Discounting invariant FTAP for large financial markets

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in honour of Yuri Kabanov

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Outline

Introduction

Large market

Definition

A large market is a sequence $(\mathbb{B}^n, \mathbf{S}^n, T^n)_n$ *of small markets, where a small market* $(\mathbb{B}, \mathbf{S}, T)$ *consists of:*

- $\mathbb{B} = (\Omega, \mathcal{F}, \mathbb{F}, P)$ *filtered probability space (with usual conditions),*
- $\mathbf{S} = (\mathbf{S}_t) = (\mathbf{S}_t^{(1)}, \dots, \mathbf{S}_t^{(N)})$ *modelling asset price process (with* $\mathbf{S} \geq 0$ *and* $\mathbf{1} \cdot \mathbf{S} > 0$, $\mathbf{1} \cdot \mathbf{S}_- > 0$*),*
- T *general,* $[0, \infty]$*-valued stopping time.*

We write:

- N^n number of assets in the nth market.
- $\vec{\vartheta} = (\vartheta^n)_n$ large strategy, where each $\vartheta^n \in L_+^{sf}(\mathbf{S}^n)$ meaning

$$V(\vartheta^n)[\mathbf{S}^n] := \vartheta^n \cdot \mathbf{S}^n = \vartheta_0^n \cdot \mathbf{S}_0^n + \int \vartheta^n d\mathbf{S}^n \geq 0 \qquad P\text{-a.s.}$$

- $\vec{0} := (0^n)_n$.

Large markets - overview

Origin: *Kabanov and Kramkov '94*.

Application: financial markets with infinitely many assets, such as bond markets (with maturities e.g. in an infinite set).

Developments:

- Absence of arbitrage "weak notions" ("NA1/NUPBR-type"): *Kabanov and Kramkov, Klein and Schachermayer*, later *Rokhlin*.
- Absence of arbitrage "strong notions" ("NFL-type"): *Klein* in more consecutive papers, recently *Cuchiero, Klein, Teichmann*.
- Utility maximization: *De Donno, Guasoni, Pratelli*, recently *Rásonyi*.
- Hedging and pricing: *Baran*, recently *Roch*.
- Transaction costs, Heston-type models, etc.

Absence of arbitrage - classical concept

The following definition is due to *Kabanov & Kramkov '94*.

Definition

A large market $(\mathbb{B}^n, \mathbf{S}^n, T^n)_n$ *with* $T^n < \infty$ P^n*-a.s. for each* n *admits no asymptotic arbitrage (NAA) if for any large strategy* $\vec{\vartheta}$ *satisfying* $\lim_n V_0(\vartheta^n)[\mathbf{S}^n] = 0$, *we have* $\limsup_n P^n[V_{T^n}(\vartheta^n)[\mathbf{S}^n] \geq 1] = 0$.

Drawbacks of NAA:

- Not stable with respect to discounting (see next slide).
- Time horizon in each small market restricted to be finite $T^n < \infty$.
- Does not imply any absence of arbitrage property for the small markets.

Black Scholes - unstability of NAA

Consider:

$$Y_t^{(1)} = e^{rt},$$
$$Y_t^{(2)} = e^{mt+\sigma W_t - \frac{1}{2}\sigma^2 t},$$

where $r \in \mathbb{R}$, $m \in \mathbb{R}$, $\sigma > 0$ constants and W one-dimensional BM.

Then the following are two discounting possibilities:

- by $Y^{(1)}$: $\mathbf{S} := (1, X) := Y / Y^{(1)}$,
- by $Y^{(2)}$: $\mathbf{S}' := (X', 1) := Y / Y^{(2)}$.

Consider the large markets (for an adequate stochastic basis $\mathbb{B}$):

- $\mathcal{L} := (\mathbb{B}, \mathbf{S}_{.\wedge n}, n)_n$,
- $\mathcal{L}' := (\mathbb{B}, \mathbf{S}'_{.\wedge n}, n)_n$.

