Dynamic Cournot-Nash equilibrium via Causal Optimal Transport

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joint work with Julio Backhoff-Veraguas

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Problem formulation

Given:

- → a population of agents whose **type evolves in time**
- → agents select their own **actions/strategies in time**
- → agents face a **cost** that depends on their own type, action, and on the mean-field interaction with the rest of the population

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Aim:

- → characterize **equilibria** for games in this setting
- → develop & exploit connection with **causal optimal transport**

Setting

- Discrete time $t = 1, ..., T$; game played at time $t = 1$.
- $\mathbb{X}^T$ = path-space of types, and $\mathbb{Y}^T$ = path-space of actions
- $\eta^i \in \mathcal{P}(\mathbb{X}^T)$: type distribution for player $i = 1, ..., N$ (known)

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Cost function:

$$\begin{array}{ccc} & F(x, y, \nu) & \\ \nearrow & \uparrow & \nwarrow \\ \text{type} & \text{action} & \text{actions} \\ x\in\mathbb{X}^T & y\in\mathbb{Y}^T & \text{distribution} \\ & & \nu\in\mathcal{P}(\mathbb{Y}^T) \end{array}$$

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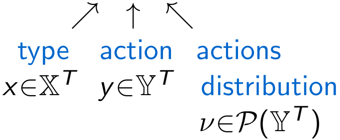
Cost function: $F(x, y, \nu)$



type $x \in \mathbb{X}^T$

action $y \in \mathbb{Y}^T$

actions distribution $\nu \in \mathcal{P}(\mathbb{Y}^T)$

Usually: $F(x, y, \nu) = \sum_{t=1}^{T} F_t(x_{1:t}, y_{1:t}, \nu_{1:t})$

Example 1

Route planning

$$\mathbb{X} = \{\text{possible destinations}\}, \quad \mathbb{Y} = \{\text{possible routes}\}$$

- Population: holiday makers in the same region.
- Type: next destination.
- Action: which route to take to reach the next destination.
- Mean-field interaction: traffic.
- Cost: takes into account distance/tolls relative to the chosen destination, and the congestion effect.

Example 2

Consumption/investment planning

$$\mathbb{X} = \mathbb{R}_+, \quad \mathbb{Y} = \mathbb{R}^n \times \mathbb{R}_+$$

- Population: investors in a given market with n risky assets and 1 riskless asset.
- Type: $x =$ consumption appetite/need.
- Action: consumptions c, # shares in each risky asset.
- Mean-field interaction: via price impact.
- Cost: takes into account the relation x/c, and the expected terminal wealth (price impact effect).

Pure Nash equilibrium

$\mathcal{F}_t$: all agents' type up to time t (common knowledge)

Pure strategy: $\mathcal{F}$-adapted $\mathbb{Y}^N$-valued process $(Y^1, \ldots, Y^N)$

Cost faced by player i for every pure strategy $(Y^1, \ldots, Y^N)$:

$$J^i(Y^1, \ldots, Y^N) := \int_{\mathbb{X}^{N\times T}} F\left(X^i, Y^i, \tfrac{1}{N}\textstyle\sum_{k=1}^N \delta_{Y^k}\right) \bar{\eta}(dX),$$

where $\bar{\eta} := \otimes_{i\leq N}\, \eta^i$ (average over all possible type evolutions)

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Definition (Pure Nash equilibrium)

$(Y^1, ..., Y^N)$ is a Pure Nash equilibrium if, for all i and all $\mathcal{F}$-adapted $\mathbb{Y}$-valued processes $\widetilde{Y}^i$:

$$J^i(Y^1, ..., Y^N) \leq J^i(Y^1, .., Y^{i-1}, \widetilde{Y}^i, Y^{i+1}, .., Y^N).$$

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$\rightarrow$ Pure equilibria rarely exists $\Rightarrow$ consider randomized strategies

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Mixed-strategy: measurable $Z : \mathbb{X}^{N\times T} \to \mathcal{P}(\mathbb{Y}^{N\times T})$ s.t. $\forall t$:

$$\int_{\mathbb{Y}^{N\times T}} f\left(\left\{Y_s^k : s \leq t, k \leq N\right\}\right) Z(dY)$$

is $\mathcal{F}_t$-measurable, for all bounded Borel functions $f : \mathbb{Y}^{N\times t} \to \mathbb{R}$.

