Utility Maximization in a Multi-dimensional Market with Small Nonlinear Price Impact

Thomas Cayé

Dublin City University, School of Mathematics
joint work (in progress) with Erhan Bayraktar
and Ibrahim Ekren

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Transaction Costs

- Classical paradigm: idealized frictionless market
- Trading on real financial markets induces costs:
  - Fixed costs (access to market, broker,...)
  - Proportional transaction costs (bid-ask spread, per unit fees)
  - Price impacts: large trades affects price adversely
- Trade-off between
  - Displacement from frictionless optimal strategy (strategy = choice of risk/return tradeoff)
  - Trading costs induced by frictions (price impacts)

  $\longrightarrow$ Solutions take different forms: impulse control, singular control, regular control.

How to Model Price impacts ?

- Price impacts increase with size and speed of trading
  - Model of Almgren ['03]: proportional to power α of trading rate
  - Buying dx_t during dt increases price by $\left|\frac{dx_t}{dt}\right|^\alpha$
    $\longrightarrow$ additional cost $\propto \left|\frac{dx_t}{dt}\right|^\alpha \frac{dx_t}{dt} dt$
    $\Longrightarrow$ Transaction costs produced by price impacts are proportional to power $\alpha + 1$ of trading rate.
- Linear price impacts: $\alpha = 1$
- Practitioners: square root law ($\alpha = \frac{1}{2}$), Almgren et al.['05]: $\alpha = 0.6$, Lillo et al.['03]: α between 0.2 and 0.5
- In multidimensional markets (several assets): possibility of cross-impacts
  $\longrightarrow$ α-homogeneous vector field

Overview of Recent Literature on Price Impact

- Optimal execution (Bertsimas & Lo['98], Almgren & Chriss['99,'01], Alfonsi, Fruth & Schied ['10], Predoiu, Shaiket & Shreve['11], Obizhaeva & Wang ['13],...)
- Linear price impact (quadratic transaction cost)
  - Portfolio choice in Black-Scholes (1 or multiple assets): Guasoni & Weber ['15, '16]
  - Portfolio choice in a factor model: Garleanu & Pedersen ['13], Collin-Dufresne et al.['14]
  - Option hedging in a Bachelier market: Bank, Soner & Voss['15], Almgren & T.M. Li ['15]
  - General Markovian Itô diffusion: Moreau, Muhle-Karbe & Soner ['14]

Overview of Recent Literature

- Nonlinear price impacts
  - Black-Scholes model with price impact proportional to a power of a volume-normalization of trading rate: Guasoni & Weber ['15]
  - Proportional and quadratic costs: Liu, Muhle-Karbe & Weber ['14]
  - Target tracking in general setting: Cai, Rosenbaum & Tankov ['15]
  - Utility Maximization for general one-dimensional Itô diffusion prices: C., Herdegen, Muhle-Karbe ['18]

What We Do

- Nonlinear price impact in a multi-dimensional market
  - Utility maximization of consumption over a finite horizon T
  - Investor with constant relative risk aversion (CRRA): power utility
  - Market with
    - A bank account with stochastic (local) rate
    - d risky assets following Markovian Itô dynamics
- Result
  - Expansion of the problem value function around the frictionless solution
  - Family of candidates for asymptotically optimal strategies

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Utility Maximization Without Frictions

- Market dynamics

$$\frac{dS_t^j}{S_t^j} = \mu^j(S_t)dt + \sigma^j(S_t) \cdot dB_t, \text{ for } 1 \leqslant j \leqslant d$$
$$dS_t^0 = S_t^0 r(S_t)dt$$

with μ, r, σ continuous and $\sigma\sigma^\top$ invertible.

