VON NEUMANN–GALE DYNAMICS AND CAPITAL GROWTH IN FINANCIAL MARKETS WITH FRICTIONS

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▸ The focus of the study is on **capital growth theory** for financial markets (Shannon, Kelly, Cover and others).

▸ The main goal is to extend the classical capital growth theory, pertaining to frictionless markets, to models of financial **markets with frictions**: transaction costs and portfolio constraints.

▸ The mathematical framework for the analysis is based on stochastic **von Neumann-Gale dynamical systems**.

CLASSICAL CAPITAL GROWTH THEORY

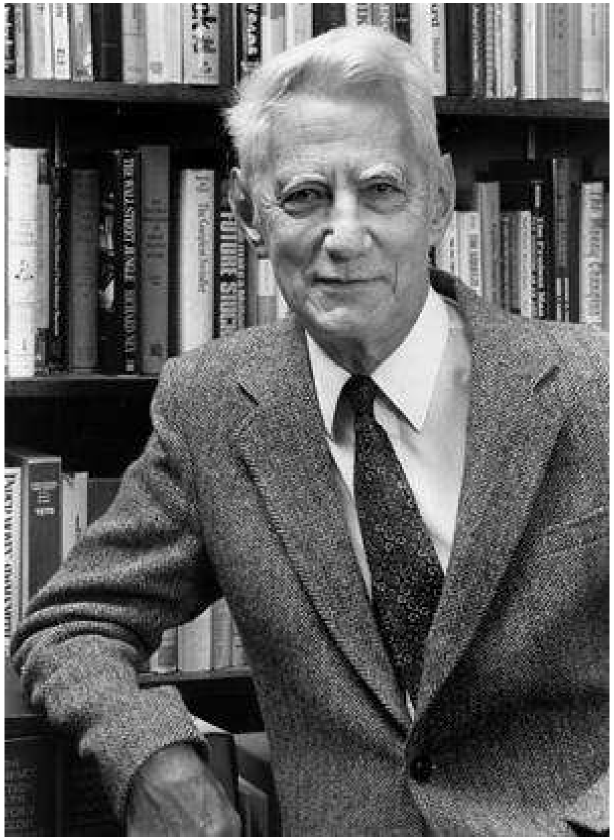
▸ Capital growth theory is a fascinating subject having a rich and peculiar history.

▸ Initially, its roots were in mathematical information theory. It received remarkable applications in Finance.

▸ Recently it served as the basis for new cutting-edge directions of research such as Evolutionary and Behavioral Finance.

Founders

Claude Shannon – one of the greatest mathematicians of the 20th century, the founding father of *information theory*.



Early days of classical capital growth theory and the role of Shannon are described in:

Thomas M. Cover, **Shannon and investment**, *IEEE Information Theory Society Newsletter*, Summer 1998, Special Golden Jubilee Issue, pp. 10–11.

John L. **Kelly** – the author of the celebrated *Kelly portfolio rule*.

The main question: How to construct an investment strategy guaranteeing almost surely the fastest growth rate of wealth in the long run?

Answer (for frictionless markets): allocate your wealth across assets over each time period between $t-1$ and t so as to maximize the conditional expectation of the logarithm of the portfolio return.

This principle is referred to as the **Kelly criterion** or the **Kelly portfolio rule**.

Where to read about it?

The most complete (*advanced*) exposition of capital growth theory for frictionless markets is given in the book:

T. **M**. **Cover and J**. **Thomas**, **Elements of Information Theory**, Wiley, 2006, 2nd Edition. **Chapter 16** "**Information theory and portfolio theory**".

There are several *popular* books describing the main ideas and the history of classical capital growth theory, e.g.,

W. **Poundstone**, **Fortune**’**s Formula**: The Untold Story of the Scientific Betting System That Beat the Casinos and Wall Street, Hill and Wang Publ., 2006.

The only textbook where capital growth theory is systematically presented at the *undergraduate level* is

I.**V**. **Evstigneev**, **T**. **Hens and K**.**R**. **Schenk**-**Hoppé**, **Mathematical Financial Economics**: A Basic Introduction, Springer, 2015.

