Lebesgue's Convergence Theorem and Fatou's Lemma for Varying Propabilities

Eugene A. Feinberg

Stony Brook University

Innovative Research in Mathematical Finance
Conference in Honor of Yuri Kabanov

Marseille, France
September 7, 2018

This talk is based on joint papers with Pavlo Kasyanov and Yan Liang.

Plan of the Talk

1. Classic facts
2. Fatou's lemma for varying probabilities including uniform Fatou's lemma
3. Lebesgue's convergence theorem for varying probabilities
4. Applications

Fatou's Lemma: Classic Facts

1. Fatou's lemma
2. Lebesgue's dominated convergence theorem
3. Monotone convergence theorem

Fatou's Lemma

For a probability measure p and a sequence of measurable functions, the inequality

$$\liminf_{n\to\infty} \int f_n(x)p(dx) \geq \int \liminf_{n\to\infty} f_n(x)p(dx)$$

holds, if one of the following conditions is satisfied:

(i) *each function f_n is nonnegative;*

(ii) *there exists a measurable function g such that $g \leq f_n$ and $\int g(x)p(dx) > -\infty$;*

(iii) *the sequence $\{f_n^-\}$, where $f_n^- := -\min\{f_n, 0\}$, is uniformly integrable, that is,*

$$\lim_{K\to+\infty} \inf_{n=1,2,\dots} \int_{\mathbb{X}} f_n(x)\mathbf{I}\{x \in \mathbb{X} : f_n(x) \leq -K\}p(dx) \geq 0.$$

Stronger results hold when the sequence of functions satisfies some additional conditions. Lebesgue's dominated convergence theorem and monotone convergence theorem are corollaries from Fatou's lemma.

Extensions of Fatou's lemma

A. Fatou's lemma for converging probabilities (weak convergence, setwise convergence, convergence in total variation)

B. Uniform Fatou's lemma

C. Lower bounds on possibly negative functions

   (i) existence of a sequence of minorant functions

   (ii) uniform integrability with respect to a sequence of probability measures

D. Convergence and continuity properties of functions

   (i) lower semi-equicontinuity

   (ii) semi-convergence in probability

Convergence of Probabilities

We are interested in the situation when there is a sequence of converging probabilities on a metric space $\mathbb{X}$. Recall the definition of the following three types of convergence:

(i) weak convergence (convergence in probability)

$p_n \to p$ *if for every bounded continuous function* f

$$\int f(x)p_n(dx) \to \int f(x)p(dx) \tag{1}$$

(ii) setwise convergence

$p_n \to p$ *if* $p_n(E) \to p(E)$ *for each measurable subset* E*. Equivalently,* (1) *holds for every bounded measurable function* f*.*

(iii) convergence in total variation

$p_n \to p$ *if* $\rho_{TV}(p, p_n) \to 0$*, where* $\rho_{TV}(p, p_n) = 2\sup\{|p(B) - p_n(B)| : B \in \mathcal{B}(\mathbb{X})\}$.

For discrete random variables, these three types of convergence coincide.

Weak convergence is the most natural and general form of convergence of probabilities, $\text{(iii)} \Rightarrow \text{(ii)} \Rightarrow \text{(i)}$.

Fatou's Lemma for Varying Probabilities

Theorem (Serfozo 1981)

For probabilities converging weakly $p_n \to p$ and nonnegative measurable functions f_n,

$$\liminf_{n\to\infty} \int f_n(x) p_n(dx) \geq \int \liminf_{y\to x, n\to\infty} f_n(y) p(dx).$$

Theorem (Royden 1963)

For probabilities converging setwise $p_n \to p$ and nonnegative measurable functions f_n,

$$\liminf_{n\to\infty} \int f_n(x) p_n(dx) \geq \int \liminf_{n\to\infty} f_n(x) p(dx).$$

Uniform Fatou's Lemma

Let $(\mathbb{X}, \mathcal{F})$ be a measurable space. The inequality in Fatou's lemma can be rewritten as

$$\inf_{X \in \mathcal{F}} \Big\{ \liminf_{n\to\infty} \int_X f_n(x) p_n(dx) - \int_X [\liminf_{n\to\infty} f_n(x)] p(dx) \Big\} \geq 0. \qquad (2)$$

Uniform Fatou's lemma (F., Kasyanov, and Zgurovsky 2016) is a stronger inequality:

$$\liminf_{n\to\infty} \inf_{X \in \mathcal{F}} \Big\{ \int_X f_n(x) p_n(dx) - \int_X [\liminf_{n\to\infty} f_n(x)] p(dx) \Big\} \geq 0. \qquad (3)$$

This is a stronger inequality and examples support this.

The difference between is (2) and (3) is similar to the difference between convergence and uniform convergence.

