# ***Price Dynamics and Repeated games***

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CIRM

Innovative Research in Mathematical Finance

in honor of Yuri KABANOV

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- *The price process should be a CMMV (Continuous Martingale of Maximal Variation)*

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  *(ex. Shareholder meeting.)*

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- *The message* $m$ *can be identified with* $L(m)$*.*
  $\mu$ *=law of* $L(m)$*.*

# *The game* $\Gamma_n(\mu)$*:*

- *Stage 0:*

  *Nature chooses* $L \sim \mu$

  *P1 is informed of* $L$ *not P2.*

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  $I$, $J$*=P1's and P2's action spaces.*

  $T : I \times J \to \mathbb{R}^2$*.*

  *If choices=*$(i, j)$*,* $T(i, j) = (A_{ij}, B_{ij})$ *where* $A_{ij}$ *and* $B_{ij}$ *are the numbers of* $R$ *and* $N$ *shares that P2 gives to P1.*

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  $(i_q, j_q)$ *is then publicly announced.*
  $y_q = (y_q^R, y_q^N)$*= P1's portfolios after* $q$
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  *=Equilibrium strategy in the 0-sum game where a risk neutral P2 aims to maximize the liquidation value of his final portfolio.*

# Natural exchange mechanism

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- *Numéraire scale invariance*
- *Invariance with respect to the riskless part of the risky asset.*
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- *Positive value of information.*
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*A trading mechanism* $\langle I, J, T \rangle$ *is natural if*

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  - $\Rightarrow \forall \alpha > 0, \forall X : V_1([\alpha \cdot X]) = \alpha \cdot V_1([X])$
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  - $\exists p \in [1, 2[, \exists A$ s. th. $\forall$ v.a. $X, Y$:
    $|V_1([X]) - V_1([Y])| \leq A\|X - Y\|_{L^p}$

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  where $\Pi$ is a martingale with $\Pi_1 \sim \mu$ then $\Pi^n \stackrel{Law}{\to} \Pi^\mu$ as $n \to \infty$

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- *If the exchange mechanism is natural*
- *if,* $\forall n$*,* $(\sigma^n, \tau^n)$ *is an equilibrium in* $\Gamma_n(\mu)$
- *if* $L_q^n := E_{\pi_{(\mu,\sigma^n,\tau^n)}}[L|i_s, j_s; s \leq q]$ *and* $\Pi_t^n := L_{[\![nt]\!]}^n$

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# *Theorem:*

*This result suggests that, in a risk neutral market, the price process should be a CMMV.*

- *Where does the B.M. comes from?*
- *Why a CMMV?*
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  - *It also appears if player 2 is risk averse (De Meyer- Fournier*

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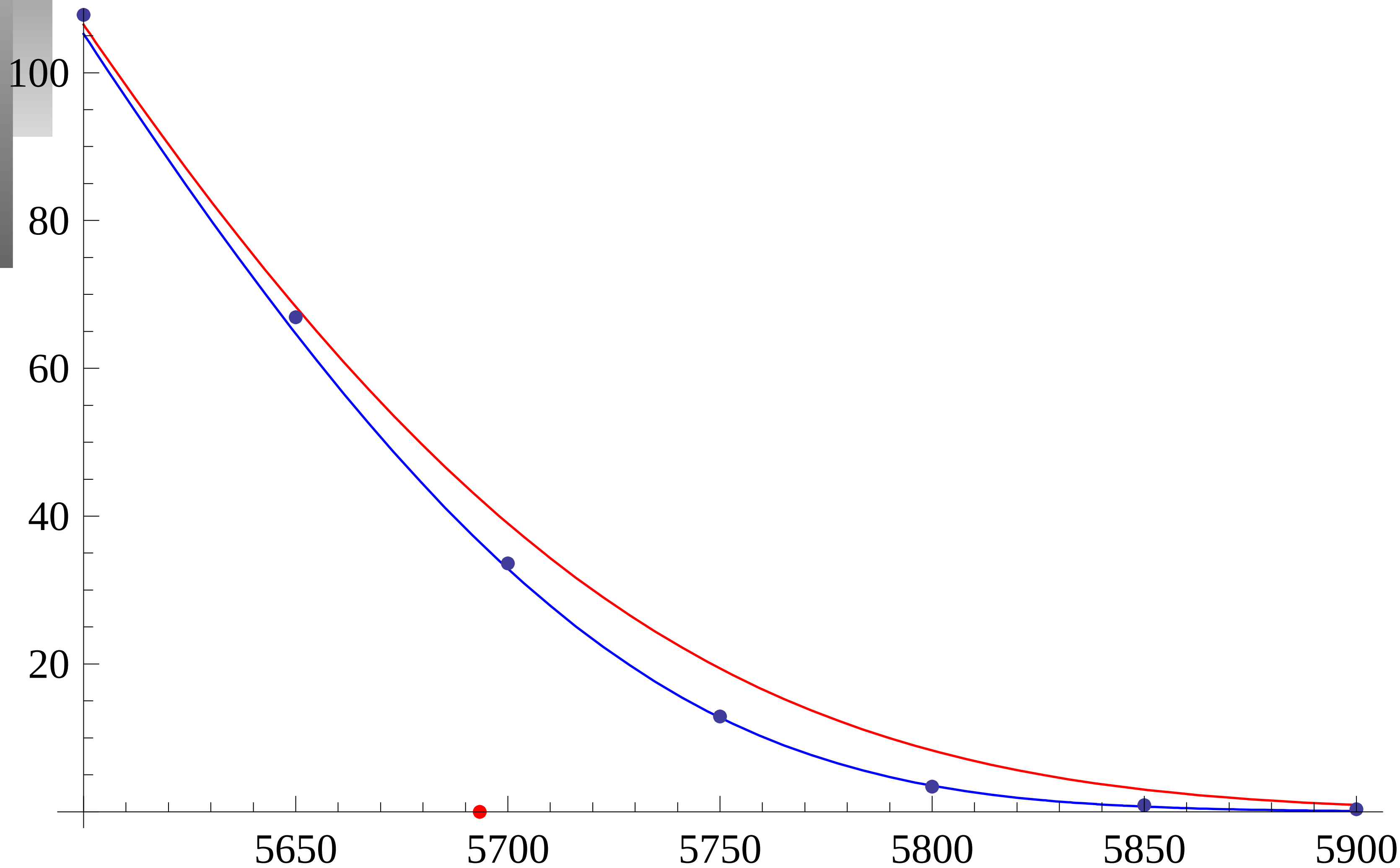
# Example: European Call on CAC40



