cdot S - \kappa \Lambda \cdot \|\hat{X}\|,$$

where $\hat{X}$ is an adapted process of finite variation and $\|\hat{X}\|$ is its total variation process. Λ is a given adapted process (say, $\Lambda = S$).

- $\hat{X}$ is a trading strategy of finite proportional transaction costs.
- Here one can consider $\alpha \in [-2, 0)$ as well.
- $\alpha = -2$ corresponds to the Stratonovich integral.

A class of control strategies

We consider $\hat{X} = \hat{X}^\kappa$ of the form

$$\mathrm{d}\hat{X}_t = \frac{1}{\kappa}\mathrm{sgn}(Z_t)c(|Z_t|, F_t)\mathrm{d}\langle X\rangle_t - \kappa \mathrm{d}L_t + \kappa \mathrm{d}R_t,$$

where

- $Z = Z^\kappa = (X - \hat{X})/\kappa$, a scaled deviation.
- $c(z, f)$ is a nonnegative piecewise $C^{0,2}$ function with

$$(z - z')(-\mathrm{sgn}(z)c(|z|, f) + \mathrm{sgn}(z')c(|z'|, f)) \leq L|z - z'|^2.$$

- F is a multi-dimensional continuous semimartingale.
- L and R are adapted, non-decreasing with

$$\mathrm{d}L_t = 1_{\{Z_t = -G_t\}}\mathrm{d}L_t, \quad \mathrm{d}R_t = 1_{\{Z_t = G_t\}}\mathrm{d}R_t, \quad |Z_t| \leq G_t.$$

- G is a positive continuous semimartingale.

The existence of the strategy

For a given continuous semimartingale (X, F, G), $G > 0$, there exists unique solution $(\hat{Z}, \hat{L}, \hat{R})$ of the Skorokhod equation

$$\begin{aligned} \mathrm{d}\hat{Z} = & \frac{\mathrm{d}X}{\kappa G} + \frac{1}{\kappa}\mathrm{d}\langle X, G\rangle + \mathrm{d}\hat{L} - \mathrm{d}\hat{R} \\ & - \frac{1}{\kappa^2 G}\mathrm{sgn}(\hat{Z})c(|\hat{Z}|G, F)\mathrm{d}\langle X\rangle + \hat{Z}G\mathrm{d}\frac{1}{G} \end{aligned}$$

with

$$\mathrm{d}\hat{L} = 1_{\{\hat{Z}=-1\}}\mathrm{d}\hat{L}, \quad \mathrm{d}\hat{R} = 1_{\{\hat{Z}=1\}}\mathrm{d}\hat{R}, \quad |\hat{Z}| \leq 1.$$

(a fixed point argument using the one-sided Lipschitz continuity)
The strategy $\hat{X}$ is then well-defined by

$$\mathrm{d}L = G\mathrm{d}\hat{L}, \quad \mathrm{d}R = G\mathrm{d}\hat{R}, \quad \hat{X} = X - \kappa G\hat{Z}.$$

Tracking error

$$\mathcal{E}^\kappa := (X-\hat{X})\cdot S + \kappa\Lambda\cdot\|\hat{X}\| - \frac{1}{\alpha}\langle X,S\rangle$$
$$= \kappa Z\cdot S + (\Lambda c(Z,F))\cdot\langle X\rangle + \kappa^2\Lambda\cdot(L+R) - \frac{1}{\alpha}\langle X,S\rangle.$$

Theorem : Assume

- $1_{\{\Gamma=0\}}\cdot\langle S\rangle = 0$ a.s.,
- Λ and Γ are continuous semimartingales.