Uniform Fatou's Lemma

Uniform Fatou's Lemma (F., Kasyanov, and Zgurovsky 2016)

Let $(\mathbb{X}, \mathcal{F})$ *be a measurable space, a sequence of probability measures* $\{p_n\}_{n=1,2,\dots}$ *converge in total variation to a probability measure* p *on* $\mathbb{X}$, $f \in L^1(\mathbb{X}; p)$, *and* $f_n \in L^1(\mathbb{X}; p_n)$ *for each* $n = 1, 2, \dots$. *Then the inequality*

$$\liminf_{n\to\infty} \inf_{X\in\mathcal{F}} \left(\int_X f_n(x) p_n(dx) - \int_X f(x) p(dx) \right) \geq 0$$

holds if and only if the following two statements hold:

(i) for each $\epsilon > 0$

$$p(\{x \in \mathbb{X} : f_n(x) \leq f(x) - \epsilon\}) \to 0 \text{ as } n \to \infty,$$

and, therefore, there exists a subsequence $\{f_{n_k}\}_{k=1,2,\dots} \subseteq \{f_n\}_{n=1,2,\dots}$ *such that*

$$\liminf_{k\to\infty} f_{n_k}(x) \geq f(x) \qquad \text{for } p\text{-a.s. } x \in \mathbb{X};$$

(ii) the following inequality holds:

$$\lim_{K\to+\infty} \inf_{n=1,2,\dots} \int_{\mathbb{X}} f_n(x) \mathbf{I}\{x \in \mathbb{X} : f_n(x) \leq -K\} p_n(dx) \geq 0.$$

Observation: As a special case, uniform Fatou's lemma holds for

$$f(x) := \liminf_{n\to\infty} f_n(x).$$

Uniform Lebesgue's Convergence Theorem

Uniform Lebesgue's Convergence Theorem (F., Kasyanov, and Zgurovsky 2016)

Let $(\mathbb{X}, \mathcal{F})$ *be a measurable space, a sequence of probability measures* $\{p_n\}_{n=1,2,\ldots}$ *converge in total variation to a probability measure* p *on* $\mathbb{X}$, $f \in L^1(\mathbb{X}; p)$, *and* $f_n \in L^1(\mathbb{X}; p_n)$ *for each* $n = 1, 2, \ldots$. *Then the equality*

$$\lim_{n\to\infty} \sup_{X\in\mathcal{F}} \left| \int_X f_n(x) p_n(dx) - \int_X f(x) p(dx) \right| = 0$$

holds if and only if the following two statements hold:

(i) the sequence of functions $\{f_n\}_{n=1,2,\ldots}$ *converges in probability* p *to* f; *and*

(ii)

$$\lim_{K\to+\infty} \sup_{n=1,2,\ldots} \int |f_n(x)| \mathbf{I}\{x \in \mathbb{X} : |f_n(x)| \geq K\} p_n(dx) = 0.$$

Uniform Integrability of a Sequence of Functions

Uniformly integrable (u.i.) with respect to (w.r.t.) $\{p_n\}$

$$\lim_{K\to+\infty}\sup_{n=1,2,\ldots}\int |f_n(x)|\mathbf{I}\{x\in\mathbb{X}:|f_n(x)|\ge K\}p_n(dx)=0.$$

Asymptotically uniformly integrable (a.u.i.) w.r.t. $\{p_n\}$

$$\lim_{K\to+\infty}\limsup_{n\to\infty}\int |f_n(x)|\mathbf{I}\{x\in\mathbb{X}:|f_n(x)|\ge K\}p_n(dx)=0.$$

Theorem

A sequence of functions $\{f_n\}_{n=1,2,\ldots}$ is a.u.i. w.r.t. $\{p_n\}_{n=1,2,\ldots}$ if and only if there exists $\tilde{N}=0,1,2,\ldots$ such that $\{f_{n+\tilde{N}}\}_{n=1,2,\ldots}$ is u.i. w.r.t. $\{p_{n+\tilde{N}}\}_{n=1,2,\ldots}$ (Kartashov 2008 for $p_n=p$).

Uniform Fatou's Lemma

Uniform Fatou's Lemma (F., Kasyanov, and Zgurovsky 2016)

Let $(\mathbb{X}, \mathcal{F})$ *be a measurable space, a sequence of probability measures* $\{p_n\}_{n=1,2,\ldots}$ *converge in total variation to a probability measure* p *on* $\mathbb{X}$, $f \in L^1(\mathbb{X}; p)$, *and* $f_n \in L^1(\mathbb{X}; p_n)$ *for each* $n = 1, 2, \ldots$ *. Then the inequality*

$$\liminf_{n\to\infty} \inf_{X\in\mathcal{F}} \Big(\int_X f_n(x) p_n(dx) - \int_X f(x) p(dx) \Big) \geq 0$$

holds if and only if the following two statements hold:

(i) for each $\epsilon > 0$

$$p(\{x \in \mathbb{X} : f_n(x) \leq f(x) - \epsilon\}) \to 0 \text{ as } n \to \infty,$$

and, therefore, there exists a subsequence $\{f_{n_k}\}_{k=1,2,\ldots} \subseteq \{f_n\}_{n=1,2,\ldots}$ *such that*

$$\liminf_{k\to\infty} f_{n_k}(x) \geq f(x) \qquad \text{for } p\text{-a.s. } x \in \mathbb{X};$$

(ii) the following inequality holds:

$$\lim_{K\to+\infty} \liminf_{n\to\infty} \int_{\mathbb{X}} f_n(x) \mathbf{I}\{x \in \mathbb{X} : f_n(x) \leq -K\} p_n(dx) \geq 0.$$

Uniform Lebesgue's Convergence Theorem

Uniform Lebesgue's Convergence Theorem (F., Kasyanov, and Zgurovsky 2016)