Cost: $L^i(Z) := \int \int F\left(X^i, Y^i, \frac{1}{N}\sum_{k=1}^N \delta_{Y^k}\right) Z(X)(dY)\, \bar{\eta}(dX)$

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Definition (Mixed Nash equilibrium)

A mixed strategy Z is called a Mixed Nash equilibrium if, for all i,

$$L^i(Z) \le L^i(\widetilde{Z}) \quad \text{for all mixed strategies } \widetilde{Z} \text{ s.t.}$$

$$\int_{\mathbb{Y}^{N\times T}} f(\{Y^k : k \ne i\}) Z(dY) = \int_{\mathbb{Y}^{N\times T}} f(\{Y^k : k \ne i\}) \widetilde{Z}(dY)\ \bar{\eta}\text{-a.s.}$$

for every bounded Borel $f : \mathbb{Y}^{(N-1)\times T} \to \mathbb{R}$.

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- in particular η^i "converge" to some $\eta \in \mathcal{P}(\mathbb{X}^T)$

→ We study **asymptotic problem** for a type-distribution η

Our toolkit: Optimal Transport

- **Optimal Transport:** given two Polish spaces $(\mathcal{X}, \mu), (\mathcal{Y}, \nu)$, and a cost function $c : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$, minimize cost of transportation of μ into ν:

$$\inf \{\mathbb{E}^{\pi}[c(x, y)] : \pi \in \Pi(\mu, \nu)\}$$

$$\Pi(\mu, \nu) := \{\pi \in \mathcal{P}(\mathcal{X} \times \mathcal{Y}) : \mathcal{X}\text{-marginal } \mu,\ \mathcal{Y}\text{-marginal } \nu\}$$

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$$\pi(dy_t | dx_1, \cdots, dx_T) = \pi(dy_t | dx_1, \cdots, dx_t) \quad \forall t$$

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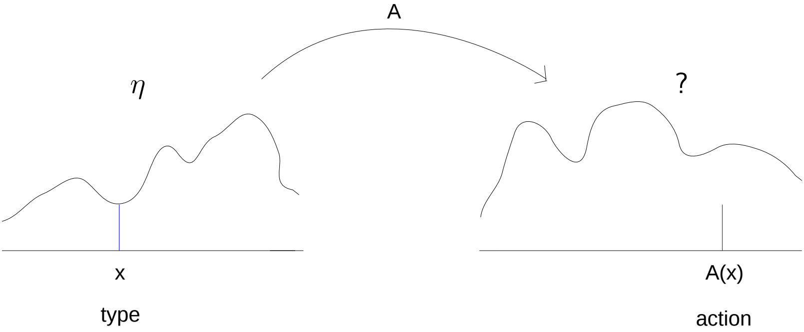
$$\pi(dy_t|dx_1,\cdots,dx_T)=\pi(dy_t|dx_1,\cdots,dx_t)\quad\forall t$$

