APPROXIMATION AND CALIBRATION OF LAWS OF SOLUTIONS TO STOCHASTIC DIFFERENTIAL EQUATIONS

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Outline

Introduction

Our motivation comes from the following problem:
Assume that you have an exact diffusion model with high complexity coefficients.
How to select a simplified diffusion model within a class of models under the constraint that the probability distribution of the exact model is preserved as much as possible?
We need to consider a distance which metrizes the weak topology. The Wasserstein distance $\mathcal{W}^2$ metrizes the weak topology on the set of probability measures on square integrable paths. Unfortunately numerical computation of $\mathcal{W}^2$ on an infinite dimensional space is not possible.

INTRODUCTION

We introduce A NEW DISTANCE on the set of probability distributions of strong solutions to stochastic differential equations. This new distance is defined by RESTRICTING THE SET OF POSSIBLE COUPLING MEASURES.

Like the classical Wasserstein distance, THIS NEW DISTANCE METRIZES THE WEAK TOPOLOGY.

This new distance is the value function of a stochastic control problem.

Classical results do not apply to the corresponding Hamilton-Jacobi-Bellman equation especially because the differential operator is degenerate.

Nevertheless we prove that THIS H.J.B. EQUATION HAS A SMOOTH SOLUTION.

We finally exhibit an optimal coupling measure.

Outline

Definition

$$\mathbf{P} = \{\mathbb{P}_x^{\mu,\sigma}, \mu, \sigma \text{ Lipschitz}, \sigma \text{ uniformly strongly elliptic}, \ x \in \boldsymbol{R}^d\}$$

where $\mathbb{P}_x^{\mu,\sigma}$ is the probability distribution of the unique strong solution to the stochastic differential equation with coefficients μ and σ and initial condition x.

$$X_t = x + \int_0^t \mu(X_s)\, ds + \sigma(X_s)\, dW_s$$

Set of coupling measures

Definition

Given two probability measures $\mathbb{P}_x^{\mu,\sigma}$ and $\mathbb{P}_{\bar{x}}^{\overline{\mu},\overline{\sigma}}$ belonging to $\mathbf{P}$, let $\widetilde{\Pi}(\mathbb{P}_x^{\mu,\sigma};\mathbb{P}_{\bar{x}}^{\overline{\mu},\overline{\sigma}})$ be the set of the probability law $\widetilde{\mathbb{P}}$ of $(X^{\mathcal{C}},\overline{X})$ solution to the following system of SDEs:

$$\begin{cases} dX_s^{\mathcal{C}} = \mu(X_s^{\mathcal{C}})\, ds + \sigma(X_s^{\mathcal{C}})\, (\mathcal{C}_s\, d\overline{W}_s + \mathcal{D}_s\, dW_s), \\ d\overline{X}_s = \overline{\mu}(\overline{X}_s)\, ds + \overline{\sigma}(\overline{X}_s)\, d\overline{W}_s, \end{cases} \tag{1}$$

with initial condition $(x,\bar{x})$, with (C_s) predictable with values in correlation matrices and $\mathcal{D}_s = \sqrt{\mathrm{Id}_d - \mathcal{C}_s\,\mathcal{C}_s^\intercal}$ for any $0 \le s$.

set of d-dimensional correlation matrices:

$$\mathbf{C}_d := \{C \in \mathbf{M}_d;\ \text{there exist } \boldsymbol{R}^d \text{ valued centered random variables } X \text{ and } Y \text{ s.t.}\\ \mathbb{E}(XX^\intercal) = \mathbb{E}(YY^\intercal) = \mathrm{Id}_d,\ \ C = \mathbb{E}(XY^\intercal)\}.$$

Definition of $\widetilde{\mathcal{W}}^2$

Definition

$$\widetilde{\mathcal{W}}^2(\mathbb{P}_x^{\mu,\sigma};\mathbb{P}_{\overline{x}}^{\overline{\mu},\overline{\sigma}}) := \left\{ \inf_{\widetilde{\mathbb{P}} \in \widetilde{\Pi}(\mathbb{P}_x^{\mu,\sigma};\mathbb{P}_{\overline{x}}^{\overline{\mu},\overline{\sigma}})} \int_\Omega \int_0^T |\omega_s - \overline{\omega}_s|^2 \, ds \, \widetilde{\mathbb{P}}(d\omega, d\overline{\omega}) \right\}^{\frac{1}{2}}. \quad (2)$$

Proposition

$\widetilde{\mathcal{W}}^2$ metrizes the weak topology on the set of probability distributions $\mathbb{P}_x^{\mu,\sigma}$ with coefficients μ and σ uniformly Lipschitz bounded, σ uniformly strictly elliptic and x in a compact set.