Let $(\mathbb{X}, \mathcal{F})$ *be a measurable space, a sequence of probability measures* $\{p_n\}_{n=1,2,\ldots}$ *converge in total variation to a probability measure* p *on* $\mathbb{X}$, $f \in L^1(\mathbb{X}; p)$, *and* $f_n \in L^1(\mathbb{X}; p_n)$ *for each* $n = 1, 2, \ldots$. *Then the equality*

$$\lim_{n\to\infty} \sup_{X\in\mathcal{F}} \left| \int_X f_n(x)p_n(dx) - \int_X f(x)p(dx) \right| = 0$$

holds if and only if the following two statements hold:

(i) the sequence of functions $\{f_n\}_{n=1,2,\ldots}$ *converges in probability* p *to* f; *and*

(ii)

$$\lim_{K\to+\infty} \limsup_{n\to\infty} \int |f_n(x)| \mathbf{I}\{x \in \mathbb{X} : |f_n(x)| \geq K\} p_n(dx) = 0.$$

Dunford-Pettis Theorem

Dunford-Pettis Theorem

Let $(\mathbb{X}, \mathcal{F})$ *be a measurable space,* p *be a probability measure on* $\mathbb{X}$*, and* $\{f_n\}_{n=1,2,\dots} \subset L^1(\mathbb{X}; p)$ *be a sequence of measurable functions on* $\mathbb{X}$*.*
Then the following statements are equivalent:

(i) *there exists* $\{f_{n_k}\}_{k=1,2,\dots} \subset \{f_n\}_{n=1,2,\dots}$ *such that* $f_{n_k} \to f$ *weakly in* $L^1(\mathbb{X}; p)$ *for some* $f \in L^1(\mathbb{X}; p)$*;*

(ii) *there exists* $\tilde{N} = 1, 2, \dots$ *such that* $\{f_{n+\tilde{N}}\}_{n=1,2,\dots}$ *is u.i.;*

(iii) $\{f_n\}_{n=1,2,\dots}$ *is a.u.i.*

In view of Eberlein-Šmulian theorem, statements (i) and (ii) are equivalent due to the Dunford-Pettis theorem.

Minorant Condition for a Sequence of Functions

Consider the following condition (depending on how p_n converges to p):

Minorant condition (when $p_n \to p$ weakly)

Let there exists a sequence of measurable functions $\{g_n\}$ such that $g_n \leq f_n$ and

$$-\infty < \int \limsup_{n\to\infty, y\to x} g_n(y)p(dx) \leq \liminf_{n\to\infty} \int g_n(x)p_n(dx).$$

Minorant condition (when $p_n \to p$ setwise)

Let there exists a sequence of measurable functions $\{g_n\}$ such that $g_n \leq f_n$ and

$$-\infty < \int \limsup_{n\to\infty} g_n(x)p(dx) \leq \liminf_{n\to\infty} \int g_n(x)p_n(dx).$$

When $p_n = p$ and $g_n = g$ for some measurable function $g,$ the minorant condition in the case when $p_n \to p$ setwise becomes the standard condition for classic Fatou's lemma that there exists a measurable function g such that $g \leq f_n$ and $\int g(x)p(dx) > -\infty.$

Fatou's Lemma for Varying Probabilities

Theorem

For probabilities converging weakly $p_n \to p$ and a sequence of measurable functions $\{f_n\}$, if one of the following conditions holds:

(i) *each function f_n is nonnegative (Serfozo 1981 and F., Kasyanov, Zadoianchuk 2014);*

(ii) *$\{f_n\}$ satisfies the minorant condition (F., Kasyanov, Zadoianchuk 2014);*

(iii) *$\{f_n^-\}$ is a.u.i. w.r.t. $\{p_n\}$;*

then

$$\liminf_{n\to\infty} \int f_n(x) p_n(dx) \geq \int \liminf_{y\to x, n\to\infty} f_n(y) p(dx).$$

Theorem

For probabilities converging setwise $p_n \to p$ and a sequence of measurable functions $\{f_n\}$, if one of the following conditions holds:

(i) *each function f_n is nonnegative (Royden 1963);*

(ii) *$\{f_n\}$ satisfies the minorant condition (F., Kasyanov, Zadoianchuk 2014);*

(iii) *$\{f_n^-\}$ is a.u.i. w.r.t. $\{p_n\}$;*

then

$$\liminf_{n\to\infty} \int f_n(x) p_n(dx) \geq \int \liminf_{n\to\infty} f_n(x) p(dx).$$

Fatou's Lemma in the Standard Form for Weakly Converging Probabilities

Lower semi-equicontinuity

A family $\mathcal{H}$ of real-valued functions on a metric space $\mathbb{X}$ is called lower semi-equicontinuous at the point x if for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$h(y) > h(x) - \varepsilon \qquad \text{for all } y \in B_\delta(x) \text{ and for all } h \in \mathcal{H},$$

where $B_\delta(x)$ is the open ball of radius δ centered at x.

If the family of functions $\{f_n\}_{n=1,2,\dots}$ is lower semi-equicontinuous at x, then

$$\liminf_{n\to\infty, y\to x} f_n(y) = \liminf_{n\to\infty} f_n(x),$$

Equicontinuity

A family $\mathcal{H}$ of real-valued functions on a metric space $\mathbb{X}$ is called equicontinuous at the point x if for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|h(y) - h(x)| < \varepsilon \qquad \text{for all } y \in B_\delta(x) \text{ and for all } h \in \mathcal{H},$$

where $B_\delta(x)$ is the open ball of radius δ centered at x.

