Risk aversion of insider and asymmetric information.

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(based on joint work P. Shi)

Innovative Research in Mathematical Finance
In honour of 70th anniversary of Yuri Kabanov
September 7th 2018

Review of Literature

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- Back (1992), Back and Pedersen (1998), Wu (1999), D. (2010), Campi, Çetin, D. (2013); Subrahmanyam (1991), Back, Cao, and Willard (2000), Çetin and D. (2016); Collin-Dufresne and Fos (2016).

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- Baruch (2002), Cho (2003).
- Literature on Markov bridges: Chaumont and Bravo (2011), Fitzsimmons, Pitman and Yor (1993).

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Continuous trading on $[0, 1]$, at time 1 dividends are paid and market terminates.

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Assumptions on f

- f is increasing, bounded, differentiable and has bounded derivative that vanishes at infinity.
- Wlog the range of f is an interval $[b,d]$.
- Wlog $\mathbb{E}[V] = 0$

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$$\sup_{X \in \mathcal{A}(H)} \mathbb{E}^{0,v}\left[-e^{-\gamma W_1^\theta}\right] = \sup_{X \in \mathcal{A}(H)} \mathbb{E}^{0,v}\left[-e^{-\gamma\left[(V-S_1)\theta_1+\int_0^1 \theta_s dS_s\right]}\right],$$

where $\mathbb{E}^{0,v}$ is the expectation using the probability measure of the insider who is given the realisation $V = v$.

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- Market maker: Observes $\mathcal{F}_t^Y$ where $Y_t = \theta_t + B_t$ and sets the price

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  We will look for S satisfying $dS_t = w(t, S_t)dY_t$.

On the form of the pricing rule

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$\Rightarrow$ we choose $H(t, x) = x$. It is wlog since

$$dH(t, \xi_t) = H_x(t, \xi_t) w(t, \xi_t) dY_t$$

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3. (S, V) is a Markov process wrt $((\mathcal{F}_t^I), \mathbb{P}^{0,v})$.

Further we call it inconspicuous if $\mathbb{E}[\theta|\mathcal{F}_t^Y] = 0$ for every $t \in [0,1]$.

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2. Insider optimality condition: given w^*, θ^* solves the insider optimization problem:

$$\mathbb{E}^{0,v}\left[u\left(W_1^{\theta^*}\right)\right] = \sup_{\theta \in \mathcal{A}} \mathbb{E}^{0,v}\left[u\left(W_1^{\theta}\right)\right].$$

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We focus on inconspicuous equilibrium.

Theorem

Suppose the admissible pricing rule w satisfies

1. $$\frac{w_t(t,\xi)}{w^2(t,\xi)} + \frac{w_{\xi\xi}(t,\xi)}{2} = -\gamma.$$
2. θ^* *admissible and satisfies*

$$\xi_1^* = v, \ \mathbb{P}^{0,v} \ a.s..,$$

where ξ^* is the strong solution to

$$\xi_t = \int_0^t w(s,\xi_s)d(B_s + \theta_s^*).$$

Then θ^* is the optimal strategy.

Proof

Define function

$$\phi(t,\xi) = \int_V^{\xi} \frac{y-V}{w(t,y)}dy + \frac{1}{2}\int_t^1 w(s,V)ds.$$

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$$\phi(1,\xi_1) = \int_V^{\xi_1} \frac{y-V}{w(1,y)}dy \geq 0.$$

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And

$$\phi(t,\xi_t) = \phi(0,0) + \frac{\gamma}{2}\int_0^t (\xi_s - V)^2 ds + \int_0^t (\xi_s - V)d(\theta_s + B_s).$$

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In particular

$$\begin{aligned} -W_1^{\theta} &= (\xi_1 - V)\theta_1 - \int_0^1 \theta_s d\xi_s = \int_0^1 (\xi_s - V)d\theta_s \\ &= \phi(1,\xi_1) - \phi(0,0) - \int_0^1 \frac{\gamma}{2}(\xi_s - V)^2 ds - \int_0^1 (\xi_s - V)dB_s \end{aligned}$$

Proof, ctd.