Then,

$$Y^\kappa := \frac{1}{\kappa}\left(\mathcal{E}^\kappa - \frac{\Lambda}{\xi(F,G)}\cdot\langle X\rangle + \frac{1}{\alpha}\langle X,S\rangle\right) \to W_Q$$

stably in law on $C[0,T]$, where W is an independent BM,

$$Q = \eta(F,G,\Lambda\Gamma)\cdot\langle S\rangle, \quad \text{and...}$$

$$\xi(f,g) = 2\int_0^g m(x,f)\mathrm{d}x,$$

$$m(x,f) = \exp\left(-2\int_0^{|x|} c(z,f)\mathrm{d}z\right),$$

$$\eta(f,g,\lambda) = \frac{2}{\xi(f,g)}\int_0^g (x - \lambda h(x,f,g))^2 m(x,f)\mathrm{d}x,$$

$$h(x,f,g) = \frac{2\mathrm{sgn}(x)}{m(x,f)}\int_0^{|x|}\left(c(z,f) - \frac{1}{\xi(f,g)}\right) m(z,f)\mathrm{d}z$$

Remark : $m(\cdot,f)$ is the speed measure density, $[-g,g]$ is the state space, $\xi(f,g)$ is the total mass of the speed measure, and h solves $h(-g,f,g) = 1$, $h(g,f,g) = -1$ with

$$-\mathrm{sgn}(x)c(x,f)h(x,f,g) + \frac{1}{2}\frac{\partial h}{\partial x}(x,f,g) = c(x,f) - \frac{1}{\xi(f,g)}.$$

Proof

- Jacod's theorem of stable convergence : a sequence of continuous semimartingales Y^κ converges to W_Q stably in law on $C[0,T]$ if $\langle Y^\kappa, S\rangle \to 0$, $\langle Y^\kappa \rangle \to Q$ and Q is continuous.
- Averaging lemma : if

$$\int_{-g}^{g} \psi(x, f, g, \lambda, \gamma) m(x, f) \mathrm{d}x = 0$$

for each (f, g, λ, γ), then

$$\sup_{\tau \in [0,T]} \left| \int_0^\tau \psi(Z_t^\kappa, F_t, G_t, \Lambda_t, \Gamma_t) \mathrm{d}\langle X \rangle_t \right| \to 0$$

as $\kappa \to 0$.

Homogenization

Recall $Z = Z^{\kappa} = (X - \hat{X})/\kappa$ and

$$\mathrm{d}Y^{\kappa} = Z^{\kappa}\mathrm{d}S + \frac{\Lambda}{\kappa}\left(\kappa^2\mathrm{d}(L+R) + \left(c(Z,F) - \frac{1}{\xi(F,G)}\right)\mathrm{d}\langle X\rangle\right),$$

$$\mathrm{d}Z^{\kappa} = \frac{1}{\kappa}\mathrm{d}X - \frac{1}{\kappa^2}\mathrm{sgn}(Z^{\kappa})c(Z^{\kappa},F)\mathrm{d}\langle X\rangle + \mathrm{d}L - \mathrm{d}R.$$

The function $h(z,f,g)$ was so chosen that

$$\kappa^2(L+R) + \left(c(Z,F) - \frac{1}{\xi(F,G)}\right)\cdot\langle X\rangle$$
$$= -\kappa h(Z^{\kappa},F,G)\cdot X + O_p(\kappa^2),$$

so that

$$Y^{\kappa} \approx (Z^{\kappa} - \Lambda\Gamma h(Z^{\kappa},F,G))\cdot S.$$

A deterministic problem

- Can choose $c(\cdot, F)$ and G.
- $\xi(F, G)$ determines the limit of the tracking error. Say, need $\xi(F, G) = \alpha\Lambda\Gamma$ for the asymptotic replication.
- Minimize the asymptotic variance $\eta(f, g, \lambda)$ with $\xi(f, g)$ fixed.
- Changing variables, the problem is to minimize

$$\eta(f, g, \lambda) = \int_0^1 \left(y(u) + \lambda + \frac{2\lambda}{\xi(f, g)}(u - 1)y'(u) \right)^2 \mathrm{d}u$$

in the set $\mathcal{Y}$ of the increasing convex functions y on $[0, 1]$ with

$$y(0) = 0, \quad y'(0) = \frac{\xi(f, g)}{2}.$$

Explicit solution

Theorem :

$$\inf_{\xi(f,g)=\xi} \eta(f,g,\lambda) = \lim_{x\to\lambda} x^2 \eta_\dagger(\xi/x) = \begin{cases} \xi^2/12 & \text{if } \lambda = 0, \\ \lambda^2 \eta_\dagger(\xi/\lambda) & \text{if } \lambda \neq 0, \end{cases}$$