We denote $\Pi_c(\eta,\nu) := \{\pi\in\Pi(\eta,\nu) : \pi \text{ causal}\}$

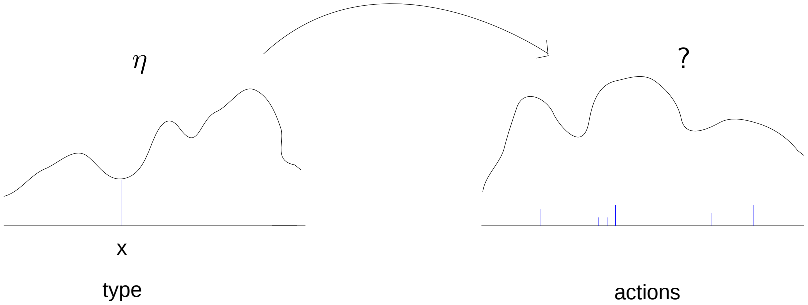
$$\Pi_c(\eta,.) := \cup_{\xi\in\mathcal{P}(\mathbb{Y}^T)}\Pi_c(\eta,\xi)$$

Pure equilibrium / Monge transport



adapted **pure** strategy = adapted **Monge** transport

Mixed-strategy equilibrium / Kantorovich transport



non-anticipative **mixed** strategy = causal **Kantorovich** transport

Cournot-Nash equilibrium

$$N \longrightarrow \infty$$

(ϵ-)Nash equilibria $\longrightarrow$ Cournot-Nash equilibria

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(ϵ-)Nash equilibria → Cournot-Nash equilibria

Definition (Cournot-Nash)

$\pi^* \in \Pi_c(\eta, .)$ is called Cournot-Nash equilibrium if:

1. π^* attains $\inf\limits_{\pi \in \Pi_c(\eta,.)} \mathbb{E}^\pi[F(x, y, \nu^*)]$,
2. π^* has second marginal ν^*

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Called pure if, $\forall t$, $y_t = g_t(x_{1:t})$ π-a.s. for some measurable g_t.

Potential games

We study the asymptotic problem in the following setting:

- separable cost: $F(x, y, \nu) = \underbrace{f(x, y)}_{\text{idiosyncratic part}} + \underbrace{V[\nu](y)}_{\text{mean-field interaction}}$
- potential game: V is the first variation of $\mathcal{E} : \mathcal{P}(\mathbb{Y}^T) \to \mathbb{R}$:

$$\lim_{\epsilon \to 0^+} \frac{\mathcal{E}(\nu + \epsilon(\xi - \nu)) - \mathcal{E}(\nu)}{\epsilon} = \int_{\mathbb{Y}^T} V[\nu] d(\xi - \nu), \quad \nu, \xi \in \mathcal{P}(\mathbb{Y}^T)$$

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  - **Congestion effect**: $V^c[\nu](y) = h\left(\frac{d\nu}{dm}(y)\right)$, with $m \in \mathcal{P}(\mathbb{Y}^T)$ reference measure, wrt which congestion measured, and $h \nearrow$ $\mathcal{E}^c(\nu) = \int_{\mathbb{Y}^T} H\left(\frac{d\nu}{dm}(y)\right) dm(y)$, where $H(u) = \int_0^u h(s)ds$
  - **Attractive effect**: $V^a[\nu](y) = \int_{\mathcal{Y}} \phi(y,z) d\nu(z)$, with ϕ cont, symmetric, convex, minimal on the diagonal $\mathcal{E}^a(\nu) = \frac{1}{2} \int_{\mathbb{Y}^T} \int_{\mathbb{Y}^T} \phi(y,z) d\nu(z) d\nu(y)$

Variational problem

Consider the variational problem

$$(VP) \qquad \inf_{\nu \in \mathcal{P}(\mathbb{Y}^T)} \left\{ \underbrace{\inf_{\pi \in \Pi_c(\eta,\nu)} \mathbb{E}^{\pi}[f(x,y)]}_{\mathrm{COT}(\eta,\nu)} + \mathcal{E}[\nu] \right\}$$

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Theorem (Equivalence CN and VP)

Let $\mathcal{E}$ be convex, then the following are **equivalent***:*

(i) π^* *is a Cournot-Nash equilibrium;*

(ii) $(p_2(\pi^*), \pi^*)$ *solves (VP).*

Note: Convexity of $\mathcal{E}$ is only needed for "$(i) \Rightarrow (ii)$", and is e.g. satisfied by $\mathcal{E}^c$.

