Infinite dimensional polynomial processes

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in Honor of Yuri Kabanov
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Outline

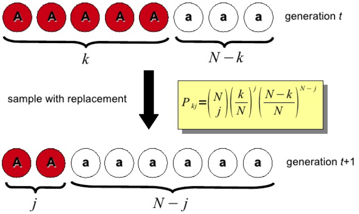
1. Introduction and motivation
   - ... from population genetics
   - ... from stochastic portfolio theory
   - ... and some connections to Yuri's work
2. The theory of infinite dimensional polynomial processes
3. Application: (rough) polynomial forward variance models

From population genetics to ...

- Neutral 2-allele Wright-Fisher Markov chain model from population genetics
  - Discrete time model of a population with constant size N with two types of alleles, denoted by A and a
  - X_t^N: number of type A individuals at time t
  - X_t^N is modeled as a Markov chain with state space $\{0, \dots, N\}$ and transition probabilities

$$P_{kj} = P(X_{t+1}^N = j | X_t^N = k) = \binom{N}{j} \left(\frac{k}{N}\right)^j \left(1 - \frac{k}{N}\right)^{N-j}.$$

  - Binomial sampling with probability X_t^N / N



... a guiding example of polynomial processes

- Diffusion approximation
  - The process $\frac{1}{N} X^N_{[Nt]}$ converges in law to the Kimura or Wright-Fisher diffusion on $[0, 1]$

$$d\lambda_t = \sqrt{(1 - \lambda_t)\lambda_t} dB_t, \quad \lambda_0 \in [0, 1],$$

where B denotes a standard Brownian motion.

$\Rightarrow$ Guiding example of a polynomial process

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  $\Rightarrow$ Guiding example of a polynomial process

- It has the following non-standard properties:
  - Non-Lipschitz property of the volatility
  - Statespace is not the whole of $\mathbb{R}$
  - Nevertheless existence of strong and pathwise unique solutions

Guiding example of standard polynomial processes

- Key property and moment formula: the expected value of polynomials of the process' marginals can be computed easily:

$$\mathbb{E}\left[\sum_{j=0}^{k} y_j \lambda_t^j\right] = \sum_{j=0}^{k} y_{j,t} \lambda_0^j,$$

where y_t solves the linear ODE in $\mathbb{R}^{k+1}$

$$\partial_t y_t = L_k y_t, \quad y_0 = (y_0, y_1, \ldots, y_k) \in \mathbb{R}^{k+1}$$

with L_k the matrix representation of the infinitesimal generator applied to the basis monomials $(1, x, \ldots, x^k)$.

$\Rightarrow$ Tractability: the Feynman-Kac PDE reduces to a linear $k+1$ - dimensional ODE, that is solved by matrix exponentiation.

Polynomial jump diffusions on $\mathcal{E} \subseteq \mathbb{R}^d$

Definition

- A linear operator $L : \mathrm{Pol}(\mathbb{R}^d) \to \mathrm{Pol}(\mathbb{R}^d)$ is called polynomial if it maps polynomials to polynomials of same or lower degree.
- Let L be a polynomial operator. Then a polynomial jump diffusion on $\mathcal{E}$ is a càdlàg $\mathcal{E}$-valued solution λ to the martingale problem for L, i.e. for all $p \in \mathrm{Pol}(\mathbb{R}^d)$

$$p(\lambda_t) - \int_0^t Lp(\lambda_s)ds = (\text{local martingale}).$$

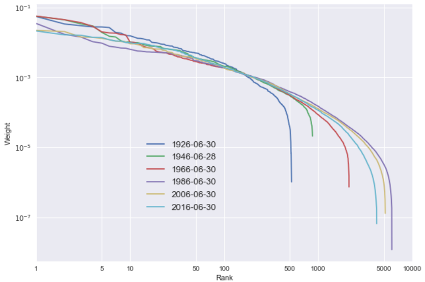
For polynomial operators the moment formula holds true and they are of the form

$$\nabla p(\lambda)^\top b(\lambda) + \frac{\mathrm{Tr}(c(\lambda)\nabla^2 p(\lambda))}{2} + \int (p(\lambda+\xi) - p(\xi) - \nabla p(\lambda)^\top \xi) K(\lambda, d\xi)$$

with $\lambda \mapsto b(\lambda)$ affine, $\lambda \mapsto c_{ij}(\lambda) + \int \xi_i \xi_j K(\lambda, d\xi)$ quadratic, $\lambda \mapsto \int \xi^{\mathbf{k}} K(\lambda, d\xi)$ polynomial up to degree $|\mathbf{k}|$ for $|\mathbf{k}| \geq 3$.