Then $\mathcal{L}$ and $\mathcal{L}'$ should have "the same economic properties".

Black Scholes (cont.)

Lemma

Let a large financial market of the form $(\mathbb{B}, (1, Z_{.\wedge n}), n)_n$ *for a fixed semimartingale* Z*. Then* $(\mathbb{B}, (1, Z_{.\wedge n}), n)_n$ *satisfies NAA if and only if* $(\mathbb{B}, (1, Z), \infty)$ *satisfies NUPBR (i.e. set of final wealths is bounded in* L^0*).*

Note $X' = 1/X$ and calculate:

$$X := \frac{Y^{(2)}}{Y^{(1)}} = e^{(m-r)t+\sigma W_t - \frac{1}{2}\sigma^2 t}$$

- Set for example $m = r$, then $X \to 0$ and $X' \to \infty$.
- Then clearly $(\mathbb{B}, (X', 1), \infty)$ does not satisfy NUPBR.
- On the other hand, X is a local-martingale and hence $(\mathbb{B}, (1, X), \infty)$ satisfies NUPBR (due to e.g. *Delbaen & Schachermayer '94*).
- Hence, by the above lemma, $\mathcal{L}$ satisfies NAA but $\mathcal{L}'$ does not.

Small markets - a short recall

Deflators - small markets

Define:

$$\mathcal{X}(\mathbf{S}) := \{V(\vartheta)[\mathbf{S}] : \vartheta \in L_+^{sf}(\mathbf{S})\}$$

Definition

- *A local-martingale deflator (LMD) D is a semimartingale satisfying $D > 0$, $D_- > 0$ and $D_0 = 1$, and s. t. X/D is a local-martingale for each $X \in \mathcal{X}(\mathbf{S})$.*
- *We write LMD^+, if in addition $\liminf_{t \to T}(\mathbf{1} \cdot \mathbf{S}_t)/D_t > 0$ P-a.s..*
- *An LMD (or LMD^+) D is tradable, if $D \in \mathcal{X}(\mathbf{S})$.*
- *Super-martingale deflators: analogously. We write SMD, SMD^+ accordingly.*

Absence of arbitrage - small markets

Definition

A strategy $\vartheta \in L_+^{sf}$ *is* *index weight maximal* *if* $\nexists \psi = (\psi_t)$ *adapted, converging* P*-a.s. to some* $\psi_\infty \in L_+^0 \setminus \{0\}$ *such that* $\forall \epsilon > 0 \ \exists \hat{\vartheta} \in L_+^{sf}$ *with* $V_0(\hat{\vartheta})[\mathbf{S}] \leq V_0(\vartheta)[\mathbf{S}] + \epsilon$

$$\liminf_{t \to T} (\hat{\vartheta}_t - \vartheta_t - \psi_t \mathbf{1}) \geq 0 \qquad P\text{-a.s.}$$

Definition

A small market $(\mathbb{B}, \mathbf{S}, T)$ *is* *dynamic index weight viable (DIWV), if the* $0 \in L_+^{sf}$ *index weight maximal.*

- DIWV is stable with respect to discounting ("$\mathbf{S} \leftrightarrow S := \mathbf{S}/D$").
- DIWV is strictly weaker than NUPBR in models with a riskless asset.

FTAP results - small markets

The following is due to *B. & Schweizer '18* (Theorem 4.7):

Proposition (FTAP I. for small markets)

If the small market $(\mathbb{B}, \mathbf{S}, T)$ *satisfies* $\mathbf{S} \geq 0$ *and* $\mathbf{1} \cdot \mathbf{S} > 0$, $\mathbf{1} \cdot \mathbf{S}_{-} > 0$, *then* $\mathbf{S}$ *satisfies DIWV if and only if there exists an LMD*$^{+}$ D.