Insider's utility is given by:

$$\begin{aligned} J &= -\frac{1}{\gamma} \inf_{\theta} \mathbb{E}^{0,v}\left[e^{-\gamma \int_0^1 (V-\xi_t) d\theta_t} \right] \\ &= -\frac{1}{\gamma} \inf_{\theta} \mathbb{E}^{0,v}\left[e^{-\gamma(\phi(0,0)-\phi(1,\xi_1))} \mathcal{E}_1(-\gamma(\xi - V)) \right] \\ &\leq -\frac{1}{\gamma} e^{-\gamma\phi(0,0)} \inf_{\theta} \mathbb{E}^{0,v}\left[\mathcal{E}_1(-\gamma(\xi - V)) \right], \end{aligned}$$

where

$$\mathcal{E}_t(X) = \exp\left\{ \int_0^t X_s dBs - \frac{1}{2}\int_0^t X_s^2 ds \right\}.$$

On PDE for weighting function

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Then value function of the insider will become:

$$\begin{aligned} J &= -\frac{1}{\gamma} \inf_{\theta} \mathbb{E}^{0,v}\left[e^{-\gamma \int_0^1 (V-\xi_t) d\theta_t} \right] \\ &= -\frac{1}{\gamma} \inf_{\theta} \mathbb{E}^{0,v}\left[e^{-\gamma(\phi(0,0)-\phi(t,\xi_1))-\gamma \int_0^1 \int_V^{\xi_t} (g(t,y)-1)(y-V) dy dt} \mathcal{E}_1 \right] \end{aligned}$$

where $\mathcal{E}_1 = \mathcal{E}_1\left(-\gamma(\xi - V)\right)$

Characterisation of Equilibrium

Theorem

A pair (w^*, θ^*) *is an inconspicuous equilibrium if:*

1. w^* *satisfies*
$$\frac{w_t^*(t,\xi)}{w^*(t,\xi)^2} + \frac{w_{\xi\xi}^*(t,\xi)}{2} = -\gamma, \tag{2}$$
2. $Y^* = B + \theta^*$ *is a standard Brownian motion in its own filtration,*
3. $\xi_1^* = v$, $\mathbb{P}^{0,v}$ *a.s. where* ξ^* *is the strong solution to*
$$\xi_t = \int_0^t w(s,\xi_s)dY_s^*.$$

To find w satisfying (2) and

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Let $\kappa_t = K_w(t,\xi_t)$, $\lambda(t,y) = K_w^{-1}(t,y)$.
Then $\lambda(1,\kappa_1) =^{\mathcal{L}} V$ and

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with λ solving Burger's equation

$$\lambda_t(t,x) + \frac{1}{2}\lambda_{xx}(t,x) = -\gamma\lambda_x(t,x)\lambda(t,x)$$

Measure change is given by:

$$\frac{d\mathbb{P}}{d\tilde{\mathbb{P}}} = e^{\int_0^1 \gamma\lambda(t,\kappa_t)d\kappa_t - \frac{\gamma^2}{2}\int_0^1 \lambda^2(t,\kappa_t)dt} = Ce^{\gamma\int_0^{\kappa_1}\lambda(1,x)dx}$$

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Thus, market maker's problem becomes:

$$\begin{aligned} P(x) &= \mathbb{P}[\kappa_1 \leq x] = C\tilde{\mathbb{E}}[e^{\gamma\int_0^{\beta_1}\lambda(1,u)du}\mathbf{1}_{\{\beta_1 \leq x\}}] \\ &= C\int_{-\infty}^{x} e^{\gamma\int_0^y f\circ\Phi^{-1}\circ P(u)du - \frac{y^2}{2}}dy \end{aligned}$$

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$\Rightarrow$ Fixed point problem.Consider recursive map $P^{n+1} = TP^n$.

$$g^n(x) = f\circ\Phi^{-1}\circ P^n(x), \quad G^n(x) = \int_0^x g^n(u)du,$$

$$c_n^* = \frac{\sqrt{2\pi}}{\int_{-\infty}^{\infty}\exp\left\{\gamma G^n(u) - \frac{u^2}{2}\right\}},$$

$$P^{n+1}(x) = \frac{c_n^*}{\sqrt{2\pi}}\int_{-\infty}^{x}\exp\left\{\gamma G^n(u) - \frac{u^2}{2}\right\}du$$