Fatou's Lemma in the Standard Form for Weakly Converging Probabilities

Theorem

For probabilities converging weakly $p_n \to p$ and a lower semi-equicontinuous family of measurable functions $\{f_n\}_{n=1,2,\ldots}$, if one of the following conditions holds:

(i) *each function f_n is nonnegative;*

(ii) *$\{f_n\}$ satisfies the minorant condition;*

(iii) *$\{f_n^-\}$ is a.u.i. w.r.t. $\{p_n\}$;*

then

$$\liminf_{n\to\infty} \int f_n(x) p_n(dx) \geq \int \liminf_{n\to\infty} f_n(x) p(dx) = \int \liminf_{y\to x, n\to\infty} f_n(y) p(dx).$$

Semi-Convergence in Probability

Lower semi-convergence in probability

A sequence $\{f_n\}_{n=1,2,\dots}$ lower semi-converges in probability p to f if for each $\varepsilon > 0$

$$p(\{x \in \mathbb{X} : f_n(x) \le f(x) - \varepsilon\}) \to 0 \text{ as } n \to \infty.$$

In particular, for

$$f^*(x) := \liminf_{n\to\infty} f_n(x), \qquad x \in \mathbb{X},$$

$\{f_n\}_{n=1,2,\dots}$ lower semi-converges to f^* in every probability measure p on $\mathbb{X}$.

If $\{f_n\}_{n=1,2,\dots}$ lower semi-converges in probability p to some function f, then there exists a subsequence $\{f_{n_k}\}_{k=1,2,\dots}$ of the sequence $\{f_n\}_{n=1,2,\dots}$ such that $f(x) \le \liminf_{k\to\infty} f_{n_k}(x)$ for p-a.s. $x \in \mathbb{X}$ (F., Kasyanov, and Zgurovsky 2016).

A sequence $\{f_n\}_{n=1,2,\dots}$ upper semi-converges in probability p to f if $\{-f_n\}_{n=1,2,\dots}$ lower semi-converges in probability p to $-f$.

A sequence $\{f_n\}_{n=1,2,\dots}$ converges in probability p to f if and only if $\{f_n\}_{n=1,2,\dots}$ lower and upper semi-converges in probability p to f.

Fatou's Lemma for Varying Probabilities

Theorem

For probabilities converging setwise $p_n \to p$ and measurable functions $\{f_n\}$ lower semi-converging in probability p to a measurable function f, if one of the following conditions holds:

(i) *each function f_n is nonnegative;* (ii) *$\{f_n\}$ satisfies the minorant condition;*
(iii) *$\{f_n^-\}$ is a.u.i. w.r.t. $\{p_n\}$;*

then

$$\liminf_{n\to\infty} \int f_n(x)p_n(dx) \geq \int f(x)p(dx).$$

Theorem

For probabilities converging weakly $p_n \to p$ and a lower semi-equicontinuous family of measurable functions $\{f_n\}$, that lower semi-converges in probability p to a measurable function f, if one of conditions (i), (ii), *and* (iii) *from the theorem above holds, then*

$$\liminf_{n\to\infty} \int f_n(x)p_n(dx) \geq \int f(x)p(dx).$$

In all the cases, we can take $f(x) := \liminf_{n\to\infty} f_n(x)$, but there are examples when

$$f(x) > \liminf_{n\to\infty} f_n(x) \quad \text{for all } x \in \mathbb{X}.$$

Lebesgue's Convergence Theorem for Varying Probabilities

Theorem

For probabilities converging setwise $p_n \to p$ *and measurable functions* $\{f_n\}$ *converging in probability* p *to a measurable function* f, *if one of the following conditions holds:*

(i) $\{f_n\}$ *is a.u.i. w.r.t.* $\{p_n\}$;

(ii) *both* $\{f_n\}$ *and* $\{-f_n\}$ *satisfy the minorant condition;*

then

$$\lim_{n\to\infty} \int f_n(x)p_n(dx) = \int f(x)p(dx).$$

Lebesgue's Convergence Theorem for Varying Probabilities

Theorem

For probabilities converging weakly $p_n \to p$ and measurable functions $\{f_n\}$ such that $\lim_{n\to\infty, y\to x} f_n(y)$ exists for p-a.s. $x \in \mathbb{X}$, if one of the following conditions holds:

(i) $\{f_n\}$ *is a.u.i. w.r.t.* $\{p_n\}$;

(ii) *both* $\{f_n\}$ *and* $\{-f_n\}$ *satisfy the minorant condition;*

then

$$\lim_{n\to\infty} \int f_n(x) p_n(dx) = \int \lim_{n\to\infty} f_n(x) p(dx).$$

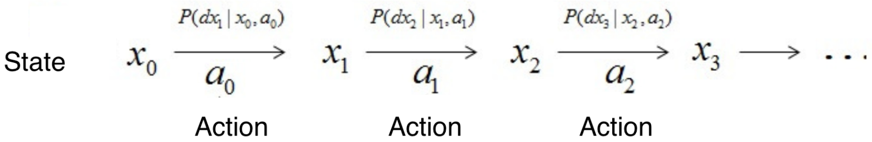
Theorem

For probabilities converging weakly $p_n \to p$ and an equicontinuous family of functions $\{f_n\}$, that converges in probability p, if one of conditions (i) *and* (ii) *from the theorem above holds, then*

$$\lim_{n\to\infty} \int f_n(x) p_n(dx) = \int \lim_{n\to\infty} f_n(x) p(dx).$$

Applications: Optimality Inequality and Equations for Average-Cost Markov Decision Processes

