Computation of systemic risk measures

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- Aggregation function $\Lambda \colon \mathbb{R}^d \to \mathbb{R}$:
  - Increasing function
  - $\Lambda \circ X \in L_1^\infty$ is a quantification of the impact of the wealths to society.
  - Simple examples: $\Lambda(x) = \sum_{i=1}^d x_i$, $\Lambda(x) = -\sum_{i=1}^d x_i^-$
  - More sophisticated examples to be considered: Eisenberg, Noe '01 and Rogers, Veraart '13 models.

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    Eisenberg, Noe '01 and Rogers, Veraart '13 models.
- Scalar convex risk measure $\rho\colon L_1^\infty \to \mathbb{R}$ to test the acceptability of $\Lambda \circ X$:

$$\rho(Z) = \sup_{\mathbb{S} \ll \mathbb{P}} \left(\mathbb{E}^{\mathbb{S}}\left[-Z\right] - \alpha(\mathbb{S}) \right), \quad \alpha(\mathbb{S}) = \sup_{Z \in L_1^\infty} \left(\mathbb{E}^{\mathbb{S}}\left[-Z\right] - \rho(Z) \right)$$

- e.g. negative expectation, average-value-at-risk, optimized certainty equivalents, entropic risk measure, utility-based shortfall risk measures

Systemic risk measures

- Systemic risk measure insensitive to capital levels (Chen *et al.* '13):

$$\rho^{\text{ins}}(X) = \rho(\Lambda \circ X) = \inf\left\{\sum_{i=1}^{d} y_i \mid \rho\left(\Lambda \circ X + \sum_{i=1}^{d} y_i\right) \leq 0\right\}.$$

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- R^{sen} is a set-valued risk measure with dual representation (A., Rudloff '16):

$$R^{\text{sen}}(X) = \bigcap_{\mathbb{Q} \in \mathcal{M}_d(\mathbb{P}), w \in \mathbb{R}^d_+ \setminus \{0\}} \mathbb{E}^{\mathbb{Q}}[-X] + \left\{z \in \mathbb{R}^d \mid w^{\mathsf{T}} z \geq -\alpha^{\text{sys}}(\mathbb{Q}, w)\right\},$$

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where α^{sys} is the systemic penalty function given by

$$\alpha^{\text{sys}}(\mathbb{Q}, w) = \inf_{\mathbb{S} \approx \mathbb{P}}\left(\alpha(\mathbb{S}) + \mathbb{E}^{\mathbb{S}}\left[g\left(w_1 \frac{d\mathbb{Q}_1}{d\mathbb{S}}, \ldots, w_d \frac{d\mathbb{Q}_d}{d\mathbb{S}}\right)\right]\right).$$

  - $\mathbb{S} \approx \mathbb{P}$ probability measure of society
  - $\mathbb{Q}_i \ll \mathbb{P}$ probability meaure of bank i
  - $g(y) = \sup_{x \in \mathbb{R}^d}\left(\Lambda(x) - x^{\mathsf{T}} y\right)$ conjugate function
  - multivariate g-divergence

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- Suppose there are g groups. Use a 0-1 matrix $B \in \mathbb{R}^{d \times g}$ so that for a capital allocation vector $z \in \mathbb{R}^g$ for groups

$$Bz = (z_1, \ldots, z_1; \ldots; z_g, \ldots, z_g)^{\mathsf{T}} \in \mathbb{R}^d$$

  gives the capital allocation vector for banks.
- From now on, let us redefine $R^{\text{sen}}(X)$ as

$$R^{\text{sen}}(X) = \left\{ z \in \mathbb{R}^g \mid \rho\left(\Lambda \circ (X + Bz)\right) \leq 0 \right\}.$$

  (The case $d = g$ with $B = I$ recovers the earlier definition.)