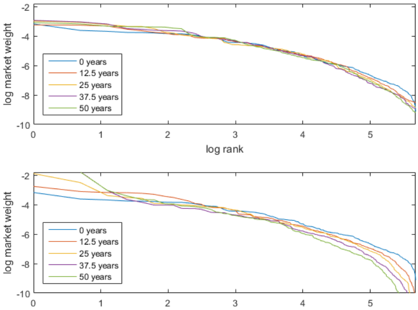
From stochastic portfolio theory to...

- Stochastic portfolio theory (SPT) (introduced by Robert Fernholz, Ioannis Karatzas, etc.) analyzes high dimensional stock markets, typically the constituents of large equity indices like S&P 500 and the capital distribution curves.
- That is, the mapping $\log k \mapsto \log(\mu_t^{(k)})$, where $\mu_t^{(1)}, \ldots, \mu_t^{(d)}$ are the ordered market weights of the d considered companies.
- They are of remarkable stability between 1926 -2016 for the US stock market (see the graph by J. Ruf below).



...to polynomial models for the market weights

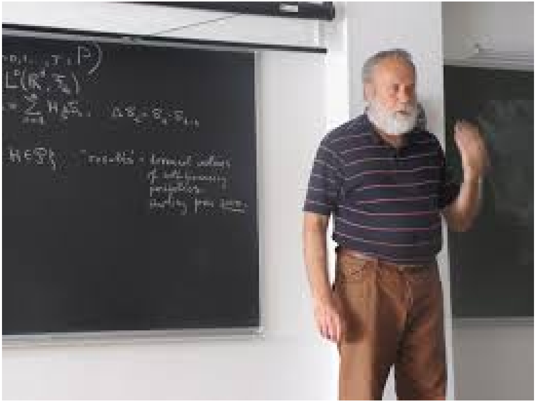
- Polynomial models allow to overcome certain shortcomings of existing models (see [C.'18; Polynomial processes in stochastic portfolio theory]).
- Within the Financial Maths Team Challenge 2016 South Africa, we performed a calibration of a 300 dimensional polynomial process.
- Shape preservation and "correct" dynamic behavior over 50 years.
  - Comparison between the polynomial model and a Black & Scholes model:



From large financial markets in SPT...

- Consider a set of stocks with market capitalizations $S_t^1, \dots, S_t^d$ and the corresponding market weights $\mu_t^i = \frac{S_t^i}{S_t^1 + \cdots + S_t^d}$ taking values in the unit simplex $\Delta^d = \{z \in [0,1]^d \colon z_1 + \cdots + z_d = 1\}$, i.e. the space of probability measures on a set of d points.
- Large financial market as $d \to \infty$, e.g. for analyzing the capitalization curves.
- $\Rightarrow$ Probability measure valued setting with an (uncountably) infinite dimensional underlying space.
- Possible approach: Linear factor models, i.e. view $(\mu^1, \dots, \mu^d)$ as the projection of a single tractable infinite dimensional model.
  - Let X be a probability measure valued polynomial process.
  - For functions $g_i^d \geq 0$ such that $g_1^d + \ldots + g_d^d \equiv 1$, set $\mu_t^{i,d} = \int g_i^d(x) X_t(dx)$. $\Rightarrow$ much richer class than polynomial models on Δ^d but similar tractability.
  - Consider limits as $d \to \infty$.