To extend the classical capital growth theory to financial markets with frictions we use the mathematical framework of von Neumann-Gale dynamical systems.

Multivalued dynamical systems

- *Multivalued dynamical system.* Given:
- Set X_t (*state space* at time t), $t = 0, 1, 2, \ldots$
- Set-valued mapping (*transition mapping*):

$$x \mapsto A_t(x), \quad x \in X_{t-1}, \quad A_t(x) \subseteq X_t \ .$$

- *Paths* or *trajectories,* of the dynamical system: sequences $x_0, x_1, \ldots$ such that

$$x_t \in A_t(x_{t-1}).$$

Von Neumann-Gale dynamical systems

They are defined as follows:

- The state spaces X_t are **cones** in $\mathbf{R}^n$.
- For each t, the graph of the transition mapping $A_t(\bullet)$,

$$Z_t = \{(x, y) \in X_{t-1} \times X_t : y \in A_t(x)\},$$

is a **cone**.

Equivalent description in terms of transition cones:

- Given: *transition cones* Z_t.
- *Paths* are sequences $x_0, x_1, \ldots$ such that

$$(x_{t-1}, x_t) \in Z_t.$$

Autonomous systems: X_t and $A_t(\bullet)$ (or Z_t) do not depend on t.

Von Neumann's (1937) model of economic growth

J. **von Neumann** (1937) Über ein ökonomisches Gleichungssystem und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes, in: *Ergebnisse eines Mathematischen Kolloquiums*, No. **8**, 1935-1936, Franz-Deuticke, Leipzig und Wien, pp. 73-83. [An English translation: A model of general economic equilibrium, *Rev. Econ. Studies* **13** (1945-1946), 1-9.]

D. **Gale** (1956) A closed linear model of production, in: H. W. Kuhn *et al.* (eds.), *Linear Inequalities and Related Systems,* Ann. of Math. Studies, vol. 38, Princeton Univ. Press, Princeton, pp. 285-303.

Von Neumann – *polyhedral cones*; Gale – *general cones*.

Stochastic von Neumann–Gale dynamical systems

Pioneering work of **Eugene Dynkin**, **Roy Radner** and their research groups in the 1970s.

In a stochastic von Neumann-Gale dynamical system we are given:

- a probability space $(\Omega, \mathcal{F}, P)$;
- filtration $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \ldots \subseteq F_t \subseteq \ldots \subseteq \mathcal{F}$;
- for each $t = 0, 1, 2, \ldots,$ an $\mathcal{F}_t$-measurable random cone (random *state space* $X_t(\omega) \subseteq \mathbf{R}^n$;
- set-valued *transition mappings* $(\omega, a) \mapsto A_t(\omega, a)$ assigning to each $\omega \in \Omega$ and $a \in X_{t-1}(\omega)$ a set

$$A_t(\omega, a) \subseteq X_t(\omega)$$

such that the graph

$$Z_t(\omega) := \{(a, b) : b \in A_t(\omega, a)\}$$

of the mapping $A_t(\omega, \cdot)$ is a closed convex cone (*transition cone*) depending $\mathcal{F}_t$-measurably on ω.

Paths (**trajectories**): $x_0(\omega), x_1(\omega), \ldots$

$$x_t(\omega) \in A_t(\omega, x_{t-1}(\omega)) \text{ (a.s.)},$$

or, equivalently,

$$(x_{t-1}(\omega), x_t(\omega)) \in Z_t(\omega) \text{ (a.s.)}$$

and x_t is F_t-measurable. Technical assumption: $x_t \in L^\infty$.

Canonical von Neumann-Gale systems. Those dynamical systems for which

$$X_t(\omega) = \mathbf{R}^n_+$$

are called *canonical*.

Von Neumann-Gale dynamics in application to Economics

Initially, von Neumann-Gale dynamical systems (both deterministic and stochastic) were applied to the modeling of economic dynamics.