MDP diagram



Applications: Optimality Inequality and Equations for Average-Cost Markov Decision Processes

Let us consider a Markov decision process (MDP) defined by $\{\mathbb{X}, \mathbb{A}, P, c\}$, where

(i) $\mathbb{X}$ is a state space;

(ii) $\mathbb{A}$ is an action space;

(iii) $P(dy|x,a)$ is the transition probability;

(iv) $c(x,a)$ is the one-step cost function.

The objective is to minimize

(i) finite-horizon costs: $v^{\pi}_{N,\alpha}(x) := \mathbb{E}^{\pi}_{x}\left[\sum_{t=0}^{N-1} \alpha^t c(x_t,a_t)\right]$, where $N = 1, 2, \ldots$ is the horizon length and $\alpha \in [0,1)$ is the discount factor;

(ii) infinite-horizon costs: $v^{\pi}_{\alpha}(x) := \mathbb{E}^{\pi}_{x}\left[\sum_{t=0}^{\infty} \alpha^t c(x_t,a_t)\right]$; or

(iii) average costs per unit time: $w^{\pi}(x) := \limsup_{N\to\infty} \frac{1}{N}\mathbb{E}^{\pi}_{x}\left[\sum_{t=0}^{N-1} c(x_t,a_t)\right]$.

For each function $V^{\pi}(x) = v^{\pi}_{N,\alpha}(x)$, $v^{\pi}_{\alpha}(x)$, or $w^{\pi}(x)$, define the optimal value

$$V(x) := \inf_{\pi\in\Pi} V^{\pi}(x),$$

where Π is the set of all policies.

Applications: Optimality Inequality and Equations for Average-Cost Markov Decision Processes

Continuity assumptions on transition probabilities and costs:

Assumption W*:

1. The transition probability P is weakly continuous.
2. The cost function c is $\mathbb{K}$-inf-compact and bounded below.

Definition

A function $f : \mathbb{X} \times \mathbb{A} \to \mathbf{R}$ is called $\mathbb{K}$-inf-compact, if for every compact set $K \subseteq \mathbb{X}$, the function $f : K \times \mathbb{A} \to \mathbf{R}$ is inf-compact.

Suppose Assumption **W*** holds. Then discounted optimality equation holds (F., Kasyanov, and Zadoianchuk 2012):

$$v_\alpha(x) = \min_a \left\{ c(x,a) + \alpha \int v_\alpha(y) P(dy|x,a) \right\}.$$

Vanishing discount factor approach for average-cost MDPs:
Let $m_\alpha := \inf_x v_\alpha(x)$ and $u_\alpha(x) := v_\alpha(x) - m_\alpha$. Then

$$u_\alpha(x) + (1-\alpha)m_\alpha = \min_a \left\{ c(x,a) + \alpha \int u_\alpha(y) P(dy|x,a) \right\}.$$

Optimality Inequality for Average-Cost Markov Decision Processes

The optimality equation for discounted-cost MDP implies

$$(1-\alpha)m_\alpha + u_\alpha(x) = \min_a \left\{ c(x,a) + \alpha \int u_\alpha(y) P(dy|x,a) \right\}.$$

For a sequence of discount factors $\{\alpha_n \uparrow 1\}$ and for every sequence $(x_n, a_n) \to (x,a)$, Fatou's lemma for weakly converging probabilities implies

$$\liminf_{n\to\infty} \int u_{\alpha_n}(y) P(dy|x_n,a_n) \geq \int \liminf_{n\to\infty, y\to x} u_{\alpha_n}(y) P(dy|x,a).$$

Assumption B. The following conditions hold:
(i) $w^* := \inf_{x\in\mathbb{X}} w(x) < +\infty$;
(ii) $\sup_{\alpha<1} u_\alpha(x) < \infty$ for all $x \in \mathbb{X}$.

For $\alpha \in [0,1)$ consider $\underline{w} = \liminf_{\alpha\uparrow 1} (1-\alpha)m_\alpha$ and $\overline{w} = \limsup_{\alpha\uparrow 1} (1-\alpha)m_\alpha$.