- Wealth of the investor W^0 satisfies

$$dW_t^0 = (r_t W_t^0 - c_t)dt + \sum_{j=1}^{d} H_t^j S_t^j (\mu^j(S_t) - r(S_t))dt$$
$$+ \sum_{i,j=1}^{d} H_t^j S_t^j \sigma_i^j(S_t) dB_t^i,$$

Utility Maximization Without Frictions

- Investor chooses optimally $c \geqslant 0$ and H to maximize

$$\mathbb{E}\left[\int_0^T U(c_t)dt + U(c_T)\middle| W_0 = w, S_0 = s\right] \to \max\ !$$

  under the constraint $W^0 \geqslant 0$,
- Utility function $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$, $\gamma \in (0,\infty)\backslash\{1\}$
- Value function (state variables are: time t, wealth w and stock price s)

$$V^0(t,w,s) = \sup_{c,H} \mathbb{E}\left[\int_t^T U(c_t)dt + U(c_T)\middle| W_t = w, S_t = s\right]$$

- Admissible strategies $(c, H) \in \mathcal{A}^0$: $W^0 \geqslant 0$, $c^0 \geqslant 0$ on $[0, T]$

Utility Maximization Without Frictions

- V^0 satisfies the Hamilton-Jacobi-Bellman PDE:

$$\begin{cases} V^0_t - \tilde{U}(V^0_w) - \mathcal{L}^s V^0 - V^0_w rw \\ \qquad\qquad - \sup_h \left\{ Q(t, w, s, h, V^0_w, V^0_{ww}, V^0_{ws}) \right\} = 0 \\ V^0(T, w, s) = U(w), \end{cases}$$

- $\tilde{U}(y) = \sup_{x \in \mathbb{R}^d} \{U(x) - xy\} = \frac{R}{1-R} y^{1-\frac{1}{R}}$ convex conjugate,
- Q functional quadratic in h.

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Trading With Price Impact

- The investor now faces price impact: absolutely continuous strategy H^ε
$$dH_t^\varepsilon = \theta_t^\varepsilon dt$$
$\longrightarrow$ Execution price: $\tilde{S}_t^j = S_t^j + f^j(S_t, \varepsilon\theta_t)$
- f price impact function: homogeneous of degree α in the 2nd variable, $\alpha \in (0,1)$
$\longrightarrow$ cf. Guasoni & Weber['15], C., Herdegen, Muhle-Karbe ['18]:
$\tilde{S}_t = S_t + \lambda\Lambda_t \mathrm{sgn}(\theta_t)|\theta_t|^\alpha$.
- May depend on S, and trading speed in all assets θ^ε
- ε small parameter
- Impose that $\theta \cdot f(s, \varepsilon\theta) > 0$, for $\theta \neq 0$: positive cost

Trading With Price Impact

- Limiting cases:
  - For $\alpha \to 1 \longrightarrow$ Linear price impact: Guasoni & Weber['16], Moreau, Muhle-Karbe & Soner['17],
  - For $\alpha \to 0 \longrightarrow$ Proportional transaction costs: Possamai, Soner & Touzi ['13, '15]
- For $\alpha \in (0, 1)$, one-dimensional problem (with power utility) solved Guasoni & Weber, (with exponential utility) and C., Herdegen & Muhle-Karbe['18]

$\longrightarrow$ Multidimensional case is new.

Trading With Price Impact

- Marked-to-market wealth W^{ε} with dynamics

$$dW_t^{\varepsilon} = (r_t W_t^{\varepsilon} - c_t)dt + \sum_{j=1}^{d} H_t^j S_t^j((\mu_t^j - r_t)dt + \sigma_t^j dB_t)$$
$$- \sum_{j=1}^{d} \theta_t^j f^j(S_t, \varepsilon\theta_t)dt$$

- Need new state variables: vector of positions h
- Investor chooses optimally processes θ and $c \geqslant 0$ to maximize

$$\mathbb{E}\left[\int_0^T U(c_t)dt + U(c_T) \middle| W_0 = w, H_0 = h, S_0 = s\right] \rightarrow \text{ max!}$$

- Constraints: $W^{\varepsilon} \geqslant 0$, $c \geqslant 0$, and $\frac{H^j S^j}{W^{\varepsilon}} \in [0, 1]$, for all j

Value Function with Price Impact

- Value function now depends on position as well

$$V^{\varepsilon}(t,w,s,h) = \sup_{c,\theta} \mathbb{E}\left[\int_t^T U(c_s)ds + U(c_T) \middle| W_t = w, H_t = h, S_t = s\right]$$