- In such models,

$$X_t(\omega) = \mathbf{R}^n_+,$$

(canonical systems),

- elements of X_t are *commodity vectors*,
- $(x, y) \in Z_t$ are *input-output* pairs,
- Z_t are *technology sets.*

Von Neumann-Gale dynamics in application to Finance

The idea of applying von Neumann-Gale dynamics to Finance (primarily to asset pricing and hedging) was put forward in the paper:

M.A.H. Dempster, I.V. Evstigneev and M.I. Taksar, Asset pricing and hedging in financial markets with transaction costs: An approach based on the von Neumann-Gale model, 2006, *Annals of Finance*, v. 2, 327-355.

The above paper dealt with a finite probability space. The general case was considered in:

I.V. Evstigneev and M.V. Zhitlukhin, Controlled random fields, von Neumann-Gale dynamics and multimarket hedging with risk, 2013, *Stochastics*, v. 85, 652-666.

Novel aspects:

- *multi-market hedging*: controlled random fields on graphs;
- *"soft" hedging (with risk)* in terms of risk measures.

VON NEUMANN-GALE ASSET MARKET MODEL

- n **assets** (securities).
- **Portfolio** of assets

$$x_t(\omega) = (x_t^1(\omega), \ldots, x_t^n(\omega)),\ \mathrm{F}_t\text{-measurable.}$$

Note: x_t^i denotes the amount of money invested in asset i (portfolio positions are described in *monetary terms*, not in terms of "physical units").

- **Admissible portfolio**: $x_t(\omega) \in X_t(\omega)$ (a.s.), where $X_t(\omega)$ is the given random F_t-measurable cone specifying *portfolio constraints*.
- **Investment/trading strategy**: a sequence of admissible portfolios $x_0, x_1, x_2, \ldots$ such that

$$(x_{t-1}(\omega), x_t(\omega)) \in Z_t(\omega) \text{ (a.s.),}$$

where $Z_t(\omega) \subseteq X_{t-1}(\omega) \times X_t(\omega)$ is the given closed cone depending F_t-measurably on ω and describing the *self-financing constraints*.

- **Von Neumann-Gale dynamical system** is defined by the cones $X_t(\omega)$ and $Z_t(\omega)$. *The state spaces:* the cones $X_t(\omega)$ of admissible portfolios. The *transition cones*: $Z_t(\omega)$. *Paths in the system:* (self-financing) strategies.
- **Terminology**. "*Paths*" and "*strategies*" will be used interchangeably.

Specialized model: a standard example

Transaction cost rates. Let $\lambda^+_{t,i}(\omega) \geq 0$ and $1 > \lambda^-_{t,i}(\omega) \geq 0$ be the *transaction cost rates* for buying and selling asset i, respectively.

Asset returns. Let $R^i_t(\omega)$ $[= S^i_t(\omega)/S^i_{t-1}(\omega)]$ be the (gross) *return* on asset i.

The cone of admissible portfolios $X_t(\omega)$ consists of those portfolios $x = (x^1, \ldots, x^n)$ which satisfy the *margin requirement*

$$M \sum_{i=1}^{n} (1 + \lambda^+_{t,i})(-x^i)_+ \leq \sum_{i=1}^{n} (1 - \lambda^-_{t,i})(x^i)_+ \quad [a_+ = \max\{a, 0\}],$$

where $M > 1$: the long positions must cover the short ones with excess.

The transition cone $Z_t(\omega)$ consists of pairs of portfolios (x, y) satisfying

$$\sum_{i=1}^{n} (1 + \lambda^+_{t,i})(y^i - R^i_t x^i)_+ \leq \sum_{i=1}^{n} (1 - \lambda^-_{t,i})(R^i_t x^i - y^i)_+,$$

meaning that *asset purchases are made only at the expense of sales* of available assets (under transaction costs).

Extensions of this model are developed that take into account: (a) *several currencies*; (b) *dividend-paying assets*; (c) *different interest rates* for lending and borrowing.

