Perspectives on real geometry at CIRM

On multi-Harnack smoothings of real plane branches

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Based on a joint work with Jean-Jacques Risler.

Outline

- Introduction
- Harnack curves in toric surfaces
- Semi-quasi-homogeneous deformations and Viro method
- A sequence of toric maps
- Constructing msqh-deformations
- Multi-Harnack smoothings

Let $C \subset \mathbf{C}P^2$ be a **real algebraic curve** with **real part** $\mathbf{R}C \subset \mathbf{R}P^2$.

16^{th} **Hilbert problem:**

Determine the topological type of the pairs $(\mathbf{R}C, \mathbf{R}P^2)$, for smooth C of degree d.

Harnack bound: The number of connected components of C is $\leq \frac{1}{2}(d-1)(d-2)+1$.

C is an **M-curve** if $\mathbf{R}C$ has the maximal number of components.

A germ $(C, 0)$ of real plane curve singularity is a **real plane branch** if it is analytically irreducible in $(\mathbf{C}^2, 0)$.
It has a Newton Puiseux parametrization:

$$\begin{cases} x(t) = t^n, \\ y(t) = \sum_{i \geq n} \eta_i t^i, \quad \text{with } \eta_i \in \mathbf{R}. \end{cases}$$

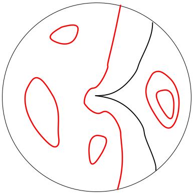
Local version of 16^{th} Hilbert problem

Let $(C, 0) \subset (\mathbf{C}^2, 0)$ be a real algebraic plane **curve singularity**. We denote by $\mathbf{R}C$ its real part.

Let B be a **Milnor ball** for $(C, 0)$.

A **smoothing** of C is a real analytic family C_t with $C_0 = C$ and $C_t \cap B$ smooth and transversal to ∂B, for $0 < t \ll \epsilon$.

Problem: Determine the possible topological types of smoothings $(\mathbf{R}C_t, \mathbf{R}B)$.



- The number of **non-compact components** of a smoothing is equal to the number $r_{\mathbf{R}}$ of real branches of C.

- The compact components of the smoothing are called **ovals**.

Local Harnack bound: the number of **ovals** of a smoothing of $(C, 0)$ is $\leq$

$$\begin{cases} \frac{1}{2}(\mu(C) - r + 1) \text{ if } r_{\mathbf{R}} \geq 1, \\ \frac{1}{2}(\mu(C) - r + 3) \text{ if } r_{\mathbf{R}} = 0, \end{cases}$$

where:

- $\mu(C)$ is the Milnor number of C.
- r is the number of branches of C viewed in $(\mathbf{C}^2, 0)$.

Definition: C_t is a M-**smoothing** if the number of ovals is equal to the local Harnack bound.

- Risler proved that if $(C, 0)$ is a real branch a M-smoothing of C exists and it can be constructed by the blowing up construction.
- M-smoothings do not always exists (Kharlamov, Orevkov, Shustin).
- Other classes of M-smoothings of real plane curves singularities were obtained by Kharlamov, Risler, Shustin and Chevalier.

- Risler's motivation was to study to which extent Mikhalkin's rigidity property of Harnack curves in toric surfaces, generalizes to the local case.

Some definitions: Positions of curves and lines in $\mathbf{P}^2$

- C **is in maximal position with respect to a line** L if there exists an arc $\mathbf{a} \subset \mathbf{R}C$ such that

$$\mathbf{a} \cap L = C \cap L, \text{ transversally.}$$

- C has **good oscillation** with respect to L if in addition the points in $\mathbf{a} \cap L$ appear in the same order on the arc $\mathbf{a}$ and on the line L.

- C is in **maximal position with respect to lines** $L_1, L_2, \ldots, L_r$ if there exists disjoint arcs $\mathbf{a}_1, \ldots, \mathbf{a}_r$ contained in the same component of $\mathbf{R}C$ such that

$$\mathbf{a}_i \cap L = C \cap L_i \text{ transversally, for } i = 1, \ldots, r.$$

Let Θ be a two dimensional integral polytope in $\mathbf{R}^2_{\geq 0}$.

$Z(\Theta)$ is the **toric surface** defined by Θ and $\mathbf{R}Z(\Theta)$ is its real part.