Note that $\mathcal{W}^2 \leq \widetilde{\mathcal{W}}^2$.

OUTLINE

INTERPRETATION OF $\widetilde{\mathcal{W}}^2$

This section is aimed to prove
- that the value of $\widetilde{\mathcal{W}}^2(\mathbb{P}_x^{\mu,\sigma};\mathbb{P}_{\overline{x}}^{\overline{\mu},\overline{\sigma}})$ can be obtained by solving a stochastic control problem.
- and that the corresponding HJB equation admits a regular solution.

For $0 \leq t \leq T$, let $\mathbf{Ad(t,T)}$ denote the set of processes $(C_\theta)_{t\leq\theta\leq T}$, predictable with respect to the filtration generated by $((W_u - W_t), (\overline{W}_u - \overline{W}_t))_{t\leq u\leq T}$ which take values in the set of correlation matrices.

INTERPRETATION OF $\widetilde{\mathcal{W}}^2$

There exists a unique strong solution $(X^{\mathcal{C}}_\theta(t,x,\overline{x}), \overline{X}_\theta(t,x))$ to

$$\forall t \leq \theta \leq T, \begin{cases} X^{\mathcal{C}}_\theta = x + \int_t^\theta \mu(X^{\mathcal{C}}_s)\, ds + \int_t^\theta \sigma(X^{\mathcal{C}}_s)\, (\mathcal{C}_s\, d\overline{W}_s + \mathcal{D}_s\, dW_s), \\ \overline{X}_\theta = \overline{x} + \int_t^\theta \overline{\mu}(\overline{X}_s)\, ds + \int_t^\theta \overline{\sigma}(\overline{X}_s)\, d\overline{W}_s, \end{cases} \tag{3}$$

where $\mathcal{D}_s\, \mathcal{D}_s^\intercal + \mathcal{C}_s\, \mathcal{C}_s^\intercal = \mathrm{Id}_d$ for any $t \leq s \leq T$.
We consider the objective function

$$\min_{(\mathcal{C}_\theta) \in \mathbf{Ad(t,T)}} \mathbb{E} \int_t^T |X^{\mathcal{C}}_\theta(t,x,\overline{x}) - \overline{X}_\theta(t,\overline{x})|^2 d\theta. \tag{4}$$

INTERPRETATION OF $\widetilde{\mathcal{W}}^2$

The corresponding Hamilton–Jacobi–Bellman equation is the following:

$$\begin{cases} \partial_t V(t,x,\bar{x}) + \mathcal{L}V(t,x,\bar{x}) + H(t,x,\bar{x},V) = -|x-\bar{x}|^2, \; 0 \leq t < T, \\ V(T,x,\bar{x}) = 0, \end{cases} \tag{5}$$

$$\mathcal{L}V(t,x,\bar{x}) := \sum_{i=1}^{d} \mu_i(x)\partial_{x_i}V(t,x,\bar{x}) + \sum_{i=1}^{d} \overline{\mu}_i(\bar{x})\partial_{\bar{x}_i}V(t,x,\bar{x})$$

$$+ \frac{1}{2}\sum_{i,j=1}^{d} (\sigma\sigma^{\intercal}(x))^{ij}\partial^2_{x_i,x_j}V(t,x,\bar{x}) + \frac{1}{2}\sum_{i,j=1}^{d} (\overline{\sigma}(\bar{x})\overline{\sigma}(\bar{x})^{\intercal})^{ij}\partial^2_{\bar{x}_i,\bar{x}_j}V(t,x,\bar{x})$$

$$\text{and} \;\; H(t,x,\bar{x},V) := \min_{C \in \mathbf{C}_d} \sum_{i,j=1}^{d} (\sigma(x)C\overline{\sigma}(\bar{x})^{\intercal})^{ij}\partial^2_{x_i,\bar{x}_j}V(t,x,\bar{x}).$$