Asymptotic growth-optimality

Investor's wealth. Let $w_t(x) = w_t(\omega, x)$ be a function of $\omega \in \Omega$ and $x \in X_t(\omega)$ measuring the *wealth* of the investor possessing the portfolio x at time t. This might be, for example, the *value of the portfolio* x in the market prices

$$w_t(x) = \sum_{i=1}^n x^i$$

or its *liquidation value*

$$w_t(x) = \sum_{i=1}^n [(1 - \lambda_{t,i}^-)x^i{}_+ - (1 + \lambda_{t,i}^+)(-x^i)_+].$$

It is assumed that $c|x| \leq w_t(x) \leq C|x|$ for $x \in X_t$, where $c, C > 0$ and $|x| = \sum |x^i|$.

Definition. An investment strategy $x_0, x_1, \ldots$ is called **asymptotically growth-optimal** if for any other investment strategy $y_0, y_1, \ldots$ we have

$$\frac{w_t(y_t)}{w_t(x_t)} \leq \alpha_t \text{ (a.s.)},$$

where α_t is a *supermartingale*. This implies:

$$\sup \frac{w_t(y_t)}{w_t(x_t)} < \infty \text{ (a.s.)}, \ \limsup \frac{1}{t} \ln \frac{w_t(y_t)}{w_t(x_t)} \leq 0 \text{ (a.s.)},$$

$$\sup_\tau E \frac{w_\tau(y_\tau)}{w_\tau(x_\tau)} < \infty, \ \sup_\tau E \ln \frac{w_\tau(y_\tau)}{w_\tau(x_\tau)} < \infty \quad [\tau \text{ stopping times}].$$

Rapid paths

A central notion in the theory of von Neumann-Gale dynamical systems is the notion of a *rapid path*. Such paths are important, in particular, because they turn out to be asymptotically growth-optimal.

Rapid paths. A path $x_0, x_1, \ldots$ is called *rapid* if there is a sequence of integrable random vectors $p_1, p_2, \ldots$ such that p_t is F_t-measurable,

$$p_{t+1}(\omega) \in X_t^*(\omega) \text{ (a.s.)}, \tag{1}$$

where $X_t^*(\omega)$ is the *dual cone*;

$$E(p_{t+1}|\mathrm{F}_t)y \leq p_t x \text{ (a.s.) if } (x(\omega), y(\omega)) \in Z_t(\omega) \text{ (a.s.)}, \tag{2}$$

and

$$p_{t+1}x_t = 1 \text{ (a.s.)} \tag{3}$$

Dual paths. A sequence (p_t) satisfying (1) and (2) is called a *dual path*. Coordinates p_t^i are interpreted as *consistent discount factors*. If (3) holds, we say that the dual path (p_t) *supports* the path (x_t). Thus,

a path is rapid if and only if it is supported by a dual path.

Supermartingale property. By virtue of (1) and (2), for any path (y_t), the sequence $(p_{t+1}y_t)$ is a *nonnegative supermartingale*.

Rapid paths, numeraire portfolios and benchmark strategies

Why "**rapid**"? Because a rapid path (x_t) grows, in a sense, faster than any other path (y_t) both in the short and in the long run.

In the short run: for every t,

$$E\frac{p_{t+1}y_t}{p_t y_{t-1}} \leq E\frac{p_{t+1}x_t}{p_t x_{t-1}} = 1, \quad E\ln\frac{p_{t+1}y_t}{p_t y_{t-1}} \leq E\ln\frac{p_{t+1}x_t}{p_t x_{t-1}} = 0,$$

i.e. (x_t) maximizes the expected growth rate and the expected logarithm of the growth rate (measured in terms of (p_t)) over each time period $t-1, t$.

In the long run: rapid paths possess the property of asymptotic growth-optimality as defined above.

Analogues of rapid paths. In the financial context, rapid paths generalize **numeraire portfolios**,

Long, **J**.**B**., The numeraire portfolio, *Journal of Financial Economics* **26** (1990) 29–69,

or **benchmark strategies**, see

Platen, **E**. **and Heath**, **D**., *A benchmark approach to quantitative finance*, Springer, 2006,

and references therein.