The moment map: $(\mathbf{R}^*)^2 \to \mathrm{int}(\Theta)$ induces a stratified diffeomorphism

$$\mathbf{R}Z(\Theta) \longrightarrow \bigsqcup_{\rho \in \mathcal{S}} \rho(\Theta)/\sim,$$

where $\mathcal{S} \cong \mathbf{Z}_2^2$ is the group of symmetries of $(\mathbf{R}^*)^2$, and $\sim$ is a natural identification between the edges.

- Let $F \in \mathbf{R}[x, y]$ with **Newton polygon** Θ.

- We denote by C_F the real alg. curve defined by $F = 0$ in $Z(\Theta)$.

- C_F does not pass through the intersection points of the toric coordinate axes.

Definition (Mikhalkin). C_F is a simple **Harnack curve** in $Z(\Theta)$ if it is a M-curve and it is **cyclically in maximal position with good oscillation** with respect to the toric coordinate lines of $Z(\Theta)$.

Theorem. (Mikhalkin) If C is a Harnack curve in $Z(\Theta)$ then the topological type of the triple:

$$(\mathbf{R}Z(\Theta), \mathbf{R}C, (\mathbf{R}^*)^2)$$

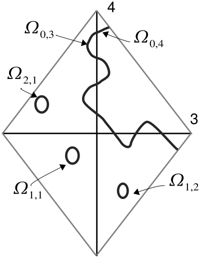
is uniquely determined by Θ.

Remarks.

- Harnack curves in $Z(\Theta)$ can be constructed by patchworking.
- There is only one component meeting the coordinate axes.
- There are $\#(\mathrm{int}(\Theta) \cap \mathbf{Z}^2)$ other components which are ovals.

Example: Θ is the triangle with vertices $(0,0)$ $(3,0)$ and $(0,4)$.

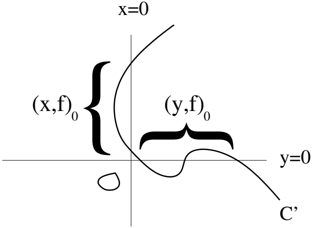
By using the **moment map** the real part of a Harnack curve $C_F \subset Z(\Theta)$ can be represented in the figure:



Question: Does Mikhalkin's Theorem generalize to Harnack smoothings?

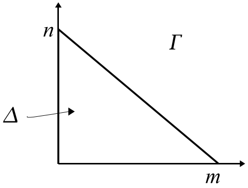
Definition. A **Harnack smoothing** is a M-smoothing such that it has maximal position with respect to the coordinate axes.

Example. A Harnack smoothing of the cusp $y^2 - x^3 = 0$.



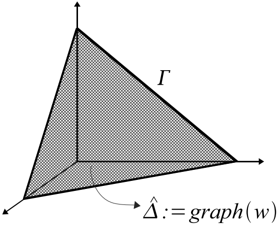
Semi-quasi-homogeneous deformations and Viro method

Let $f(x,y) = \sum a_{ij} x^i y^j \in \mathbf{R}\{x,y\}$ with **local Newton polygon** of the form:



$f_\Gamma = \prod_{k=1}^{e} (y^p - \theta_k x^q)$ is **quasi-homogeneous** $(\mathrm{gcd}(p,q)=1)$.
The $\theta_k \in \mathbf{C}$ are called the **peripheral roots** of f.

Set $w(i,j) = nm - ni - mj$ for $(i,j) \in \Delta$ and $\hat{\Delta} := \text{graph } (w)$.



Definition. $F_t(x,y)$ defines a **semi-quasi-homogeneous** deformation of $f(x,y)$ if $F_0(x,y) = f(x,y)$ and the local Newton polyhedron of $F = F_t(x,y) \in \mathbf{R}[t]\{x,y\}$ is $\hat{\Delta} + \mathbf{R}^3_{\geq 0}$.