Existence and uniqueness

Corollary (Existence)

Let f be l.s.c. and bounded below. Then

- $V = V^c$ *and growth condition on* $h \Rightarrow \exists$ *CN equilibria*;
- $V = V^a$ *and growth condition on* $f \Rightarrow \exists$ *CN equilibria*.

Growth conditions ensure existence of a solution ν^* to (VP), and COT(η, ν^*) admits a solution π^* easily. Apply previous theorem.

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Corollary (Uniqueness)

If $\mathcal{E}$ *strictly convex* $\Rightarrow$ *unique optimal distribution of actions (all CN equilibria have same second marginal* ν^**).*

Indeed, $\nu \mapsto$COT(η, ν) convex, hence $\mathcal{E}$ strictly convex implies unique solution ν^* for (VP). Then apply previous theorem.

Structure of equilibria: first thoughts

Let $\mathcal{X} = \mathcal{Y} = \mathbb{R}^T$. Assume

- η has independent increments, and
- $f(x, y) = f_1(x_1, y_1) + \sum_{t=2}^{T} f_t(\Delta x_t - \Delta y_t)$, with f_t convex.

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Then:

- CN equilibria are **Knothe-Rosenblatt rearrangements** (and uniquely determined by the second marginal).
- If moreover η has a density, all CN equilibria are pure.

The Knothe-Rosenblatt map

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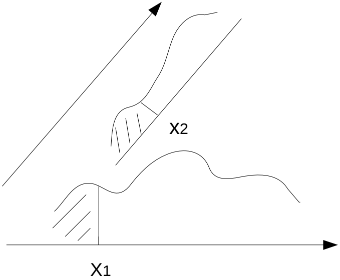

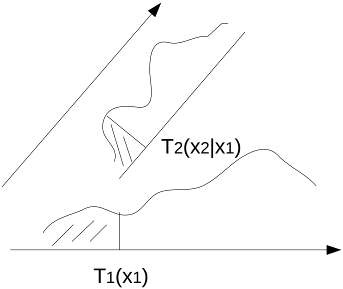

Competitive vs cooperative equilibria

- Cooperative equilibria: minimize average cost in N-player game. Asymptotically this becomes:

$$\inf_{\pi \in \Pi_c(\eta,.)} \mathbb{E}^{\pi}[F(x, y, p_2(\pi))]$$

$\rightarrow$ for competitive equilibria we had a fixed point problem

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- In the separable case:

$$\inf_{\nu \in \mathcal{P}(\mathbb{Y}^T)} \{\mathrm{COT}(\eta, \nu) + \mathbb{E}^{\nu}[V[\nu]]\}$$

$\rightarrow$ here equivalence always true with the above variational problem, while for competitive equilibria we needed potential games, and $\mathcal{E}$ convex

Conclusions

We have seen:

- A characterization of competitive equilibria via causal optimal transport;
- Existence and uniqueness results in the potential case;
- First structural results via K-R rearrangements;
- Hint to cooperative equilibria.

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Work in progress:

- Develop numerics for equilibria & price of anarchy.
- Which form of transports/equilibria do we expect when the K-R requirements are not fulfilled?
- Exploit transport-typical concepts, such as displacement convexity, e.g. to obtain uniqueness.

Some literature

Competitive equilibrium with a continuum of agents, static case:

- Schmeidler (1973)
- Mas-Colell (1984)
- . . .
- Blanchet and Carlier (2015), Lacker and Ramanan (2017)

Optimal Transport, and Causal OT:

- Monge (1781)
- Kantorovich (1942)
- . . .
- Lassalle (2013), Backhoff, Beiglböck, Lin, Zalashko (2016), A., Backhoff, Zalashko (2016), A., Backhoff, Carmona (2018)

Thank you for your attention and Happy Birthday Yuri!