Assumptions

We write $|a|$ for the norm $\sum|a^i|$ of the vector $a = (a^1, \ldots, a^n)$. The ball of a radius r with the center at a is denoted by $\mathrm{B}(a, r)$.

(**A1**) For every $t = 0, 1, \ldots$, there exists an F_t-measurable random vector $q_t(\omega)$ satisfying such that $q_t(\omega)a \geq 0$ for $a \in X_t(\omega)$ and

$$H_t^{-1}|a| \leq q_t(\omega)a \leq H_t|a|,\ a \in X_t(\omega),\ \omega \in \Omega,$$

where $H_t \geq 1$.

(**A2**) For every $t = 1, 2, \ldots$, $\omega \in \Omega$ and $a \in X_{t-1}(\omega)$, there exists $b \in X_t(\omega)$ such that $(a, b) \in Z_t(\omega)$.

(**A3**) There exist constants K_t ($t = 1, 2, \ldots$) such that $|b| \leq K_t|a|$ for any $(a, b) \in Z_t(\omega)$ and $\omega \in \Omega$.

(**A4**) For each $t = 1, 2, \ldots$, there exists a bounded F_t-measurable vector function $\mathring{z}_t = (\mathring{x}_t, \mathring{y}_t)$ such that for all $\omega \in \Omega$, we have

$$(\mathring{x}_t(\omega), \mathring{y}_t(\omega)) \ \in Z_t(\omega) \text{ and } \mathrm{B}(\mathring{y}_t(\omega), \varepsilon_t) \subseteq X_t(\omega),$$

where $\varepsilon_t > 0$ is some constant.

Meaning: (**A1**) margin requirement; (**A2**) no deadlocks; (**A3**) boundedness of returns; (**A4**) uniformly interior point.

Central results

Fix some portfolio $x_0(\omega)$ such that $\mathrm{B}(x_0(\omega), \varepsilon_0) \subseteq X_0(\omega)$ (a.s.) for some $\varepsilon_0 > 0$.

Finite time horizon.

Theorem 1. *For each $N > 1$, there exists a finite rapid path $x_0, \ldots, x_N$ with the initial state x_0. It can be constructed by maximizing the logarithmic functional $E \ln q_N x_N$.*

The maximization of the logarithmic functional shows connections with the classical capital growth theory (the Kelly portfolio rule). The proof is based on duality results for convex stochastic optimization problems.

For canonical von Neumann-Gale systems (i.e. when $X_t(\omega) = \mathbf{R}^n_+$), Theorem 1 was proved in

I.V. Evstigneev and S.D. Flåm, Rapid growth paths in multivalued dynamical systems generated by homogeneous convex stochastic operators, 1998, *Set-Valued Analysis*, v. 6, 61-82.

For models with general cones $X_t(\omega)$, its proof is given in

E. Babaei, I.V. Evstigneev and K.R. Schenk-Hoppé, Log-optimal and rapid paths in von Neumann-Gale dynamical systems, Working paper, 2018.

Infinite time horizon.

Theorem 2. *There exists an infinite rapid path* $x_0, x_1, x_2, \ldots$ *with the initial state* x_0.

For canonical von Neumann-Gale systems, Theorem 1 was proved in

W. Bahsoun, I.V. Evstigneev and M.I. Taksar, Rapid paths in von Neumann-Gale dynamical systems, 2008, *Stochastics*, v. 80, 129-142.

For models with general cones $X_t(\omega)$, its proof is given in

E. Babaei, I.V. Evstigneev and K.R. Schenk-Hoppé, Von Neumann-Gale dynamics and capital growth in financial markets with frictions, Working paper, 2018.

An infinite rapid path is constructed by passing to the limit from finite ones with the help of a new version of the Schmeidler-Fatou lemma in several dimensions

Schmeidler, D., Fatou's lemma in several dimensions, *Proceedings of the American Mathematical Society* **24** (1970) 300–306.