One dimensional case

We choose $\sigma,\ \overline{\sigma} \geq 0$. Consider the family of stochastic differential equations

$$\forall t \leq \theta \leq T, \qquad \begin{aligned} X^*_\theta &= x + \int_t^\theta \mu(X^*_s)\,ds + \int_t^\theta \sigma(X^*_s)\,d\overline{W}_s. \\ \overline{X}_\theta &= \overline{x} + \int_t^\theta \overline{\mu}(\overline{X}_s)\,ds + \int_t^\theta \overline{\sigma}(\overline{X}_s)\,d\overline{W}_s. \end{aligned}$$

consider the function

$$V^*(t,x,\overline{x}) := \mathbb{E}\int_t^T \left(X^*_\theta(t,x) - \overline{X}_\theta(t,\overline{x})\right)^2 d\theta.$$

This function is the unique classical solution to the parabolic PDE

$$\begin{cases} \frac{\partial V^*}{\partial t} + \mu(x)\,\frac{\partial V^*}{\partial x} + \overline{\mu}(\overline{x})\,\frac{\partial V^*}{\partial \overline{x}} + \frac{1}{2}\sigma^2(x)\,\frac{\partial^2 V^*}{\partial x^2} + \frac{1}{2}\overline{\sigma}^2(\overline{x})\,\frac{\partial^2 V^*}{\partial \overline{x}^2} \\ \qquad + \sigma(x)\overline{\sigma}(\overline{x})\frac{\partial^2 V^*}{\partial x \partial \overline{x}} = -(x-\overline{x})^2, \\ V^*(T,x,\overline{x}) = 0. \end{cases}$$

One dimensional case

$$\frac{\partial^2 V^*}{\partial x \partial \overline{x}}(t,x,\overline{x}) = -2\int_t^T \mathbb{E}\left[\frac{d}{dx}X_s^*(t,x)\ \frac{d}{d\overline{x}}\overline{X}_s(\overline{x})\right]\ ds.$$

The derivative of the flow is a stochastic exponential. It follows that

$$\forall t, x, \overline{x},\ \ \frac{\partial^2 V^*}{\partial x \partial \overline{x}}(t,x,\overline{x}) < 0.$$

Therefore we have exhibited a classical solution $V^*(t,x,\overline{x})$ to the Hamilton-Jacobi-Bellman equation

$$\begin{cases} \frac{\partial V}{\partial t} + \mu(x)\ \frac{\partial V}{\partial x} + \overline{\mu}(\overline{x})\ \frac{\partial V}{\partial \overline{x}} + \frac{1}{2}\sigma^2(x)\ \frac{\partial^2 V}{\partial x^2} + \frac{1}{2}\overline{\sigma}^2(\overline{x})\ \frac{\partial^2 V}{\partial \overline{x}^2} \\ \qquad + \min_{C\in[-1,1]}\left(C\,\sigma(x)\ \overline{\sigma}(\overline{x})\ \frac{\partial^2 V}{\partial x \partial \overline{x}}\right) = -(x-\overline{x})^2, \\ V(T,x,\overline{x}) = 0. \end{cases}$$

One dimensional case

note that the differential operator $\mathcal{L}V + H(V)$ associated to his H.J.B. equation is

$$Tr\left(\begin{pmatrix} \sigma^2(x) & \sigma(x)\overline{\sigma}(\overline{x}) \\ \sigma(x)\overline{\sigma}(\overline{x}) & \overline{\sigma}^2(\overline{x}) \end{pmatrix}\begin{pmatrix} \frac{\partial^2 V}{\partial x^2} & \frac{\partial^2 V}{\partial x \partial \overline{x}} \\ \frac{\partial^2 V}{\partial x \partial \overline{x}} & \frac{\partial^2 V}{\partial \overline{x}^2} \end{pmatrix}\right)$$

Thus this differential operator is degenerate.