- HJB equation much more involved: asymptotic expansion of value function $\longrightarrow$ ansatz for V^{ε}

$$\begin{aligned} V^{\varepsilon}(t,w,s,h) =& V^0(t,w,s) - \varepsilon^{2m^*} u(t,w,s) \\ & - \varepsilon^{4m^*} \varpi(t,w,s,\varepsilon^{-m^*}(h-h^0)) \end{aligned}$$

(similar to Soner & Touzi['13], and Moreau, Muhle-Karbe & Soner['17])
$\longrightarrow$ identify corrector equation satisfied by u and ϖ

Value Function With Price Impact

- Convex conjugate of the cost functional:

$$\Phi(s,x) = \sup_{\theta \in \mathbb{R}^d} \left\{ x \cdot \theta - \sum_{j=1}^{d} \theta_j f^j(s,\theta) \right\}$$

  Already showed up in Guasoni & Rásonyi['15], C., Herdegen & Muhle-Karbe['18]

- Appears in
  - HJB equation for V^ε
  - The first corrector equation governing ϖ
  - The candidates for asymptotically optimal policy (Φ_x)
- Is homogeneous of degree $m = 1 + \frac{1}{\alpha} > 2$ in the second variable.

Corrector Equations

- In HJB equation for ansatz we identify two corrector equations
  - Fast variable $\xi = \frac{h - h^0(t,w,s)}{\varepsilon^{m^*}}$: renormalized displacement, similar as in litterature (proportional, nonlinear & quadratic costs)
  - One equation for $(t, w, s, \xi) \mapsto \varpi(t, w, s, \xi)$

$$-\frac{V^0_{ww}}{2}\left\|\sum_{j=1}^{d}\xi_j s_j \sigma^j\right\|^2 - (V^0_w)^{1-m}\Phi\left(s, -\varpi_\xi\right) + \frac{1}{2}\mathrm{Tr}\left(c^{h^0}\varpi_{\xi\xi}\right) = a$$

$\longrightarrow$ Similar to stationary PDE obtained by Ichihara['12] in the context of ergodic control

  - One for u: $-E(t, w, s, u, \partial u, \partial^2 u) = a(t, w, s)$
    $\longrightarrow$ does not depend on position h !

First Corrector Equation

$$-\frac{V^0_{ww}}{2}\left\|\sum_{j=1}^{d}\xi_j s_j \sigma^j\right\|^2 - (V^0_w)^{1-m}\Phi\left(s, -\varpi_\xi\right) + \frac{1}{2}\mathrm{Tr}\left(c^{h^0}\varpi_{\xi\xi}\right) = a$$

- Unknown: $\xi \mapsto \varpi$ and a, for each (t, w, s).
  - Ichihara ['12]: unique solution (ϖ, a)!
  - Numerically, Cacace & Camilli ['16] provides Newton gradient-type methods to solve for both (ϖ, a) together.
- For fixed (t, w, s), V^0_w, V^0_{ww} and c^{h^0} are constants:
  $\longrightarrow$ shape of equation does not depend on utility function

First Corrector Equation: 1-Dimensional Case

$$-\frac{V_{ww}^0}{2}\left\|\sum_{j=1}^{d}\xi_j s_j \sigma^j\right\|^2 - (V_w^0)^{1-m}\Phi\left(s, -\varpi_\xi\right) + \frac{1}{2}\mathrm{Tr}\left(c^{h^0}\varpi_{\xi\xi}\right) = a$$

- One dimensional ODE of Guasoni & Weber ['15], and C., Herdegen & Muhle-Karbe is a particular case:

$$s'_\alpha(z) = \frac{\alpha}{(\alpha+1)^{1+\frac{1}{\alpha}}}|s_\alpha(z)|^{1+\frac{1}{\alpha}} - z^2 + c_\alpha$$

  - Unknown: s_α and c_α, unique given growth conditions
- Sign of s_α is the sign of z: rescaled displacement from frictionless optimizer
  $\longrightarrow$ Command the mean-reverting speed in function of the displacement.
- ϖ commands speed and direction (with Φ_x)

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Value Function Expansion

Theorem (Bayraktar, C., Ekren (2018))