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     - Efficient calculation of the convex set $R^{\text{sen}}(X)$ by exploiting the structure of the constraint using scenario decompositions
  2. Rogers, Veraart '13 model: We propose a mixed integer linear programming problem to calculate $\Lambda(x)$.
     - Calculation of the nonconvex set $R^{\text{sen}}(X)$ in the risk-neutral case $\rho = -\mathbb{E}$

Eisenberg-Noe '01 Model

- Banks: nodes $1, \dots, d$

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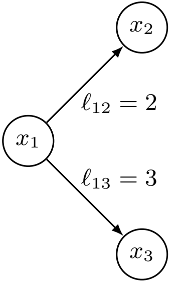

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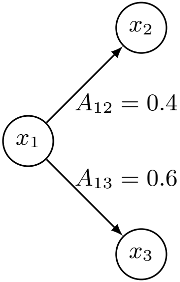
- Matrix of nominal liabilities: $(\ell_{ij})_{1 \leq i,j \leq d}$
- Total liability of entity i: $\bar{p}_i = \sum_{j=0}^{d} \ell_{ij}$
- Relative liability of i to j: $A_{ij} = \frac{\ell_{ij}}{\bar{p}_i}$



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$$p_i = \bar{p}_i \wedge \left(x_i + \sum_{j=1}^{d} A_{ji} p_j \right), \quad i \in \{1, \ldots, d\},$$

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- LP formulation: p can be computed as an optimal solution of the linear program

$$\text{maximize } \mathbf{1}^{\mathsf{T}} p \text{ subject to } p \leq x + A^{\mathsf{T}} p, \quad 0 \leq p \leq \bar{p}.$$

- Clearing vector exists since the LP is bounded (or by Brouwer's fixed point theorem).
- Aggregation function: Define $\Lambda(x)$ to be the optimal value of the LP.
  - Total debt paid at clearing.
  - Other possibilities for $\Lambda(x)$: payment received by a special node ("society"), number of nondefaulting banks, etc.

Back to computation of $R^{\mathsf{sen}}(X)$

- The aim is to compute

$$R^{\mathsf{sen}}(X) = \{z \in \mathbb{R}^g \mid \rho(\Lambda \circ (X + Bz)) \leq 0\},$$

where Λ is the increasing concave but nonsmooth function defined by

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- Vector optimization problem:

minimize z w.r.t. $\mathbb{R}^g_+$ subject to $\rho(\Lambda \circ (X + Bz)) \leq 0,\ (X + Bz \geq 0),\ z \in \mathbb{R}^g$.

- Use Benson's algorithm for convex vector optimization problems (Löhne *et al.*'14).

Convex Benson algorithm

The algorithm solves two types of scalar problems:

1. $P_1(w)$: weighted sum scalarization with weight vector $w \in \mathbb{R}^g_+ \backslash \{0\}$

$$\mathscr{P}_1(w) = \inf_{z \in R^{\text{sen}}(X)} w^\mathsf{T} z = \inf_{z \in \mathbb{R}^g} \left\{ w^\mathsf{T} z \mid \rho\left(\Lambda \circ (X + Bz)\right) \leq 0 \right\}$$

2. $P_2(v)$: scalarization by a reference variable $v \notin R^{\text{sen}}(X)$
   Find the minimum step-length $\alpha \in \mathbb{R}$ to enter $R^{\text{sen}}(X)$ from v along the direction $\mathbf{1} \in \mathbb{R}^g$

$$\begin{aligned} \mathscr{P}_2(v) &= \inf \left\{ \alpha \in \mathbb{R} \mid v + \alpha \mathbf{1} \in R^{\text{sen}}(X) \right\} \\ &= \inf \left\{ \alpha \in \mathbb{R} \mid \rho\left(\Lambda \circ (X + B(v + \alpha \mathbf{1}))\right) \leq 0 \right\} \end{aligned}$$

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- To have a concave maximization problem, pass to finite measures via $M := \gamma\frac{d\mathbb{S}}{d\mathbb{P}}$ and $\alpha(M) := \gamma\alpha(\mathbb{S})$ by slight abuse of notation.

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- Recall the definition of Λ as a $\sup$ and use the famous Theorem 14.60 in Rockafellar, Wets '97 to swap $\sup - \mathbb{E}$.

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- Last step: In the inner problem, randomize z as Z, add the constraint $Z = \mathbb{E}Z$, and Lagrange dualize it.