HÖLDER SPACE

For any $T > 0$ and $0 < \alpha < 1$ the Hölder space $\mathcal{C}^{0,\alpha}([0,T] \times \boldsymbol{R}^d)$ is the set of bounded continuous functions such that $\phi(t,\cdot)$ is Hölder continuous of order α for every t in $[0,T]$ equipped with the norm

$$\|\phi\|_{0,\alpha} := \sup_{t\in[0,T]} \|\phi(t,\cdot)\|_\alpha := \sup_{t\in[0,T]} \left(\|\phi(t,\cdot)\|_\infty + \sup_{x,y\in \boldsymbol{R}^d, x\neq y} \frac{|\phi(t,x)-\phi(t,y)|}{|x-y|^\alpha} \right).$$

The Hölder space $\mathcal{C}^{\frac{\alpha}{2},0}([0,T] \times \boldsymbol{R}^d)$ is defined similarly, and

$$\mathcal{C}^{\frac{\alpha}{2},\alpha}([0,T] \times \boldsymbol{R}^d) := \mathcal{C}^{\frac{\alpha}{2},0}([0,T] \times \boldsymbol{R}^d) \bigcap \mathcal{C}^{0,\alpha}([0,T] \times \boldsymbol{R}^d)$$

Finally, $\mathcal{C}^{1+\frac{\alpha}{2},2+\alpha}([0,T] \times \boldsymbol{R}^d)$ is the set of continuous bounded functions ϕ of class $\mathcal{C}^{1,2}([0,T] \times \boldsymbol{R}^d)$ with bounded derivatives such that $\partial_t\phi$ and $\partial^2_{x_i,x_j}\phi$ are in $\mathcal{C}^{\frac{\alpha}{2},\alpha}([0,T] \times \boldsymbol{R}^d)$ for all $1 \le i,j \le d$. We equip this space with the norm

$$\|\phi\|_{1+\frac{\alpha}{2},2+\alpha} := \|\phi\|_\infty + \sum_{i=1}^d \|\partial_{x_i}\phi\|_\infty + \|\partial_t\phi\|_{\frac{\alpha}{2},\alpha} + \sum_{i,j=1}^d \|\partial^2_{x_i,x_j}\phi\|_{\frac{\alpha}{2},\alpha}.$$

Multidimensional case: Existence of a regular solution to the HJB equation

Theorem

Suppose:

(I) *The functions μ, $\overline{\mu}$, σ and $\overline{\sigma}$ are in the Hölder space $\mathcal{C}^\alpha(\boldsymbol{R}^d)$ with* $0 < \alpha < 1$.

(II) *The matrix-valued functions* $a(x) := \sigma(x)\sigma(x)^\intercal$ *and* $\overline{a}(x) := \overline{\sigma}(x)\overline{\sigma}(x)^\intercal$ *satisfy the strong ellipticity condition*

$$\exists \lambda > 0,\ \forall \xi,\ \overline{\xi},\ x,\ \sum_{i,j=1}^{d} a^{ij}(x)\xi^i\xi^j + \sum_{i,j=1}^{d} \overline{a}^{ij}(x)\overline{\xi}^i\overline{\xi}^j \geq \lambda|(\xi|^2 + |\overline{\xi}|^2).$$

Then there exists a solution V to the HJB equation (5), $V \in \mathcal{C}^{1,2}([0,T] \times \boldsymbol{R}^d)$ *such that* $\frac{V(t,x,\overline{x})}{1+|x|^2+|\overline{x}|^2}$ *is in* $\mathcal{C}^{1+\frac{\alpha}{2},2+\alpha}([0,T] \times \boldsymbol{R}^d)$

Multidimensional case: Existence of a regular solution to the HJB equation

Sketch of proof:
Recall that we want to solve the Hamilton–Jacobi–Bellman equation:

$$\begin{cases} \partial_t V(t,x,\bar{x}) + \mathcal{L}V(t,x,\bar{x}) + H(t,x,\bar{x},V) = -|x-\bar{x}|^2, \ 0 \le t < T, \\ V(T,x,\bar{x}) = 0, \end{cases}$$

$$\mathcal{L}V(t,x,\bar{x}) := \sum_{i=1}^{d} \mu_i(x)\partial_{x_i}V(t,x,\bar{x}) + \sum_{i=1}^{d} \overline{\mu}_i(\bar{x})\partial_{\bar{x}_i}V(t,x,\bar{x})$$
$$+ \frac{1}{2}\sum_{i,j=1}^{d} (\sigma\sigma^{\intercal}(x))^{ij}\partial^2_{x_i,x_j}V(t,x,\bar{x}) + \frac{1}{2}\sum_{i,j=1}^{d} (\overline{\sigma}(\bar{x})\overline{\sigma}(\bar{x})^{\intercal})^{ij}\partial^2_{\bar{x}_i,\bar{x}_j}V(t,x,\bar{x})$$

$$\text{and } \ H(t,x,\bar{x},V) := \min_{C \in \mathbf{C}_d} \sum_{i,j=1}^{d} (\sigma(x)C\overline{\sigma}(\bar{x})^{\intercal})^{ij}\partial^2_{x_i,\bar{x}_j}V(t,x,\bar{x}).$$