Theorem (F., Kasyanov, and Zadoianchuk 2012)

Suppose Assumptions **W*** *and* **B** *hold. Consider a sequence* $\{\alpha_n \uparrow 1\}_{n=1,2,\dots}$ *of nonnegative discount factors. Then there exist a measurable function*

$$u(x) := \liminf_{n\to\infty, y\to x} u_{\alpha_n}(y), \qquad x \in \mathbb{X},$$

and a stationary policy ϕ *such that ACOI holds:*

$$\overline{w} + u(x) \geq c(x,\phi(x)) + \int u(y) P(dy|x,\phi(x)), \quad x \in \mathbb{X},$$

where ϕ *is average-cost optimal, and*

$$w(x) = w^\phi(x) = \lim_{\alpha\uparrow 1}(1-\alpha)v_\alpha(x) = \underline{w} = \overline{w}, \quad x \in \mathbb{X}.$$

Optimality Equations for Average-Cost Markov Decision Processes

Recall that
$$(1-\alpha)m_\alpha + u_\alpha(x) = \min_a \left\{ c(x,a) + \alpha \int u_\alpha(y) P(dy|x,a) \right\}.$$

For a sequence of discount factors $\{\alpha_n \uparrow 1\}$ and for every sequence $(x_n, a_n) \to (x,a)$, if Lebesgue's convergence theorem for weakly converging probabilities holds:

$$\lim_{n\to\infty} \int u_{\alpha_n}(y) P(dy|x_n,a_n) = \int \lim_{n\to\infty} u_{\alpha_n}(y) P(dy|x,a),$$

then average cost optimality equations hold:

$$w + u(x) = c(x,\phi(x)) + \int_{\mathbb{X}} u(y) P(dy|x,\phi(x)) = \min_{a\in\mathbb{A}} [c(x,a) + \int_{\mathbb{X}} u(y) P(dy|x,a)]. \qquad (4)$$

Theorem

Suppose Assumptions **W*** *and* **B** *hold. Consider a sequence $\{\alpha_n \uparrow 1\}_{n=1,2,\dots}$ of nonnegative discount factors. If the sequence $\{u_{\alpha_n}\}_{n=1,2,\dots}$ satisfies:*

- (i) *the cost function $c(x,a)$ is continuous in x for each a;*
- (ii) *the family of functions $\{u_{\alpha_n}\}_{n=1,2,\dots}$ is equicontinuous;*
- (iii) *$\{u_{\alpha_n}\}_{n=1,2,\dots}$ is a.u.i. w.r.t. any sequence of probabilities defined by $\{P(\cdot|x_n,a_n)\}$ such that $(x_n,a_n) \to (x,a)$.*

then there exist a stationary policy ϕ and a subsequence $\{\alpha_{n_k}\}_{k=1,2,\dots}$ of $\{\alpha_n\}_{n=1,2,\dots}$ such that the ACOEs (4) hold with the function $u(x)$ satisfying

$$u(x) := \lim_{k\to\infty, y\to x} u_{\alpha_{n_k}}(y) = \lim_{k\to\infty} u_{\alpha_{n_k}}(x).$$

Applications to Inventory Control

We consider an inventory control problem defined by the following parameters:

(i) $K > 0$ is a fixed ordering cost;

(ii) $\bar{c} > 0$ is the per unit ordering cost;

(iii) $\{D_t, t = 1, 2, \dots\}$ is a sequence of i.i.d. nonnegative finite random variables representing the demand at periods $0, 1, \dots;$

(iv) $h(x)$ is the holding/backlog cost per period, which is assumed to be a convex real-valued function on $\mathbf{R}$ with $h(x) \to \infty$ as $|x| \to \infty$.

The dynamics of the system are defined as

$$x_{t+1} = x_t + a_t - D_{t+1}, \qquad t = 0, 1, \dots,$$

where

(i) x_t is the amount of inventory at the end of epoch t,

(ii) a_t is the amount of inventory ordered at the end of epoch t, where $a_t \geq 0$,

(iii) D_t is the demand during the epoch t; $D_t \geq 0$ and $\{D_t\}$ are i.i.d.

The one-step costs are

$$c(x_t, a_t) = K1_{\{a_t \neq 0\}} + \bar{c}a_t + \mathbb{E}h(x_t + a_t - D_{t+1}).$$

Assumptions **W*** and **B** hold for the MDP corresponding to this problem. The cost function $c(x, a)$ is continuous in x for each a.

Applications to Inventory Control

For a sequence of discount factors $\{\alpha_n \uparrow 1\}_{n=1,2,\ldots}$, the sequence of functions $\{u_{\alpha_n}\}_{n=1,2,\ldots}$ is equicontinuous and a.u.i. w.r.t. any sequence of probabilities defined by $\{P(\cdot|x_n, a_n)\}$ such that $(x_n, a_n) \to (x, a)$.

For the average-cost problem, there exist a sequence of discount factor $\{\alpha_n\}_{n=1,2,\ldots}$ and a stationary policy φ such that ACOEs hold:

$$w + u(x) = K1_{\{\varphi(x)>0\}} + H(x + \varphi(x)) - \bar{c}x = \min\{\min_{a\geq 0}[K + H(x + a)], H(x)\} - \bar{c}x,$$

where

$$u(x) = \lim_{n\to\infty, y\to x} u_{\alpha_n}(y) = \lim_{n\to\infty} u_{\alpha_n}(x) \tag{5}$$

and

$$H(x) := \bar{c}x + \mathbb{E}h(x - D) + \mathbb{E}u(x - D).$$

Definition of K-convexity

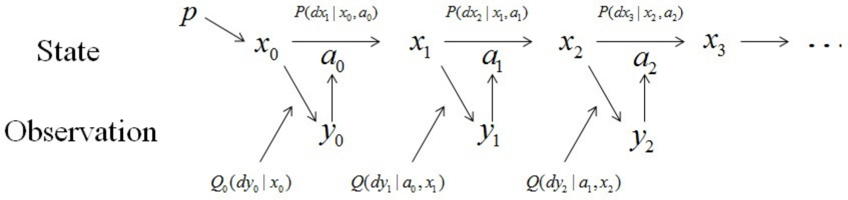
A function $G : \mathbb{X} \to \mathbf{R}$ is called K-convex, where $K \geq 0$, if for $x \leq y$ and for $\lambda \in (0, 1)$,

$$G((1 - \lambda)x + \lambda y) \leq (1 - \lambda)G(x) + \lambda G(y) + \lambda K.$$

Furthermore, the function u_α is K-convex. Formula (5) implies that the function u is also K-convex because the limit preserves K-convexity. This is useful for computing optimal policies.