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- *Metaphore*

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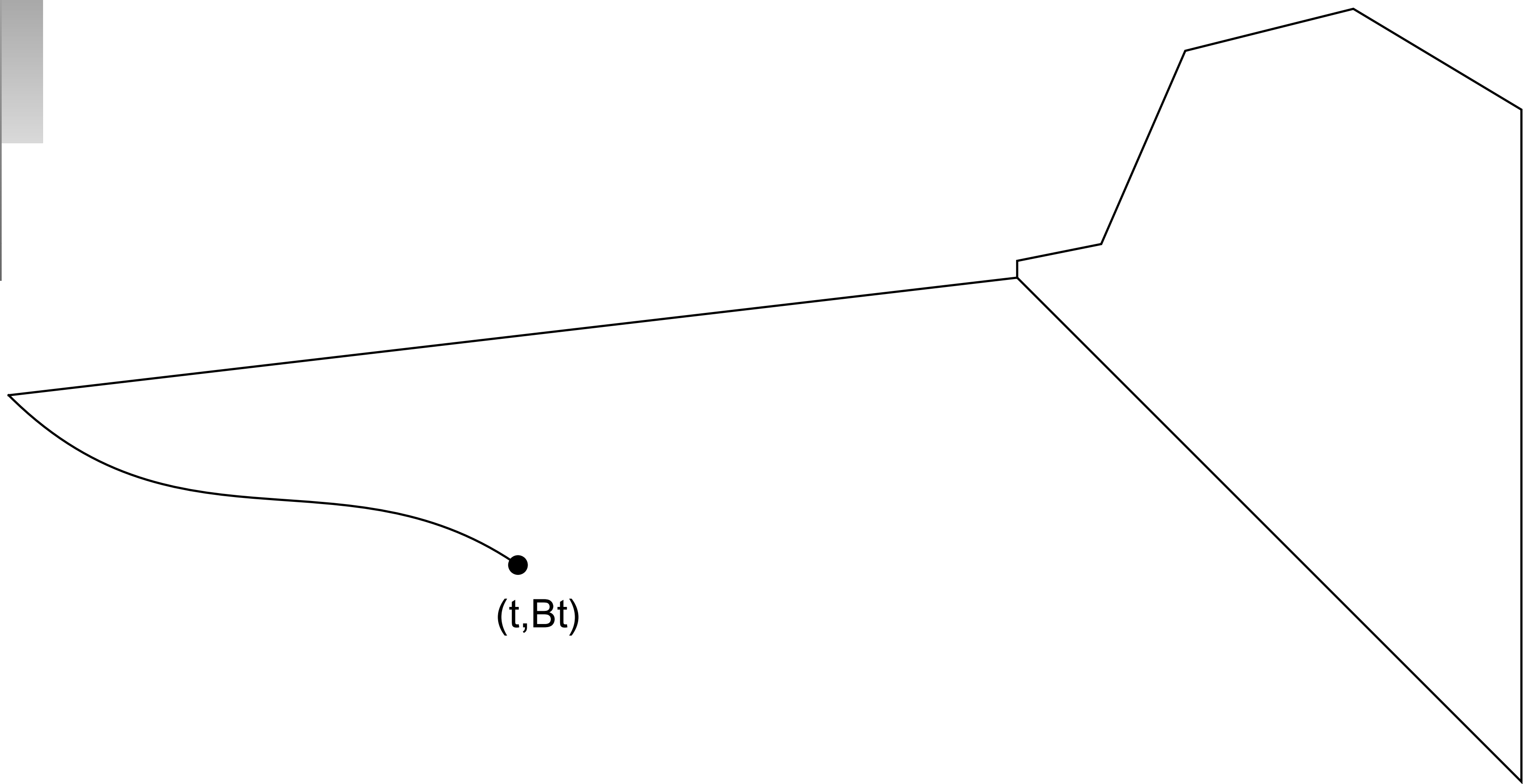
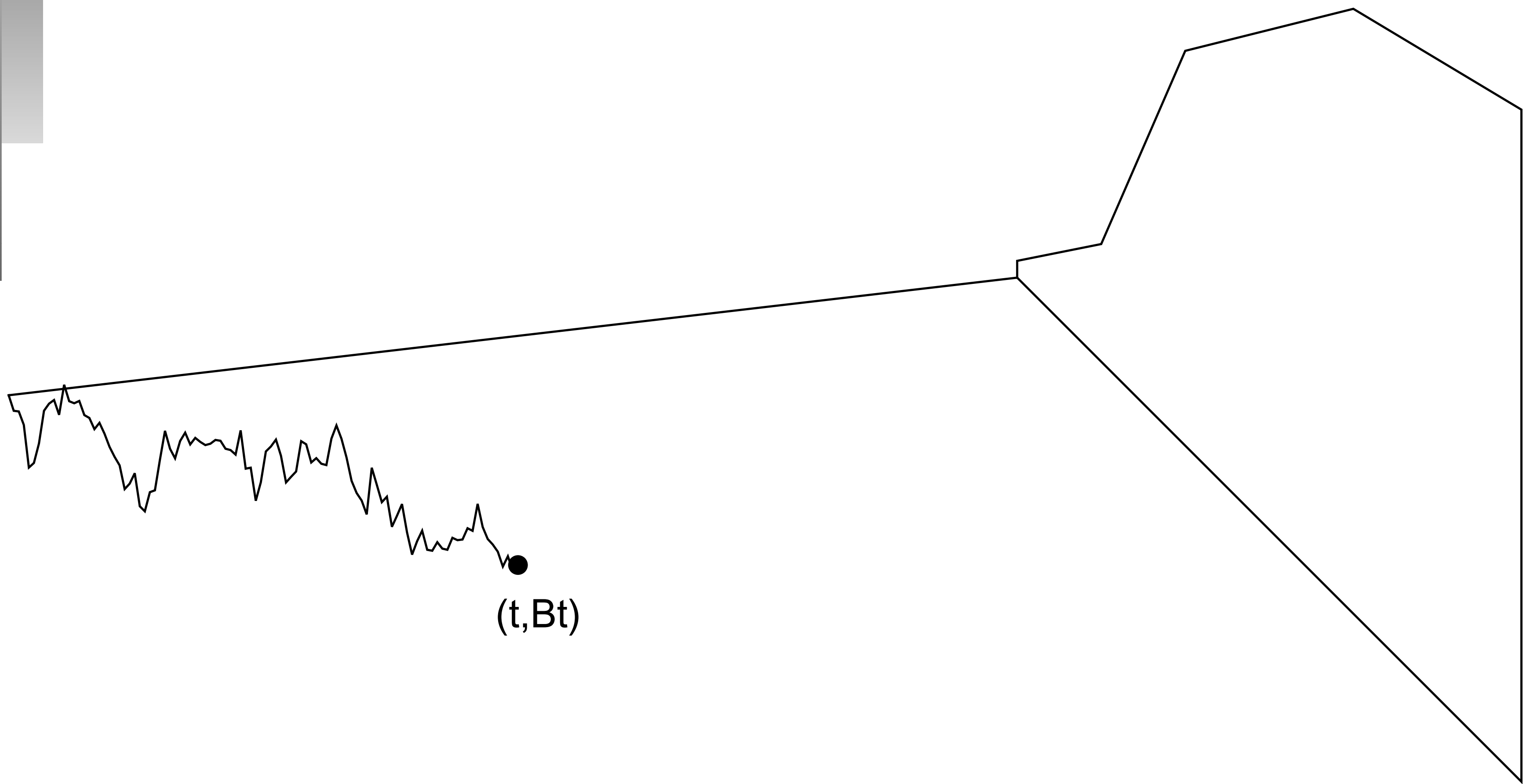


# Sunset over a foggy sea...
# ...with a drunk captain



# Theorem

*Let $\Pi^1$ and $\Pi^2$ be two distinct CMMV (i.e. with distinct $f$),*
*Let $\epsilon > 0$,*
*Let $\nu^i$ denote the probability measure induced by $\Pi^i$ on $\mathcal{C}[0, \epsilon]$*
*then $\nu^1$ and $\nu^2$ are mutually singular.*

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# ***Thank you!***

Happy Birthday Yuri!