Multidimensional case: Existence of a regular solution to the HJB equation

Let v be a function such that $\frac{v(t,x,\bar{x})}{1+|x|^2+|\bar{x}|^2}$ belongs to $\mathcal{C}^{1+\frac{\alpha}{2},2+\alpha}([0,T]\times \boldsymbol{R}^{2d})$.
Let $\widetilde{v}(t,x,\bar{x}) := \frac{v(T-t,x,\bar{x})}{1+|x|^2+|\bar{x}|^2}$
This leads to consider the following PDE

$$\begin{cases}\partial_t\widetilde{V}(t,x,\bar{x}) + \widetilde{\mathcal{L}}\widetilde{V}(t,x,\bar{x}) + \frac{H(t,x,\bar{x},(1+|x|^2+|\bar{x}|^2)\widetilde{v})}{1+|x|^2+|\bar{x}|^2} = -\frac{|x-\bar{x}|^2}{1+|x|^2+|\bar{x}|^2},\ 0\leq t<T,\\ \widetilde{V}(0,x,\bar{x}) = 0,\end{cases} \tag{6}$$

Multidimensional case: Existence of a regular solution to the HJB equation

Consider the set

$$\mathbb{K} := \{\widetilde{g} \in \mathcal{C}^{1+\frac{\alpha}{2},2+\alpha}([0,T] \times \boldsymbol{R}^d); \|\widetilde{g}\|_{1+\frac{\alpha}{2},2+\alpha}(t) \leq \widetilde{C}_0 \left(\exp(\widetilde{K}(T)t) - 1\right)\}.$$

Let $\widetilde{v}$ belong to $\mathbb{K}$. Denote $\widetilde{\psi}(\widetilde{v})$ the unique solution to the above system.
We prove that $\widetilde{\psi}$ defines a continuous map on $\mathbb{K}$ with values on $\mathbb{K}$.
Furthermore we prove that $\mathbb{K}$ is convex and compact.
We can then conclude by applying the Kakutani-Fan-Glicksberg fixed point theorem.

Multidimensional case: Existence of a regular solution to the HJB equation

Theorem

Assume that the functions μ, $\overline{\mu}$, σ and $\overline{\sigma}$ are in the Hölder space $\mathcal{C}^{\alpha}(\boldsymbol{R}^d)$ with $0 < \alpha < 1$.
Assume that $\sigma(x)\sigma(x)^{\intercal}$ and $\overline{\sigma}(x)\overline{\sigma}(x)^{\intercal}$ are uniformly strongly elliptic.
Then there exists a regular solution V to the HJB equation:

$$\begin{cases} \partial_t V(t,x,\bar{x}) + \mathcal{L}V(t,x,\bar{x}) + H(t,x,\bar{x},V) = -|x-\bar{x}|^2,\ 0 \leq t < T, \\ V(T,x,\bar{x}) = 0, \end{cases}$$

$V \in \mathcal{C}^{1,2}([0,T] \times \boldsymbol{R}^d)$ *such that* $\frac{V(t,x,\bar{x})}{1+|x|^2+|\bar{x}|^2}$ *is in* $\mathcal{C}^{1+\frac{\alpha}{2},2+\alpha}([0,T] \times \boldsymbol{R}^d)$

$$H(t,x,\bar{x},V) := \min_{C \in \mathbf{C}_d} \sum_{i,j=1}^{d} (\sigma(x) C \overline{\sigma}(\bar{x})^{\intercal})^{ij} \partial^2_{x_i,\bar{x}_j} V(t,x,\bar{x}).$$

Approximation among diffusion laws

Making use of Michael Selection Theorem, we can prove that

Proposition

For every $n \in N$ there is a continuous map $C_n(s,x,\bar{x})$ taking values in the set of correlation matrices which is $\frac{1}{n}$-optimal: $\forall s,x,\bar{x}$,

$$H(t,x,\bar{x},V) \leq \sum_{i,j=1}^{d} (\sigma(x)C_n(s,x,\bar{x})\overline{\sigma}(\bar{x})^{\mathsf{T}})^{ij}\partial^2_{x_i,\bar{x}_j}V(t,x,\bar{x}) \leq H(t,x,\bar{x},V) + \frac{1}{n}.$$

Recall that $H(t,x,\bar{x},V) = \min_{C\in\mathbf{C}_d} \sum_{i,j=1}^{d} (\sigma(x)C\overline{\sigma}(\bar{x})^{\mathsf{T}})^{ij}\partial^2_{x_i,\bar{x}_j}V(t,x,\bar{x})$.