This new version of the lemma is proved in

E. Babaei, I.V. Evstigneev and K.R. Schenk-Hoppé, A multidimensional Fatou's lemma for conditional expectations, Working paper, 2018.

STATIONARY MODELS

Autonomous von Neumann-Gale systems. All applicable stochastic models rely upon some assumptions of stationarity: "tomorrow" and "today" must be in a sense similar.

In the present context, a framework for stationary asset market models is provided by *autonomous von Neumann-Gale dynamical systems*.

In a system of this kind, we are given an automorphism T of the probability space (Ω, F, P) (*time shift*): a measure preserving one-to-one transformation of Ω.

It is assumed that

$$T^{-1}(\mathrm{F}_t) = \mathrm{F}_{t+1},\ X_t(T\omega) = X_{t+1}(\omega),\ Z_t(T\omega) = Z_{t+1}(\omega),$$

i.e. the data of the model are time-invariant (stationary).

Balanced Paths. In autonomous systems, a central role is played by *balanced paths*, i.e. paths (x_t) of the form

$$x_t(\omega) = \lambda(\omega)\lambda(T\omega)\cdots\lambda(T^{t-1}\omega)x(T^t\omega),$$

where $\lambda(\omega) > 0$ is an F_1-measurable scalar function and $x(\omega) \geq 0$ is an F_0-measurable vector function satisfying $|x(\omega)|= 1$.

A balanced path grows with stationary proportions $x(T^t\omega)$ and at a stationary rate $\lambda(T^t\omega)$. In the deterministic case, $x_t = \lambda^t x$.

Von Neumann paths and von Neumann equilibrium.

A balanced path maximizing $E \log \lambda(\omega)$ among all balanced paths is called a *von Neumann path*.

A triplet of functions $(\lambda(\omega), x(\omega), p(\omega))$ is called a *von Neumann equilibrium* if

$$x_t(\omega) = \lambda(\omega)\lambda(T\omega)\cdots\lambda(T^{t-1}\omega)x(T^t\omega)$$

is a von Neumann path and

$$p_{t+1}(\omega) = (\lambda(\omega)\lambda(T\omega)\cdots\lambda(T^{t-1}\omega))^{-1}p(T^t\omega)$$

is a dual path supporting it.

Existence of von Neumann equilibrium. Under stationary versions of assumptions (**A1**) - (**A4**), along with some further non-restrictive technical assumptions (holding for all specialized models coming from Finance), we have the following result:

Theorem 3. *A von Neumann equilibrium exists.*

Remark. Let (x, p, λ) be a von Neumann equilibrium. Consider the balanced path $(x_t)_{t=0}^{\infty}$ generated by the pair (x, λ). By the definition of an equilibrium, it is rapid. Consequently, it is asymptotically growth-optimal. Thus, by virtue of Theorem 3, there exists a balanced path that is asymptotically growth-optimal in the class of *all*, not necessarily balanced paths!

Dynkin's problem. In the deterministic case, the existence of a von Neumann equilibrium was proved in the above cited papers by von Neumann (1937) and Gale (1956).

In the 1970s Eugene Dynkin posed the problem of establishing the existence of a von Neumann equilibrium in the stochastic setting under conditions that would be fully analogous to the deterministic ones.

After a series of intermediate results, the problem was solved (for canonical von Neumann-Gale systems) in the paper

I.V. Evstigneev and K. R. Schenk-Hoppé, Stochastic equilibria in von Neumann-Gale dynamical systems, *Transactions of the American Mathematical Society,* 2008, v. 360, pp. 3345-3364.

In the general setting, Theorem 3 is proved in

E. Babaei, I. Evstigneev, K. R. Schenk-Hoppé, and M.V. Zhitlukhin, Von Neumann-Gale dynamics, market frictions, and capital growth, Working Paper, 2018.

It turned out to be possible to *deduce* the general result from the one obtained earlier for the canonical von Neumann-Gale systems.