...to some open questions inspired by Yuri's work



...to some open questions inspired by Yuri's work

- When does such a model satisfy NAA1, i.e. no (relative) asymptotic arbitrage of the first kind, as introduced in the context of large financial markets by Y. Kabanov and D. Kramkov (1994, 1998)?
  $\Rightarrow$ Existence of supermartingale deflators?
- Construction of stochastic integrals with respect to the measure valued process, in a similar spirit as in T.Björk, G. Di Masi, Y. Kabanov, W.Runggaldier (1997), by reversing the role of integrands (measure valued processes) and integrators (continuous function valued processes)?
  - Evolution of the value of a portfolio corresponding to a trading strategy which is a predictable process with values in continuous functions. The measure valued process could for instance represent an electricity forward price.
  - Functionally generated portfolios (also in a modelfree sense based on Itô-type formulas in the sense of H. Föllmer)

Infinite dimensional setting

- Y: real Banach algebra with identity element 1 for the multiplication
- Y^*: dual space equipped with the weak-$*$-topology, which is the weakest topology making all linear functionals $\lambda \mapsto \lambda(y) = \langle y, \lambda \rangle$ on Y^* continuous.
- $y_1 \otimes y_2$: symmetric tensor product for two elements $y_1, y_2 \in Y$
- We fix a (reasonable) crossnorm $\| \cdot \|_\times$ on $Y \otimes Y$, i.e. a norm $\| \cdot \|_\times$ on $Y \otimes Y$ such that
  1. $\|y_1 \otimes y_2\|_\times = \|y_1\| \|y_2\|$ for each $y_1, y_2 \in Y$, and
  2. $\sup_{y \in Y \otimes Y, \|y\|_\times \leq 1} |(\lambda_1 \otimes \lambda_2)(y)| = \|\lambda_1\|_{Y^*} \|\lambda_2\|_{Y^*}$ for each $\lambda_1, \lambda_2 \in Y^*$.
- $Y \widehat{\otimes} Y$: completion of $Y \otimes Y$ with respect to $\| \cdot \|_\times$
- $y^{\otimes k}, Y^{\otimes k}, Y^{\widehat{\otimes} k}, \lambda^{\otimes k}$ for $k \in \mathbb{N}$ are defined analogously. For $k = 0$, we identify $Y^{\widehat{\otimes} 0}$ with $\mathbb{R}$.

Guiding example and polynomials on Y^*

Example (important setting for SPT)

- Let $E \subseteq \mathbb{R}$ be compact.
- $Y := C(E)$: Banach space of continuous functions
- $Y^* = M(E)$: space of finite signed measures
- Letting $\| \cdot \|_\times$ be the supremum norm on $C(E)^{\otimes k}$, we get that $Y^{\widehat{\otimes} k}$ is the space of symmetric continuous functions $f : E^k \to \mathbb{R}$.

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- A polynomial on Y^* with coefficients $y := (y_0, \ldots, y_k) \in \bigoplus_{j=0}^k Y^{\widehat{\otimes} j}$ is defined as $p(\lambda) = \sum_{j=0}^k \langle y_j, \lambda^{\otimes j} \rangle$ with $\langle y_j, \lambda^{\otimes j} \rangle := \lambda^{\otimes j}(y_j)$ for $y_j \in Y^{\widehat{\otimes} j}$.
- $P := \{\lambda \mapsto p(\lambda) \,|\, p \text{ is a polynomial on } Y^*\}$ algebra of all polynomials on Y^*
- The space of cylindrical polynomials with coefficients in a dense linear subspace $D \subseteq Y$ is defined by

$$P^D = \{\varphi(\langle y_1, \lambda \rangle, \ldots, \langle y_k, \lambda \rangle) \,|\, \text{ polynomials } \varphi : \mathbb{R}^k \to \mathbb{R}, k \in \mathbb{N}_0, y_i \in D\}.$$

Differentiability

Definition

- A map $f : Y^* \to \mathbb{R}$ is said to be Fréchet differentiable at $\lambda \in Y^*$ if there exists a linear functional $y_\lambda^{**}(\,\cdot\,) = \langle y_\lambda^{**}, \cdot \rangle \in Y^{**}$ such that

$$\lim_{\widetilde{\lambda} \to 0} \frac{|f(\lambda + \widetilde{\lambda}) - f(\lambda) - y_\lambda^{**}(\widetilde{\lambda})|}{\|\widetilde{\lambda}\|_{Y^*}} = 0.$$

  We write $\partial f(\lambda) := y^{**}$.