Approximation among diffusion laws

Proposition

There exists a sequence $(\mathbb{P}^n)$ of solutions to martingale problems with continuous Markovian coefficients such that

$$\int_\Omega \int_0^T |\omega_s - \overline{\omega}_s|^2 \, ds \, \mathbb{P}^n(d\omega, d\overline{\omega})$$

converges to $(\widetilde{\mathcal{W}}^2(\mathbb{P}^{\mu,\sigma}; \mathbb{P}^{\overline{\mu},\overline{\sigma}}))^2$. P_n is the law of $(Y^n, \overline{Y}^n)$ weak solution to

$$\begin{cases} Y^n_\theta = x + \int_0^\theta \mu(Y^n_s)\, ds + \int_0^\theta \sigma(Y^n_s)\, [C^n(s, Y^n_s, \overline{Y}_s)\, d\overline{B}_s + D^n(s, Y^n_s, \overline{Y}_s)\, dB_s], \\ \overline{Y}^n_\theta = \overline{x} + \int_0^\theta \overline{\mu}(\overline{Y}^n_s)\, ds + \int_0^\theta \overline{\sigma}(\overline{Y}^n_s)\, d\overline{B}_s \end{cases}$$

Existence of an optimal coupling measure

Making use of Skorokhod results, we prove the existence of a process $\xi := (X^*, \overline{X}, \overline{W}, W)$ and a sequence $\xi^n := (X^n, \overline{X}^n, \overline{W}^n, W^n)$ defined on some probability space such that

- The finite-dimensional distributions of ξ^n coincide with the corresponding finite-dimensional distributions of $(Y^n, \overline{Y}, \overline{B}, B)$,

- and ξ^n_θ converges in probability to ξ_θ for every $\theta \geq 0$.

Existence of an optimal coupling measure

Theorem

There exist a predictable process C^* and an adapted and continuous solution on $[0,T]$ to the system

$$\begin{cases} X_t^* = x + \int_0^t \mu(X_s^*)\, ds + \int_0^t \sigma(X_s^*)\, C_s^*\, d\overline{W}_s + \int_0^t \sigma(X_s^*)\, D_s^*\, dW_s, \\ \overline{X}_t = \bar{x} + \int_0^t \overline{\mu}(\overline{X}_s)\, ds + \int_0^t \overline{\sigma}(\overline{X}_s)\, d\overline{W}_s, \\ C_s^* \in \underset{C \in \mathbf{C}_d}{\arg\min} \sum_{i,j=1}^d \left(\sigma(X_s^*)\, C\, \overline{\sigma}(\overline{X}_s)^T\right)^{ij} \partial^2_{x_i \bar{x}_j} V(s, X_s^*, \overline{X}_s)\right), \\ D_s^* = \sqrt{\mathrm{Id}_d - C_s^*\, C_s^{*\mathsf{T}}} \end{cases}$$

which satisfies

$$V(0,x,\bar{x}) = \mathbb{E} \int_0^T |X_t^* - \overline{X}_t|^2 dt = (\widetilde{\mathcal{W}}^2(\mathbb{P}_x^{\mu,\sigma}; \mathbb{P}_{\bar{x}}^{\overline{\mu},\overline{\sigma}}))^2.$$

Conclusion

We have introduced a Wasserstein type distance $\widetilde{\mathcal{W}}^2$ on the set of strong solutions to stochastic differential equations. This distance is defined by restricting the set of coupling probability measures to laws of stochastic differential equations.

We have proved that this distance metrizes the weak topology and that it is the value function of a stochastic control problem.

This value function is the regular solution of a H.J.B. equation.

We have constructed an optimal coupling measure. This optimal coupling measure can be approximated by diffusion laws with continuous coefficients.

THANK YOU FOR YOUR ATTENTION