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$$= \sup_{U \in L^1_g} \inf_{\substack{P \in [0,\bar{p}], \\ Z \in L^\infty_g}} \left\{ \mathbb{E}\left[w^\top Z - M \mathbf{1}^\top P \right] + \mathbb{E}\left[U^\top (Z - \mathbb{E}Z) \right] \mid (I - A)^\top P - BZ \leq X \right\}$$

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$$= \sup_{U\in L^1_g :\, \mathbb{E}U=0} \inf_{\substack{P\in[0,\bar{p}],\\ Z\in L^\infty_g}} \left\{ \mathbb{E}\left[(w+U)^\mathsf{T} Z - M\mathbf{1}^\mathsf{T} P \right] \mid (I-A)^\mathsf{T} P - BZ \leq X \right\}.$$

Solving $P_1(w)$

- Long story short: We decomposed the nonsmooth objective function of the dual $(\sup)$ problem into scenario subproblems (minus the penalty term):

$$\mathscr{P}_1(w) = \sup_{\substack{M \in L^1_+, \\ U \in L^1_g :\, \mathbb{E}[U]=0}} \left(\mathbb{E}\left[F \circ (U, M) \right] - \alpha(M) \right),$$

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  where $F \colon \Omega \times \mathbb{R}^g \times \mathbb{R} \to \bar{\mathbb{R}}$ is defined by

$$F(\omega, u, m) := \inf_{\substack{p \in \mathbb{R}^d, \\ z \in \mathbb{R}^g}} \left\{ (w+u)^\mathsf{T} z - m \mathbf{1}^\mathsf{T} p \mid (I-A)^\mathsf{T} p - Bz \leq X(\omega),\ p \in [0, \bar{p}] \right\}.$$

- $F(\omega, u, m)$ is finite if and only if $u \geq -w$. So add $U \geq -w$ as a constraint.

Solving $P_1(w)$

- Long story short: We decomposed the nonsmooth objective function of the dual $(\sup)$ problem into scenario subproblems (minus the penalty term):
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- $F(\omega, u, m)$ is finite if and only if $u \geq -w$. So add $U \geq -w$ as a constraint.
- Such problems are solved efficiently using bundle methods. In a nutshell, these methods
  - obtain piecewise-affine upper approximations of $F(\omega, \cdot, \cdot), -\alpha(\cdot)$ at a feasible point $(U^{(k)}, M^{(k)})$, call them $\tilde{F}^{(k)}, -\tilde{\alpha}^{(k)}$,
  - solve the master problem
  $$\sup_{\substack{M \in L^1_+, \\ U \in L^1_g:\, \mathbb{E}[U]=0}} \left(\mathbb{E}\left[\tilde{F}^{(k)} \circ (U, M)\right] - \tilde{\alpha}^{(k)}(M) - \text{(quadratic term)}\right)$$
  to find a “better” solution $(U^{(k+1)}, M^{(k+1)})$,
  - stop when the approximation is good enough.

Solving $P_2(v)$

- A similar dual formulation can be derived for

$$\mathscr{P}_2(v) = \inf \left\{ \alpha \in \mathbb{R} \mid \rho \left(\Lambda \circ \left(X + B(v + \alpha \mathbf{1}) \right) \right) \leq 0 \right\}.$$

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- We obtain

$$\mathscr{P}_2(v) = \sup_{\substack{M \in L^1_+, \\ Q \in L^1 :\, \mathbb{E}[Q]=0}} \left(\mathbb{E}\left[G \circ (Q, M)\right] - \alpha(M)\right),$$

where $G\colon \Omega \times \mathbb{R} \times \mathbb{R} \to \bar{\mathbb{R}}$ is defined by

$$G(\omega, q, m) = \inf_{\substack{s \in \mathbb{R}, \\ p \in \mathbb{R}^d}} \left\{(1+q)s - m\mathbf{1}^\mathsf{T} p \mid (I - A)^\mathsf{T} p - (B\mathbf{1})s \leq X(\omega) + Bv, p \in [0, \bar{p}]\right\}.$$

- $G(\omega, q, m)$ is finite if and only if $q \geq -1$. So add $Q \geq -1$ as a constraint.
- Solve efficiently using a bundle method.

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- $G(\omega, q, m)$ is finite if and only if $q \ge -1$. So add $Q \ge -1$ as a constraint.
- Solve efficiently using a bundle method.
- Overall method: Run convex Benson's algorithm with these subroutines for $P_1(w)$ and $P_2(v)$.