- Analogously, whenever it exists, we denote by $\partial^k f(\lambda)$ the element of $(Y^{\widehat{\otimes} k})^{**}$ corresponding to the k-th iterated Fréchet derivative of f at λ.

For a $p \in P^D$ of the form $p(\lambda) := \varphi(\langle y_1, \lambda \rangle, \ldots, \langle y_d, \lambda \rangle)$ we have

$$\partial^k p(\lambda) = \sum_{i_1, \ldots, i_k = 1}^{d} y_{i_1} \otimes \cdots \otimes y_{i_k} \varphi_{i_1, \ldots, i_k}(\langle y_1, \lambda \rangle, \ldots, \langle y_d, \lambda \rangle).$$

and thus $\partial^k p(\lambda) \in D^{\otimes k}$ for each $\lambda \in Y^*$.

Lévy type operator

Definition

Let $L : P^D \to P$ be a linear operator and fix $\lambda \in Y^*$. The operator L is said to be of Lévy type on $\mathcal{E} \subseteq Y^*$ at λ if $p \mapsto Lp(\lambda)$ can be represented as

$$Lp(\lambda) = -\Gamma(\lambda)p(\lambda) + B(\partial p(\lambda), \lambda) + \frac{1}{2}Q(\partial^2 p(\lambda), \lambda) + \int p(\xi) - p(\lambda) - \langle \partial p(\lambda), \xi - \lambda \rangle K(\lambda, d\xi),$$

where the quadruplet $\big(\Gamma(\lambda), B(\cdot, \lambda), Q(\cdot, \lambda), K(\lambda, \cdot)\big)$ consists of some constant $\Gamma(\lambda) \in \mathbb{R}_+$, some linear operators

$$B(\cdot, \lambda) : D \to \mathbb{R}, \qquad \text{and} \qquad Q(\cdot, \lambda) : D \otimes D \to \mathbb{R},$$

and some (nonnegative) measure $K(\lambda, \cdot)$ on $\mathcal{E} \setminus \{\lambda\}$ satisfying

$$Q(y \otimes y, \lambda) \geq 0 \qquad \text{and} \qquad \int \langle y, \xi - \lambda \rangle^2 K(\lambda, d\xi) < \infty, \quad \forall y \in D.$$

Polynomial operators

Definition

Fix $\mathcal{E} \subseteq Y^*$. A linear operator $L\colon P^D \to P$ is called $\mathcal{E}$-polynomial if for every $p \in P^D$ there is some $q \in P$ such that $q|_{\mathcal{E}} = Lp|_{\mathcal{E}}$ and $\deg(q) \leq \deg(p)$.

Proposition (C. & Svaluto-Ferro '18)

A linear operator $L\colon P^D \to P$ of Lévy type is $\mathcal{E}$-polynomial if and only if

$$\Gamma(\lambda) = const, \qquad B(y,\lambda) = B_0(y) + \langle B_1(y), \lambda\rangle$$

$$Q(y \otimes y, \lambda) + \int \langle y, \xi - \lambda\rangle^2 K(\lambda, d\xi) = \sum_{i=0}^{2} \langle Q_i(y \otimes y), \lambda^{\otimes i}\rangle$$

$$\int \langle y, \xi - \lambda\rangle^k K(\lambda, d\xi) = \sum_{i=0}^{k} \langle J_i^k(y^{\otimes k}), \lambda^{\otimes i}\rangle, \quad k \geq 3$$

for some linear operators $B_0 : D \to \mathbb{R}$, $B_1 : D \to Y$, $Q_i : D \otimes D \to Y^{\widehat{\otimes} i}$, $i \in \{0,1,2\}$, $J_i^k : D^{\otimes k} \to Y^{\widehat{\otimes} i}$, $i \in \{0,\dots,k\}$ for $k \geq 3$.

Dual operator

Definition

Let $L\colon P^D \to P$ be an $\mathcal{E}$-polynomial operator and fix $k \in \mathbb{N}$.