Define a set $\mathcal{D}$ where we pick P from as:

$$\mathcal{D} = \left\{ P \in \mathcal{C}_b(\mathbb{R}) : P \text{ a.c. cdf}, \, 0 \leq P_x(x) \leq \frac{c}{\sqrt{2\pi}} \exp\left\{ -\frac{x^2}{2\sigma^2} \right\} \right\}$$

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Then

1. $\mathcal{D}$ is convex and closed,
2. for any $P \in \mathcal{D}$ we have $TP \in \mathcal{D}$,
3. T is a continuous map wrt sup norm.

Thus: there exists w satisfying

$$\frac{w_t^*(t,\xi)}{w^*(t,\xi)^2} + \frac{w_{\xi\xi}^*(t,\xi)}{2} = -\gamma,$$

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$\Rightarrow$ Markov bridge construction. main result

Let $p(s,x;t,z)$ be a transition density of process

$$\kappa_t = \int_0^t \gamma\lambda(s,\kappa_s)ds + B_t.$$

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It satisfies

$$\lim_{t\to u}\int_{B_r^c(z)} p(t,y;u,z)p(0,x;t,y)dy = 0,\ \forall u>0,\ r>0,$$

the Chapman-Kolmogorov equations

$$p(s,x;u,y) = \int_{\mathbb{R}} p(s,x;t,z)p(t,z;u,y)dz,\ 0\le s<t\le 1,$$

and

$$\sup_{\substack{x\notin B_r(z)\\ t<1}} p(t,x;1,z) < \infty,$$

for every $z\in\mathbb{R}$ and $r>0$.

Moreover, for any $z \in \mathbb{R}$

1. $p(0, x; 1, z) > 0$.
2. For $h(t, y) = p(t, y; 1, z)$ we have $h \in C^{1,2}([0, 1) \times \overline{\mathbb{R}})$.

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Thus there exists a weak solution on $[0, 1]$ to

$$\kappa_t = \int_0^t \left\{ \gamma\lambda(u, \kappa_u) + \frac{(\nabla h(u, \kappa_u))}{h(u, \kappa_u)} \right\} du + B_t, \tag{3}$$

the law of which, $P_{0\to 1}^{x\to z}$, satisfies $P_{0\to 1}^{x\to z}(\kappa_1 = z) = 1$. Moreover, since $h(t, \cdot) > 0$ for all $t < 1$, strong uniqueness holds for the above SDE.

Construction of measures

There exists a unique weak solution to

$$X_t = x + \int_0^t \gamma\lambda(u, X_u)du + B_t$$

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h is a martingale $\Rightarrow$ X solves (3) until T under P^T ▸ main result

Tightness

It is enough to show that

$$\lim_{\delta \to 0} \limsup_{T \to 1} P^T(w(X, \delta, [0,1]) > 8c) = 0,$$

where

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Let $Z_\delta = w(X,\delta,[0,\delta])$ then $\forall T > 1-\delta$

$$[Z_\delta \circ \theta_{1-\delta} > 4c] \subset [Z_{1-T}\circ\theta_T > 2c] \cup [Z_{T-1+\delta}\circ\theta_{1-\delta} > 2c].$$

Tightness, interval $[T, 1)$

$(\Delta = 1 - T)$

$$P^T(Z_\Delta \circ \theta_T > 2c) \quad = \quad E^x\left[\mathbf{1}_{[Z_\Delta \circ \theta_T > 2c]} \frac{p(\Delta, X_T, z)}{p(1, x, z)}\right]$$

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&= \frac{E^x\left[P^{X_T}(Z_\Delta > 2c)\mathbf{1}_{[X_T \in B_r(z)]} p(\Delta, X_T, z)\right]}{p(1, x, z)} \\
&+ \frac{E^x\left[P^{X_T}(Z_\Delta > 2c)\mathbf{1}_{[X_T \notin B_r(z)]} p(\Delta, X_T, z)\right]}{p(1, x, z)}
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&+ \frac{E^x\left[P^{X_T}(Z_\Delta > 2c)\mathbf{1}_{[X_T \notin B_r(z)]} p(\Delta, X_T, z)\right]}{p(1, x, z)} \\
&\leq \sup_{y \in B_r(z)} P^y(Z_\Delta > 2c) \\
&+ \frac{1}{p(1, x, z)} \int_{B_r^c(z)} p(\Delta, y, z) p(T, x, y) m(dy),
\end{aligned}
$$