Eisenberg Noe'01 model with signed random shock

- Same setup as in Eisenberg-Noe '01 model
- d banks
- Total liabilities vector: $\bar{p} \in \mathbb{R}^d_+$
- Relative liabilities matrix: $A \in \mathbb{R}^{d \times d}$ (zero diagonal elements)

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- New feature: Assume $x \in \mathbb{R}^d$ has possibly negative entries, i.e., some banks might have external liabilities (e.g. operating costs) yielding a negative net exposure.
- Easy fix proposed in Eiseberg, Noe '01: "Those operating costs could be captured by appending to the financial system a "sink node," labeled, say, node 0."
- For us, this would mean changing the network structure both randomly and as part of the decision variable (recall $\Lambda \circ (X + Bz)$). $\rightarrow$ too complicated
- Major drawback: No seniority between interbank liabilities and external liabilities of a node.
- We propose an extension where external liabilities have a seniority over interbank liabilities, i.e., bank i pays some/all of its interbank liabilities only when $x_i + \sum_{j=1}^{d} A_{ji} p_j > 0$. (No worries when $x_i \geq 0$, as expected.)

Eisenberg Noe'01 model with signed random shock

- Clearing payment vector $p \in \mathbb{R}^d_+$ solves the fixed point problem

$$p_i = \Phi_i(p) := \begin{cases} \bar{p}_i & \text{if } \bar{p}_i < x_i + \sum_{j=1}^d A_{ji} p_j, \\ x_i + \sum_{j=1}^d A_{ji} p_j & \text{if } \bar{p}_i \geq x_i + \sum_{j=1}^d A_{ji} p_j > 0, \\ 0 & \text{if } x_i + \sum_{j=1}^d A_{ji} p_j \leq 0, \end{cases} \quad i \in \{1, \ldots, d\},$$

i.e., $p = \Phi(p) := (\bar{p} \wedge (x + A^\mathsf{T} p))^+$.

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  i.e., $p = \Phi(p) := (\bar{p} \wedge (x + A^\mathsf{T} p))^+$.
- Unlike the case $x \geq 0$, an LP characterization of a clearing vector is not possible.
- Instead: We show that a clearing vector p can be calculated as an optimal solution of the following mixed integer linear programming (MILP) problem.

$$\begin{aligned} \text{maximize} \ & \sum_{i=1}^d p_i \\ \text{s.t.} \ & p_i \leq x_i + \sum_{j=1}^d A_{ji} p_j + M(1 - s_i), \quad i \in \{1, \dots, d\} \\ & x_i + \sum_{j=1}^d A_{ji} p_j \leq M s_i, \quad i \in \{1, \dots, d\} \\ & 0 \leq p_i \leq \bar{p}_i s_i, \quad i \in \{1, \dots, d\} \\ & s_i \in \{0, 1\}, \quad i \in \{1, \dots, d\}. \end{aligned}$$

Eisenberg Noe'01 model with signed random shock

- Let $\Lambda(x)$ be the optimal value of the MILP.
- Λ is decreasing but not quasiconcave in general.
- $R^{\text{sen}}(X)$ may be a nonconvex set.

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- Let $\Lambda(x)$ be the optimal value of the MILP.
- Λ is decreasing but not quasiconcave in general.
- $R^{\mathsf{sen}}(X)$ may be a nonconvex set.
- Nevertheless, we can calculate $R^{\mathsf{sen}}(X)$ at least in the case where ρ is shifted negative expectation, that is,

$$R^{\mathsf{sen}}(X) = \left\{ z \in \mathbb{R}^g \mid \mathbb{E}\left[\Lambda \circ (X + Bz)\right] \geq \gamma \mathbf{1}^{\mathsf{T}} \bar{p} \right\},$$

  where $\gamma \in [0, 1]$ is the average fraction of total debt that should be paid at clearing.
- Calculate by nonconvex Benson's algorithm (Nobakhtian, Shafiei '17): solves $P_1(w)$ and $P_2(v)$ like the convex one but replaces supporting halfspaces with supporting (shifted) cones.