Under some technical assumptions (frictionless problem, solutions to the various PDEs, growth conditions and limits) the value function V^ε satisfies

$$V^\varepsilon\left(t, w, s, h^0(t, w, s)\right) = V^0(t, w, s) - \varepsilon^{2m^*} u(t, w, s) + o(\varepsilon^{2m^*})$$

where $m^* = \frac{1}{3m-2}$ and u is a viscosity solution of

$$\begin{cases} -E(t, w, s, \partial u, \partial^2 u) & = a(t, w, s) \quad \text{on } [0, T) \times \left(\mathbb{R}_+^*\right)^{d+1} \\ u(T, w, s) & = 0 \qquad\qquad\quad \text{on } \mathbb{R}_+^* \times \left(\mathbb{R}_+^*\right)^{d} \end{cases}$$

Asymptotically Optimal Strategy

Theorem (Bayraktar, C., Ekren (2018), continued)

A family of asymptotically optimal strategies is given by

$$\theta_t^\varepsilon := \frac{\left(V_w^0(t, W_t^\varepsilon, S_t)\right)^{1-m}}{\varepsilon^{m^*}} \Phi_x\left(S_t, -\varpi_\xi\left(t, W_t^\varepsilon, S_t, \frac{H_t^\varepsilon - h_t^0}{\varepsilon^{m^*}}\right)\right).$$

where ϖ is the unique solution to the first corrector equation.

Discussion

- Leading order coincides with litterature: $\varepsilon^{\alpha} = \lambda$ of Guasoni & Weber and C., Herdegen & Muhle-Karbe

$$\varepsilon^{2m^*} = \lambda^{\frac{2}{\alpha+3}}.$$

- Candidate strategies: driven by ϖ_ξ, multi-dimensional version of equation found in the one-dimensional models
- Form of first corrector equation does not depend on utility of investor
  $\longrightarrow$ universal to nonlinear price impact problems

Discussion

- The functions Φ_x and ϖ_ξ give the direction of trading
  $\longrightarrow$ frictionless strategy tracked according to $\Phi_x(S, -\varpi_\xi(\cdot))$
  - In one dimension: trade towards frictionless optimizer
    $\longrightarrow$ mean-reversion of displacement
  - In d-dimensional market, it is not true coordinate-wise unless $\Phi_x\left(s, -\varpi_\xi\left(\left(\sigma\sigma^\top\right)^{-\frac{1}{2}}\cdot\right)\right)$ can be separated.
    $\longrightarrow$ cf. Garleanu & Pedersen ['16], Guasoni & Weber ['16] (*principal portfolios*)

- In d-dimensional market:
  $\longrightarrow \varpi(t, W^\varepsilon, S, \frac{H^\varepsilon - H^0}{\varepsilon^{m^*}})$ is tracking 0: not same speed in every direction
- Counter-intuitive situation: investor might trade in the "wrong" direction for a given asset !

Discussion

- Static properties of u with respect to t, w and s difficult to assess
  $\longrightarrow$ Toolbox of invariant distributions for 1-dim diffusions can't be used !
- Feynman-Kac representation

$$u(t, w, s) = \mathbb{E}\left[\int_t^T a(s, W_s^\varepsilon, S_t)ds \mid W_t^0 = w, S_t = s\right]$$

$\longrightarrow$ Need to know a: simple model or numerical analysis.

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Assumptions

- Frictionless problem has a solution: V^0, h^0 and c^{h^0} are regular enough functions.
- V^0, V^ε are viscosity solutions of their respective HJB equations
- First corrector equation has classical $\mathcal{C}^2$ solution (ϖ, a) (regular in (t, w, s) as well !)
- Second corrector equation admits a viscosity solution, and satisfies comparison for a class of functions (cf. Moreau, Muhle-Karbe & Soner)
- Rescaled deviation of value function locally uniformly bounded: for all $(t^0, w^0, s^0) \in$ Domain

  $$0 \leqslant \frac{V^0(t,w,s) - V^\varepsilon(t,w,s,h)}{\varepsilon^{2m^*}} \leqslant C$$

  for $0 < \varepsilon < \varepsilon_0$, and $(t,w,s,h) \in B_{r_0}(t^0, w^0, s^0, h^0(t^0, w^0, s^0))$
- Integrability conditions (asymptotic optimality of candidate)

The end

Thank you for your attention.

Questions ?