The following is a generalization of *Karatzas & Kardaras '07* (Thm. 4.12):

Proposition (FTAP II. for small markets)

If the small market $(\mathbb{B}, \mathbf{S}, T)$ *satisfies* $\mathbf{S} \geq 0$ *and* $\mathbf{1} \cdot \mathbf{S} > 0$, $\mathbf{1} \cdot \mathbf{S}_{-} > 0$, *then* $\mathbf{S}$ *satisfies DIWV if and only if there exists a tradable SMD*$^{+}$ $\bar{D}$.
Moreover, $\bar{D}$ *is unique.*

Remarks on Karatzas & Kardaras '07

Karatzas & Kardaras '07 has certain restrictions in comparison:

- Imposes stronger assumptions:
  - $0 < \inf_t \mathbf{1} \cdot \mathbf{S}_t \ P$-a.s.,
    (instead of $0 < \inf_{t \leq N} \mathbf{1} \cdot \mathbf{S}_t \ P$*-a.s.* $\forall N \in \mathbb{R}$*).*
  - $\mathbf{S} > 0 \ P$-a.s..
    (as opposed to $\mathbf{S} \geq 0 \ P$*-a.s.).*
- Excludes strategies, which can default, i.e. works with the set of strategies $\mathcal{Y} := \{X \in \mathcal{X} : X > 0 \,\&\, X_- > 0\} \subsetneq \mathcal{X}$ instead of $\mathcal{X}$.
- Uses NUPBR instead of DIWV.
  (Note that NUPBR is strictly stronger and not stable under discounting.)

New concept and main result

Asymptotic DIWV - ADIWV

Definition

A large market strategy $\vec{\vartheta}$ *is asymptotically index weight maximal if* $\exists p > 0$ *such that* $\forall \epsilon > 0 \; \exists n \in \mathbb{N}$, $A \in \mathcal{F}^n$ *with* $P^n[A] \geq p$ *and a strategy* $\hat{\vec{\vartheta}}$ *with*

$$V_0(\hat{\vartheta}^n)[\mathbf{S}^n] \leq V_0(\vartheta)[\mathbf{S}^n] + \epsilon(\mathbf{1} \cdot \mathbf{S}_0^n)$$

$$\liminf_{t \to T^n} (\hat{\vartheta}_t^n - \vartheta_t^n) \geq p I_A \mathbf{1}^n \qquad P\text{-a.s.}$$

Definition

A large market $(\mathbb{B}^n, \mathbf{S}^n, T^n)_n$ *is asymptotically dynamic index weight viable (ADIWV), if* $\vec{0}$ *asymptotically index weight maximal.*

ADIWV - properties

- ADIWV is clearly stable under discounting, i.e. $(\mathbb{B}^n, \mathbf{S}^n, T^n)_n$ satisfies ADIWV if and only if for any sequence of semimartingales $(D^n)_n$ with $D > 0$ and $D_- > 0$, $(\mathbb{B}^n, \mathbf{S}^n/D^n, T^n)_n$ satisfies ADIWV.
- Time horizon can be any $[0, \infty]$-valued stopping time T.
- Complete formulation: ADIWV implies DIWV for each small market.
- Consistency: If the large market $(\mathbb{B}^n, \mathbf{S}^n, T^n)_n = (\mathbb{B}, \mathbf{S}, T)_n$ is a constant sequence of small markets, then $(\mathbb{B}, \mathbf{S}, T)_n$ satisfies ADIWV if and only if $(\mathbb{B}, \mathbf{S}, T)$ satisfies DIWV.
  (In comparison: $(\mathbb{B}, \mathbf{S}, T)_n$ satisfies NAA if and only if $(\mathbb{B}, \mathbf{S}, T)$ satisfies NUPBR.)