Rogers-Veraart '13 model

- Same setup as in Eisenberg-Noe '01 model
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- New feature: A defaulting bank can use only a fraction of its liquid assets, say $\theta \in (0, 1]$ of its wealth and $\beta \in (0, 1]$ of what it receives from other banks.

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- Φ_i has a discontinuity whenever $\bar{p}_i = x_i + \sum_{j=1}^d A_{ji} p_j$.
- Existence of solution is still guaranteed by Knaster-Tarski theorem.
- Greatest clearing vector algorithm / Gaussian elimination (El Bitar, Kabanov, Mokbel '18)

Rogers-Veraart '13 model

- Unlike in Eisenberg, Noe '01 model, an LP characterization of a clearing vector is not possible due to the jumps in Φ.

Rogers-Veraart '13 model

- Unlike in Eisenberg, Noe '01 model, an LP characterization of a clearing vector is not possible due to the jumps in Φ.
- Instead: We introduce binary variables for the jumps and find a clearing vector by solving the following MILP:

$$
\begin{aligned}
\text{maximize } & \sum_{i=1}^{d} p_i \\
\text{s.t. } & p_i \leq \theta x_i + \beta \sum_{j=1}^{d} A_{ji} p_j + \bar{p}_i s_i, \quad i \in \{1, \dots, d\} \\
& p_i s_i \leq x_i + \sum_{j=1}^{d} A_{ji} p_j, \quad i \in \{1, \dots, d\} \\
& 0 \leq p_i \leq \bar{p}_i, \quad i \in \{1, \dots, d\} \\
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$$\text{maximize } \sum_{i=1}^{d} p_i$$

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$$0 \leq p_i \leq \bar{p}_i, \quad i \in \{1, \ldots, d\}$$

$$s_i \in \{0, 1\}, \quad i \in \{1, \ldots, d\}.$$

- As before the optimal value gives the total debt paid in the system. Let us call it $\Lambda(x)$, the value of the aggregation function.

Rogers-Veraart '13 model

- Λ is increasing but fails to be quasiconcave in general.
- $X \mapsto R^{\text{sen}}(X) = \left\{z \in \mathbb{R}^g_+ \mid \rho(\Lambda \circ (X + Bz)) \leq 0\right\}$ fails to be quasiconvex in general.
- Consequently, $R^{\text{sen}}(X)$ may be a nonconvex set.

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- Consequently, $R^{\text{sen}}(X)$ may be a nonconvex set.
- Nevertheless, we can calculate $R^{\text{sen}}(X)$ at least when

$$R^{\text{sen}}(X) = \left\{z \in \mathbb{R}^g \mid \mathbb{E}\left[\Lambda \circ (X + Bz)\right] \geq \gamma \mathbf{1}^{\mathsf{T}} \bar{p}\right\},$$

where $\gamma \in [0, 1]$ is the average fraction of total debt that should be paid at clearing.
- Nonconvex Benson's algorithm (Nobakhtian, Shafiei '17).

Computational study

- Two groups: big banks ("core") and small banks ("periphery")
- Random shock: Gaussian copula with gamma marginals (positive case), Gaussian random vector (signed case)
- Network structure generated as an instance of a random network with independent coin flips for connections and gamma distributed nominal liabilities
- Probabilities: core-core: high, core-periphery: medium, periphery-core: low, periphery-periphery: low
- Risk measure: Negative expectation, AVaR at 95%, entropic risk measure

Decomposition performance

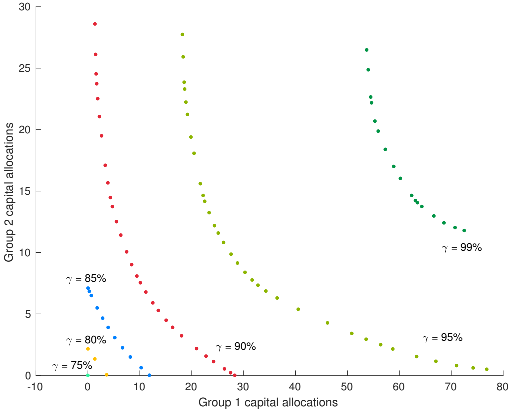
- Eisenberg-Noe model with $X \geq 0$
- Negative expectation with 15 big banks, 50 small banks

#scenarios	time (s)	#opt.	time/opt.	#bundle	#bunde/opt.
50	325	33	9.6	411	12.45
100	653	33	19.7	497	15.01
200	1376	33	41.5	545	16.52
400	2462	33	73.7	480	14.55
800	4836	33	142.2	446	13.52
1600	10339	35	277.4	458	13.01
3200	out of mem.				