- A *k-th dual operator* $L_k : \bigoplus_{j=0}^k D^{\otimes j} \to \bigoplus_{j=0}^k Y^{\hat{\otimes} j}$ is a linear operator that maps the coefficients vector of p to the coefficients vector of Lp, for each $p \in P^D$ with $\deg(p) \le k$ such that $L_k y := (L_k^0 y, \dots, L_k^k y)$ satisfies

$$Lp(\lambda) = \sum_{j=0}^{k} \langle L_k^j y, \lambda^{\otimes j} \rangle, \qquad \text{for all } \lambda \in Y^*$$

for $p(\lambda) = \sum_{j=0}^k \langle y_j, \lambda^{\otimes j} \rangle$.

Polynomial processes

Let $\mathcal{E} \subseteq Y^*$ and $L : P^D \to P$ be a linear operator. An $\mathcal{E}$-valued process $(\lambda_t)_{t\geq 0}$ defined on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is called a solution to the martingale problem for L with initial condition $\lambda_0 \in \mathcal{E}$ if

- $\lambda_0 = \lambda$ $\mathbb{P}$-a.s.,
- for every $p \in P^D$ there exists a measurable version of $(p(\lambda_t))_{t\geq 0}$ and $(Lp(\lambda_t))_{t\geq 0}$ and
- the process

$$N_t^p := p(\lambda_t) - p(\lambda_0) - \int_0^t Lp(\lambda_s)ds$$

  defines a local martingale for every $p \in P^D$.

Definition

Let L be $\mathcal{E}$-polynomial. A solution to the martingale problem for L is called an $\mathcal{E}$-valued polynomial jump diffusion if $(N_t^p)_{t\geq 0}$ is a true martingale for all $p \in P^D$.

Moment formula

Theorem (C.& Svaluto-Ferro '18)

Let $(\lambda_t)_{t\geq 0}$ be a polynomial jump diffusion corresponding to L and fix a dual operator $(L_k)_{k\in\mathbb{N}}$. Suppose that

- *the k-th dual operator L_k is closable, and that the coefficients vector $y \in \bigoplus_{j=0}^k D^{\otimes j}$ is in the domain of its closure $\overline{L}_k$.*
- *Suppose that there is a weak solution $(y_t)_{t\geq 0} = (y_{t,0}, \ldots, y_{t,k})_{t\geq 0}$ with $y_{t,j} \in Y^{\widehat{\otimes} j}$ (satisfying certain technical conditions) of the $k+1$ dimensional system of linear ODEs on $\bigoplus_{j=0}^k Y^{\widehat{\otimes} j}$*

$$\partial_t y_t = \overline{L}_k y_t, \quad y_0 = y$$

Then the following conditional moment formula holds true

$$\mathbb{E}\left[\left(\sum_{j=0}^k \langle y_j, \lambda_T^{\otimes j}\rangle\right) \mid \mathcal{F}_t\right] = \sum_{j=0}^k \langle y_{T-t,j}, \lambda_t^{\otimes j}\rangle.$$

Interpretation

- The infinite dimensional Y^* valued Feynman Kac PDE reduces to a $k+1$ dimensional systems of linear ODEs on $\bigoplus_{j=0}^{k} Y^{\widehat{\otimes} j}$.
- When $Y^* = M(E)$ for some compact set $E \subseteq \mathbb{R}$, this correpsonds to a system of linear PDES, where the j-th component is a function on E^j.
    - When E consists of one point, we are back to the one-dimensional guiding example of the Wright-Fisher diffusion and we obtain a $k+1$ dimensional ODE.

Polynomial foward variance models

- Log price dynamics: $dX_t = -\frac{1}{2}V_t dt + \sqrt{V_t} dW_t$
- Forward variance dynamics

$$\lambda_t(x) = \frac{d}{dx}\lambda_t(x)dt + dM_t^x$$

where $V_t = \lambda_t(0)$ (actually $\lambda_t(x) = \mathbb{E}[V_{t+x}|\mathcal{F}_t]$), W is a Brownian motion and (M_t^x) is a continuous martingale for all $x \in \mathbb{R}_+$.