Main result

Theorem (FTAP)

Let $(\mathbb{B}^n, \mathbf{S}^n, T^n)_n$ such that $\mathbf{S}^n \geq 0$ and $\mathbf{1} \cdot \mathbf{S}^n > 0$, $\mathbf{1} \cdot \mathbf{S}^n_- > 0$ for each n. Then the following are equivalent:

(a) *The large market $(\mathbb{B}^n, \mathbf{S}^n, T^n)_n$ satisfies ADIWV,*

(b) *There exists for each n an $\mathbf{S}^n$-tradable SMD$^+$ D^n and we have the contiguity $(P^n)_n \triangleleft ((1/\xi^n) \cdot P^n)_n$.*

Notation:

- $\xi^n := ((\mathbf{1}^n \cdot \mathbf{S}^n_0) D^n_{T^n}) / (\mathbf{1}^n \cdot \mathbf{S}^n_{T^n})$ - deflator adjusted by the market volume.
- $\xi \cdot P$ for a $\xi \in L^0_+(P)$ is the measure $(\xi \cdot P)[A] := E^P[\xi 1_A]\ \forall A \in \mathcal{F}$.
- Let $(P^n)_n$ and $(Q^n)_n$ be two sequences of probability measures. Then we have contiguity $(P^n)_n \triangleleft (Q^n)_n$ if $\forall (A^n)_n \subseteq \mathcal{F}$ with $Q^n(A^n) \to 0$ we have $P^n(A^n) \to 0$.

Proof - Technique

"$(a) \Rightarrow (b)$":

- Via tightness condition: it is straightforward to generate arbitrage (in the sense of ADIWV), if $(P^n \circ (\xi^n)^{-1})_n$ is not tight.
- If $(P^n \circ (\xi^n)^{-1})_n$ is tight, then $(P^n)_n \triangleleft ((1/\xi^n) \cdot P^n)_n$ follows easily (e.g. like in *Rokhlin '07*).

"$(b) \Rightarrow (a)$":

- Indirect: let $\vec{\vartheta}$ be an "arbitrage strategy" with $p > 0$.
- Write $\mu^n := \mathbf{S}^n / \mathbf{1} \cdot \mathbf{S}^n$. Then one can derive

$$0 \leq E^{P^n}[1/\xi^n \liminf_{t \to T^n} (\vartheta^n_t \cdot \mu^n_t)] - E^{P^n}[1/\xi^n p I_A]$$

  for some n and A with $P^n[A] > p$.
- First term on RHS is bounded by the normalized initial value of ϑ^n ("ϵ") due to super-martingale property.
- Second term on RHS is bounded away from zero due to contiguity. $\lightning$

Note that "$(b) \Rightarrow (a)$" *does not use "tradability"*.

Main result - overview

$$
\begin{array}{ccc}
\forall n\ \exists \text{ tradable LMD}^+\ \bar{D}^n & \Rightarrow & \forall n\ \exists \text{ tradable SMD}^+\ \bar{D}^n \\
\text{and } (P^n)_n \lhd ((1/\bar{\xi}^n)\cdot P^n)_n & \not\Leftarrow & \text{and } (P^n)_n \lhd ((1/\bar{\xi}^n)\cdot P^n)_n \\
\Downarrow\ \not\Uparrow & & \Updownarrow \\
\forall n\ \exists \text{ LMD}^+\ D^n & \Rightarrow & \forall n\ \exists \text{ SMD}^+\ D^n \\
\text{and } (P^n)_n \lhd ((1/\xi^n)\cdot P^n)_n & (\Leftarrow ?) & \text{and } (P^n)_n \lhd ((1/\xi^n)\cdot P^n)_n \\
 & & \Updownarrow \\
 & & \text{ADIWV}
\end{array}
$$

Remark

For $\not\Leftarrow$ and $\not\Uparrow$, the counterexample is due to Takaoka & Schweizer '14 (Remark 2.8).
$(\Leftarrow ?)$ holds for small markets, and of course without contiguity condition (see FTAP I. and II. for small markets). For large markets, it is an open question.

Application on the infinite time horizon framework

Connecting large and small markets

Consider the large market $(\mathbb{B}, \mathbf{S}^n := \mathbf{S}^n_{\cdot\wedge T^n}, T^n)_n$ where:

- Stochastic basis $\mathbb{B}$ is constant in n,
- T^n is a sequence of stopping times increasing to ∞, i.e. $T^1 \leq T^2 \leq \cdots < \infty$ with $\lim_n T^n = \infty$,
- $\mathbf{S}^n$ is the stopped process at T^n of a fixed semimartingale $\mathbf{S}$.