- Other risk measures: some numerical issues with the bundle algorithm to be fixed

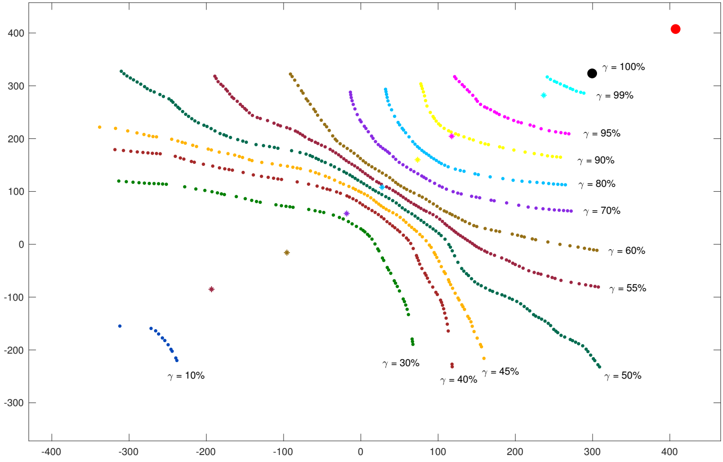
Eisenberg-Noe with positive random shock

- Two groups: 15 big banks ("core") and 50 small banks ("periphery")
- 50 scenarios, risk-neutral



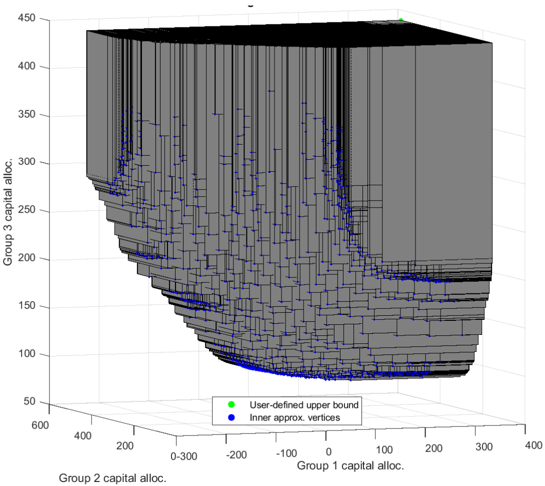
Eisenberg-Noe with signed random shock

- Two groups: 7 big banks ("core") and 8 small banks ("periphery")
- 10 scenarios



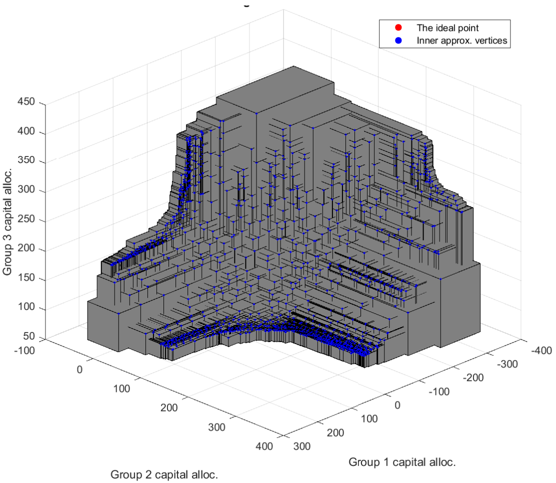
Eisenberg-Noe with signed random shock

- Three groups: 3 big banks, 5 medium banks, 10 small banks
- 20 scenarios, $\gamma = 0.82$
- Inner approximation with 1289 vertices



Eisenberg-Noe with signed random shock

- Three groups: 3 big banks, 5 medium banks, 10 small banks
- 20 scenarios, $\gamma = 0.82$
- Complement of outer approximation with 2685 vertices



Happy birthday Yuri Kabanov!