- We consider $x \mapsto \lambda_t(x)$ on the following Hilbert space Y^* (c.f. Filipovic 2001),

$$Y^* = Y = \{y \in AC(\mathbb{R}_+, \mathbb{R}) \,|\, \int_0^\infty |y'(x)|^2 \alpha(x) dx < \infty\}$$

  for a specified strictly positive weight function $\alpha > 0$.
  - Scalar product $\langle y, \lambda \rangle_\alpha = y(0)\lambda(0) + \int y'(x)\lambda'(x)\alpha(x)dx$.
  - $x \mapsto M_t^x \in Y^*$ for every $t \geq 0$ $\mathbb{P}$-a.s.
- Polynomial structure: $d[\langle y, M_t^x \rangle_\alpha, \langle y, M_t^x \rangle_\alpha] = \sum_{i=0}^2 \langle Q_i(y \otimes y), \lambda^{\otimes i} \rangle_\alpha dt$

Special case: rough polynomial variance models

- Consider an $E \subset \mathbb{R}$-valued polynomial Volterra process, i.e.

$$V_t = f(t) + \int_0^t K(t-s)\sqrt{C(V_s)}dB_s,$$

where f is a deterministic function, $C(v) = \Gamma v^2 + \gamma v + c$ for constants Γ, γ, c, B a Brownian motion and K a kernel in $L^2_{\text{loc}}(\mathbb{R}_+, \mathbb{R})$. In rough volatility models the kernel is typically fractional, i.e. $K(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $\alpha \in (\frac{1}{2}, 1)$.

- As in C. & Teichmann '18, lift the stochastic Volterra processes to Y^* and consider the following SPDE

$$d\lambda_t(x) = \frac{d}{dx}\lambda_t(x) + K(x)\sqrt{C(\langle \lambda_t, 1\rangle_\alpha)}dB_t,$$

  - Then $M_t^x = K(x)\sqrt{C(\langle \lambda_t, 1\rangle_\alpha)}B_t$.
  - $Q(y \otimes y; \lambda) = \frac{1}{2}\langle y, K\rangle_\alpha^2(\Gamma\langle \lambda, 1\rangle_\alpha^2 + \gamma\langle \lambda, 1\rangle_\alpha + c)$

Futures and options on VIX

- Define the VIX at time t via the continuous time monitoring formula

$$VIX_t^2 = \mathbb{E}\left[\frac{1}{\Delta}\int_0^{\Delta} V_{t+x}dx|\mathcal{F}_t\right] = \frac{1}{\Delta}\int_0^{\Delta}\lambda_t(x)dx$$

- The risk neutral valuation formula for an option on VIX with payoff φ (for the VIX future $\varphi(x) = \sqrt{x}$) is then

$$\mathbb{E}\left[\varphi(VIX_t^2)\right] = \mathbb{E}\left[\varphi(\frac{1}{\Delta}\int_0^{\Delta}\lambda_t(x)dx)\right].$$

- Since $\int_0^{\Delta}\lambda_t(x)dx = \langle y, \lambda_t\rangle_\alpha$ for some y, for all polynomials φ up to degree k, this expectation can be computed by solving a $k+1$ dimensional system of PDEs. $\Rightarrow$ Polynomial approximation for other payoffs.

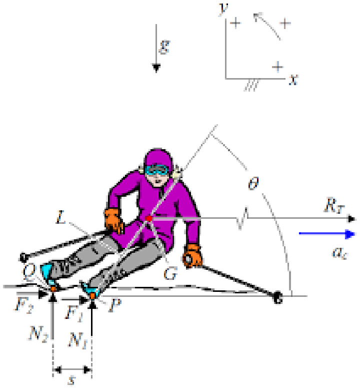
Conclusion

Framework of infinite dimensional polynomial processes for

- ...(probability) measure valued processes, e.g. for stochastic portfolio theory or population genetics;
- ...function valued processes, for term structure modeling, in particular forward variance models and rough volatility;
- ... interacting particle systems.

Thank you for your attention!

Thanks Yuri, for all your inspiring ideas, in particular for bringing each year skiing and mathematics together!



Happy birthday!