Application of the Uniform Fatou Lemma to Partially Observable Markov Decision Process (POMDP)

POMDP diagram



Application of the Uniform Fatou Lemma to Partially Observable Markov Decision Process

A *partially observable Markov decision process (POMDP)* is specified by $(\mathbb{X}, \mathbb{Y}, \mathbb{A}, P, Q, c)$, where

(i) $\mathbb{X}$ is the *state space*,

(ii) $\mathbb{Y}$ is the *observation set*,

(iii) $\mathbb{A}$ is the *action set*,

(iv) $P(dx'|x, a)$ is the *transition probability*,

(v) $Q(dy|a, x)$ is the *observation kernel*,

(vi) $c : \mathbb{X} \times \mathbb{A} \to \mathbf{R}$ is the *one-step cost*.

In addition:

$P_0(dx'|x, a)$ is the *initial state distribution on* $\mathbb{X}$,

$Q_0(dy|a, x)$ is the *distribution of the initial kernel*.

Reduction of POMDPs to COMDPs

In principle, it is known how to solve a POMDP.

In order to do this, the original states should be replaced by belief states which are probability distributions on the state space $\mathbb{X}$.

Let $\mathbb{P}(E)$ be the set of probability distributions on a set E. Then a COMDP is an MDP with the state space $\mathbb{P}(\mathbb{X})$. For example, if $\mathbb{X} = \{0, 1\}$, then $\mathbb{P}(\mathbb{X}) = [0, 1]$.

In general, if $\mathbb{X}$ is a Polish (complete, separable, metric) space then $\mathbb{P}(\mathbb{X})$ is also a Polish space if the topology of weak convergence on $\mathbb{P}(\mathbb{X})$ is considered. Once an optimal policy for the COMDP is found, it is easy to find an optimal policy for the POMDP.

Evolution of Posterior Probabilities

Given a posterior distribution $z_t \in \mathbb{P}(\mathbb{X})$ of the state $x_t \in \mathbb{X}$ and an action $a_t \in \mathbb{A}$ at time epoch $t = 0, 1, \ldots,$ the joint probability that the state $x_{t+1} \in B \in \mathcal{B}(\mathbb{X})$ and the observation $y_{t+1} \in C \in \mathcal{B}(\mathbb{Y})$ is

$$R(B \times C | z_t, a_t) := \int_{\mathbb{X}} \int_{B} Q(C|a_t, x_{t+1}) P(dx_{t+1}|x_t, a_t) z_t(dx_t)$$

and the probability that the observation $y_{t+1} \in C \in \mathcal{B}(\mathbb{Y})$ is

$$R'(C|z_t, a_t) = \int_{\mathbb{X}} \int_{\mathbb{X}} Q(C|a_t, x_{t+1}) P(dx_{t+1}|x_t, a_t) z_t(dx_t),$$

where $\mathcal{B}(\mathbb{X})$ and $\mathcal{B}(\mathbb{Y})$ are the sets of probabilities on $\mathbb{X}$ and $\mathbb{Y}$ respectively.

Then there exist transition probabilities H from $\mathbb{P}(\mathbb{X}) \times \mathbb{A} \times \mathbb{Y}$ to $\mathbb{X}$ such that

$$R(B \times C|z_t, a_t) = \int_C H(B|z_t, a_t, y_{t+1}) R'(dy_{t+1}|z_t, a_t),$$

where $B \in \mathcal{B}(\mathbb{X})$, $C \in \mathcal{B}(\mathbb{Y})$, $z_t \in \mathbb{P}(\mathbb{X})$ and $a_t \in \mathbb{A}$.

Evolution of Posterior Probabilities

Bayes' formula:

For a posterior distribution $z_t \in \mathbb{P}(\mathbb{X})$, action $a_t \in \mathbb{A}$, and an observation $y_{t+1} \in \mathbb{Y}$, the posterior distribution $z_{t+1} \in \mathbb{P}(\mathbb{X})$ is

$$z_{t+1} = H(z_t, a_t, y_{t+1}).$$

The observation y_{t+1} is not available in the COMDP model, and therefore y_{t+1} is a random variable with the distribution $R'(\cdot|z_t, a_t)$, and H maps $(z_t, a_t) \in \mathbb{P}(\mathbb{X}) \times \mathbb{A}$ to $\mathbb{P}(\mathbb{P}(\mathbb{X}))$. Thus, z_{t+1} is a random variable with values in $\mathbb{P}(\mathbb{X})$ whose distribution is defined uniquely by the transition probability

$$q(D|z,a) := \int_{\mathbb{Y}} 1_{\{H(z,a,y)\in D\}} R'(dy|z,a),$$

where $D \in \mathcal{B}(\mathbb{P}(\mathbb{X}))$, $z \in \mathbb{P}(\mathbb{X})$, $a \in \mathbb{A}$.