We investigate connections between:

$$\text{large market } (\mathbb{B}, \mathbf{S}^n := \mathbf{S}^n_{\cdot\wedge T^n}, T^n)_n \quad \longleftrightarrow \quad \text{small market } (\mathbb{B}, \mathbf{S}, \infty).$$

Note that *"their economic properties should be same"*.

Result - overview

Proposition

Let $\mathbf{S} = (1, X)$ *for a non-negative semimartingale* X*. Then:*

large market $(\mathbb{B}, \mathbf{S}^n := \mathbf{S}^n_{\cdot\wedge T^n}, T^n)_n$		small market $(\mathbb{B}, \mathbf{S}, \infty)$		dual conditions for $(\mathbb{B}, \mathbf{S}, \infty)$
NAA	$\Leftrightarrow$	*NUPBR is satisfied on* $[0, \infty)$	$\Leftrightarrow$	$\exists$ *LMD* D *with* $\lim_t D_t < \infty$ P*-a.s.*
$\Downarrow \not\Uparrow$		$\Downarrow \not\Uparrow$		$\Downarrow \not\Uparrow$
ADIWV	$\Leftrightarrow$	*DIWV*	$\Leftrightarrow$	$\exists$ *LMD*$^+$
$\Downarrow \not\Uparrow$		$\Downarrow \not\Uparrow$		$\Downarrow \not\Uparrow$
NUPBR is satisfied in each small market	$\Leftrightarrow$	*NUPBR is satisfied on* $[\![0, T^n]\!]$ *for each* n	$\Leftrightarrow$	$\exists$ *LMD*

Proof - remarks & references

- First line: stopping arguments and FTAP result from *Herdegen '14* (Thm 4.10) generalized for infinite time horizon by *B., Schweizer '18*.
- Second line: convex-unbounded sets (next slide), stopping arguments and FTAP result from *B., Schweizer '18* (Thm 4.7).
- Third line: *Chau, Cosso, Fontana, Mostovyi '17* (Prop 2.1).
- Downarrows: trivial by the third column.
  (Note: since $\mathbf{S} = (1, X)$, $\lim_t D_t < \infty$ *implies* $\inf_t \mathbf{1} \cdot \mathbf{S}_t / D_t > 0$ *and hence LMD as in the first line implies LMD*$^+$*.)*

Convex-unbounded set (technique)

How to prove DIWV $\Rightarrow$ ADIWV (in this special framework)?

Lemma (*Brannath-Schachermayer '99* and *Kardaras '10*)

Let $C \in L^0_+$ be convex. Then C is bounded in L^0 if and only if C contains no sequence $(V_n)_n$ satisfying $V_n \geq \xi$ P-a.s. for all $n \in \mathbb{N}$ and for some $\xi \in L^0_+ \setminus \{0\}$.

1. Absence of ADIWV gives us large market arbitrage strategies $\vec{\vartheta}^n$. How to build a small market arbitrage strategy (in the sense of DIWV) out of them?
2. Find an adequate payoff set $\mathcal{G}$ such that (the small market components of) $\vec{\vartheta}^n$ make it unbounded in L^0, and it is convex.
3. Use above lemma to obtain existence of small arbitrage strategies.

Conclusions & further remarks

We have proposed a concept of absence of arbitrage for large markets, which

- is stable with respect to discounting (among other favorable properties),
- can be regarded as the large market counterpart of DIWV,
- has a dual characterization (FTAP) by SMDs and contiguity.

Further research:

- Develop discounting stable absence of arbitrage notion in the "strong sense" ("NFL-type").
- Derive its dual characterization (FTAP) as in *Klein '00, '03, '06* or in *Cuchiero, Klein, Teichmann '17* (or differently).

Thank you very much for your attention!