where

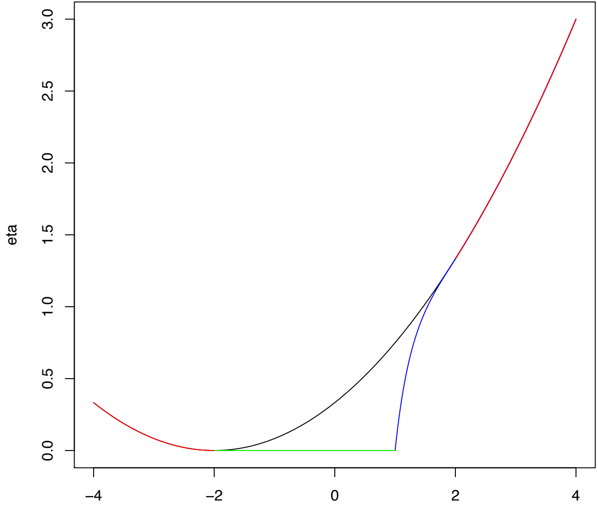
$$\eta_\dagger(x) = \begin{cases} 0 & \text{if } -2 < x \leq 1, \\ \eta_1(x) & \text{if } 1 < x < 2, \\ \eta_2(x) & \text{if } |x| \geq 2 \end{cases}$$

and

$$\eta_1(x) = \frac{4}{3} \frac{(x+2)^2(x-1)}{x^3(4-x)}, \quad \eta_2(x) = \frac{(x+2)^2}{12}.$$

Remark: For the asymptotic replication, let $\xi = \alpha\lambda$.

$\alpha \mapsto \eta_\dagger(\alpha)$



Asymptotically optimal strategy

Suppose $\Lambda\Gamma > 0$ and let $\xi(F, G) = \alpha\Lambda\Gamma$ (asymptotic replication)

- $0 < \alpha \leq 1$:

$$\mathrm{d}\hat{X} = \mathrm{sgn}(X - \hat{X})\frac{\alpha + 2}{2\alpha}\frac{1}{\kappa\Lambda\Gamma + |X - \hat{X}|}\mathrm{d}\langle X\rangle.$$

- $1 < \alpha < 2$:

$$\mathrm{d}\hat{X} = \mathrm{sgn}(X - \hat{X})\frac{4 - \alpha}{2(2 - \alpha)}\frac{1}{\kappa\Lambda\Gamma + |X - \hat{X}|}1_A\mathrm{d}\langle X\rangle,$$

  where

$$A = \left\{|X - \hat{X}| \geq 2\kappa\frac{\alpha - 1}{4 - \alpha}\Lambda\Gamma\right\}.$$

- $2 \leq \alpha$: singular control

$$|X - \hat{X}| \leq \kappa\frac{\alpha}{2}\Lambda\Gamma.$$

An open problem

After the homogenization, the system becomes

$$\mathrm{d}Y_t = Z_t \mathrm{d}W_t + c(Z_t)\gamma^2 \mathrm{d}t + \mathrm{d}L_t + \mathrm{d}R_t - \frac{\gamma}{\alpha}\mathrm{d}t,$$
$$\mathrm{d}Z_t = \gamma \mathrm{d}W_t - \mathrm{sgn}(Z_t)c(Z_t)\gamma^2 \mathrm{d}t + \mathrm{d}L_t - \mathrm{d}R_t$$

for which we get

$$\inf \lim_{T\to\infty} \frac{1}{T} E[Y_T^2] = \gamma^2 \eta_\dagger(\alpha).$$

2-dimensional degenerate singular control problem

$$\mathrm{d}Y_t = Z_t \mathrm{d}W_t + \mathrm{d}A_t - \frac{\gamma}{\alpha}\mathrm{d}t,$$
$$\mathrm{d}Z_t = \gamma \mathrm{d}W_t - \mathrm{sgn}(Z_t)\mathrm{d}A_t$$

to minimize $E[Y_T^2]$ or $E[Y_T^+]$? Explicit strategy when $T \to \infty$?