Existence of Optimal Policies for COMDPs

Theorem (F., Kasyanov, and Zgurovsky 2016)

Suppose the following conditions hold for the POMDP:

(i) *the cost function c is $\mathbb{K}$-inf-compact;*

(ii) *the transition probability $P(\cdot|x,a)$ is weakly continuous in (x,a);*

(iii) *the observation probability $Q(\cdot|a,x)$ is continuous in total variation in (a,x).*

Then, Assumption **W*** *holds for the COMDP. Furthermore, optimal policies exist and convergence of value iteration takes place for discounted cost problems.*

Setwise continuity of $Q(dy|a,x)$ is not sufficient for the existence of optimal policies. Of course, weak continuity of Q is not sufficient either (Huizhen (Janey) Yu provided an example in 2012).

The uniform Lebesgue's convergence theorem for uniformly bounded functions was understood to prove this theorem.

References

MDPs/POMDPs

E. A. Feinberg, P. O. Kasyanov, and N. V. Zadoianchuk. Average cost Markov decision processes with weakly continuous transition probabilities. *Mathematics of Operations Research*, 37(4), 591–607, 2012.

E. A. Feinberg, P. O. Kasyanov, and M. Z. Zgurovsky. Convergence of probability measures and Markov decision models with incomplete information. *Proceedings of the Steklov Institute of Mathematics*, 287(1), 96–117, 2014.

E. A. Feinberg, P. O. Kasyanov, and M. Z. Zgurovsky. Partially observable total-cost Markov decision processes with weakly continuous transition probabilities. *Mathematics of Operations Research*, 41(2), 656–681, 2016.

E. A. Feinberg and M. E. Lewis. On the convergence of optimal actions for Markov decision processes and the optimality of (s, S) inventory policies. *Naval Research Logistic*, DOI:10.1002/nav.21750, 2017.

E. A. Feinberg and Y. Liang. On the optimality equation for average cost Markov decision processes and its validity for inventory control. *Annals of Operations Research*, DOI: 10.1007/s10479-017-2561-9, 2017.

Inventory Control

E. A. Feinberg. Optimality conditions for inventory control. *Tutorials in Operations Research*, 14–44, INFORMS, 2016.

E.A. Feinberg and Y. Liang. Structure of optimal solutions to periodic-review total-cost stochastic inventory control problems. *ACM SIGMETRICS Performance Evaluation Review*, 44(2), 21–23, 2016.

E. A. Feinberg and Y. Liang. Structure of optimal solutions to periodic-review total-cost inventory control problems. *Annals of Operations Research*, DOI: 10.1007/s10479-017-2548-6, 2017.

E. A. Feinberg and Y. Liang. Stochastic setup-cost inventory model with backorders and quasiconvex cost functions. Preprint arXiv:1705.06814, http://arxiv.org/pdf/1705.06814.pdf, 2017.

References

Berge's Maximum Theorem

E. A. Feinberg, P. O. Kasyanov, and N. V. Zadoianchuk. Berge's theorem for noncompact image sets. *Journal of Mathematical Analysis and Applications*, 37(1), 255–259, 2013.

E. A. Feinberg, P. O. Kasyanov, and M. Voorneveld. Berge's maximum theorem for noncompact image sets. *Journal of Mathematical Analysis and Applications*, 413(2), 1040–1046, 2014.

E. A. Feinberg and P. O. Kasyanov. Continuity of minima: Local results. *Set-Valued and Variational Analysis*, 23(3), 485–499, 2015.

Fatou's Lemma

E. A. Feinberg, P. O. Kasyanov, and N. V. Zadoianchuk. Fatou's lemma for weakly converging probabilities. *Theory of Probability and Its Applications*, 58(4), 683–689, 2014.

E. A. Feinberg, P. O. Kasyanov, and M. Z. Zgurovsky. Uniform Fatou's lemma. *Journal of Mathematical Analysis and Applications*, 444(1), 550–567, 2016.

E. A. Feinberg, P. O. Kasyanov, and Y. Liang. Fatou's Lemma for Weakly Converging Measures under the Uniform Integrability Condition, 2018.

E. A. Feinberg, P. O. Kasyanov, and Y. Liang. Fatou's lemma and Lebesgue's convergence theorem for varying measures and some of their applications, 2018.

Games

E. A. Feinberg, P. O. Kasyanov, and M. Z. Zgurovsky. Continuity of equilibria for two-person zero-sum games with noncompact action sets and unbounded payoffs. *Annals of Operations Research* 2017, DOI 10.1007/s10479-017-2677-y.

E. A. Feinberg, P. O. Kasyanov, and M. Z. Zgurovsky. Two-person zero-sum games with unbounded payoff functions and uncertain expected payoffs. Preprint arXiv:1704.04564, http://arxiv.org/pdf/1704.04564.pdf, 2017.

E. A. Feinberg, P. O. Kasyanov, and M. Z. Zgurovsky. An example showing that A-lower semi-continuity is essential for minimax continuity theorems. *Operations Research Letters*, 46(4), 385–388, 2018.