Tightness, interval $[1-\delta, T)$

Let $T^\delta := 1-\delta$ and consider $M_t^1 := p(1-t, X_t, z)$ and $\tau^\delta := \inf\{t \geq 0 : \sup_{0\leq s\leq t} X_s - \inf_{0\leq s\leq t} X_s > 2c\} \wedge \delta \wedge 1$

$$\lim_{T\to 1} \quad P^T(Z_{T-T^\delta} \circ \theta_{T^\delta} > 2c) = \lim_{T\to 1} \frac{E^x[\mathbf{1}_{[T^\delta+\tau^\delta\circ\theta_{T^\delta}<T]} M_T]}{p(1,x,z)}$$

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$$\leq \quad \frac{1}{p(1,x,z)} \Big[E^x[\mathbf{1}_{[X_{1-\delta}\notin B_{\frac{c}{4}}(z)]} M_{1-\delta}]$$

$$+ \quad E^x[1_{[X_{T^\delta}\in B_{\frac{c}{4}}(z)]} \mathbf{1}_{[\tau_c\circ\theta_{T^\delta}<\delta]} M_{T^\delta+\tau_c\circ\theta_{T^\delta}}] \Big]$$

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$$+ \quad E^x[1_{[X_{T^\delta} \in B_{\frac{c}{4}}(z)]} \mathbf{1}_{[\tau_c \circ \theta_{T^\delta} < \delta]} M_{T^\delta + \tau_c \circ \theta_{T^\delta}}] \Big]$$

Tightness + convergence of finite dimensional distributions $\Rightarrow$ existence of the limiting measure $P_{0\to 1}^{x\to z}$ on $(C([0,1],\mathbb{R}), \mathcal{B}_1)$.

Bridge property

Observe that for any $g \in \mathbb{C}_K^\infty(\mathbb{R})$, we have that $E_{0\to 1}^{x\to z}[g(X_1)]$ can be expressed as $(\Delta = 1 - T)$

$$g(z) \quad + \quad \lim_{T\to 1} \frac{E^x\left[p(\Delta, X_T, z)(g(X_T) - g(z))\right]}{p(1, x, z)}$$

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g(z) \quad + \quad & \lim_{T\to 1} \frac{E^x\left[p(\Delta, X_T, z)(g(X_T) - g(z))\right]}{p(1,x,z)} \\
= \quad & g(z) + \lim_{T\to 1} \int_{B_r(z)} \frac{p(\Delta, y, z)p(T,x,y)}{p(1,x,z)} (g(y) - g(z))\, dy \\
& + \lim_{T\to 1} \int_{B_r^c(z)} \frac{p(\Delta, y, z)p(T,x,y)}{p(1,x,z)} (g(y) - g(z))\, dy.
\end{aligned}
$$

Theorem

There exists an equilibrium (w^*, θ^*) *where*

1. $w^*(t,\xi) = \frac{1}{\lambda_\xi^{-1}(t,\xi)}$ *be the weighting function.*
2. $\theta_t^* = \int_0^t \alpha_s^* ds$ *where* $\alpha_s^* = w^*(s,\xi_s)\frac{\rho_\xi(s,\xi_s;1,\xi_1^Z)}{\rho(s,\xi_s;1,\xi_1^Z)}$ *with* $\xi_1^Z = f(Z)$*. Moreover* ξ^* *is the unique strong solution of*

$$d\xi_t = w^*(t,\xi_t)dB_t + w^*(t,\xi_t)^2\frac{\rho_\xi(t,\xi_t;1,\xi_1^Z)}{\rho(t,\xi_t;1,\xi_1^Z)}, \xi_0 = 0,$$

where ρ *is transition density of* ξ*.*


Happy Birthday
С Днём рождения,
Юра!