Notice that $F_{\hat{\Delta}}$ is **quasi-homogeneous** in t, x, y

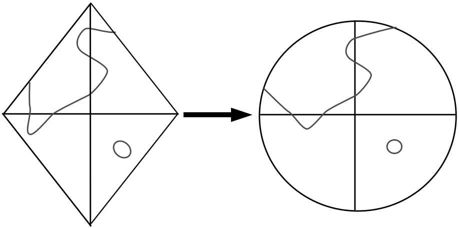
Theorem. (Viro) Let $F = F_t(x, y)$ define a **sqh-deformation** of $F_0(x, y)$ as above. If the polynomial

$$(F_{\hat{\Delta}})_{t=1}$$

is **real Newton non-degenerate** then $F_t(x, y)$ defines a smoothing of $F_0(x, y)$. The topological type of the smoothing C_t is determined by the topological type of real part of the curve C_Δ, defined by $(F_{\hat{\Delta}})_{t=1} = 0$, in $\mathbf{R}Z(\Delta)$.

Real Newton non-degenerate implies that the real part of the curve C_Δ is smooth and transversal to the toric coordinate axes.

Example. From a Harnack curve on the toric surface to a sqh-smoothing of $y^3 - x^2$



Proposition 1. *Let* $f(x,y) \in \mathbf{R}\{x,y\}$ *as above define a real germ* $(C,0) \subset (\mathbf{C}^2,0)$*. Assume that all its peripheral roots are real and of the same sign. Then:*

- there exists a Harnack sqh-smoothing C_t*.*

- the topological type of triple $(\mathbf{R}B, \mathbf{R}C_t, B \cap (\mathbf{R}^*)^2)$ *is unique.*

Remarks.

- The existence of this Harnack sqh-smoothing is obtained using Viro patchworking.

- The unicity of the topological type is based on Mikhalkin's result.

Question. Does this result generalize to real plane branches?

We cannot apply Viro method since they are not Newton non-degenerate in general.

Idea: Improve the singularity with a sequence of toric maps.

Example:
$(C, 0)$ the singularity defined by $f := (y^2 - x^3)^3 - x^{10}$
Consider the monomial map

$$\begin{array}{rcl} x & = & u_1^1 x_1^2, \\ y & = & u_1^1 x_1^3. \end{array}$$

This map is a **chart** of a proper toric map, which is a composition of point blow ups, which appears in the process of resolution of $(C, 0)$.

We have that:

$$f \circ \pi_1 \quad = \quad u_1^6 x_1^{18}\big((1-u_1)^3 - u_1^{10} x_1^{20}\big).$$

The divisor of $f \circ \pi_1$ has a **exceptional** term defined by: $u_1^6 x_1^{18} = 0$.

On this chart **only** the component $x_1 = 0$, intersects the strict transform $C^{(1)}$ of the branch C at the point $u_1 = 1$.

Set new coordinates $(x_1, y_1 := 1 - u_1)$. The **strict transform** $C^{(1)}$, defined by

$$f^{(1)}(x_1, 1 - y_1) = y_1^3 - x_1^2(1 - y_1)^4 = 0,$$

is **Newton non-degenerate** with respect to (x_1, y_1).

Idea. Construct a sqh-smoothing of $C^{(1)}$ and blow it down to get a deformation C_{t_1} of C.

Consider a sqh-deformation of $C^{(1)}$ of the form:

$$\begin{aligned} G_{t_1} = \; & y_1^3 - x_1^2(1-y_1)^4 \\ & + a_{0,0} t_1^6 + a_{1,0} y_1 t_1^4 + a_{0,2} y_1^2 t_1^2 + a_{1,1} x_1 y_1 t_1 + a_{1,0} x_1 t_1^3, \end{aligned}$$

for suitable $a_{i,j} \in \mathbf{R}$.

Problem: Blowing down this deformation we get *meromorphic* functions in general.

Instead, we build functions $M_{i,j} \in \mathbf{R}[x, y]$, for $(i, j) \in (\Delta_1 \cap \mathbf{Z}^2) \setminus \Gamma_1$ such that:

$$F_{t_1} := f(x, y) + \sum a_{i,j} t_1^{w_1(i,j)} M_{i,j} \in \mathbf{R}[t]\{x, y\}$$

defines a deformation C_{t_1} of $(C, 0)$ and:

- F_{t_1} and f have the same local Newton polygon.
- F_{t_1} is Newton non-degenerate for $0 \neq t_1 \ll 1$.
- $F_{t_1}^{(1)}(x_1, 1 - y_1)$ defines a sqh-smoothing $C_{t_1}^{(1)}$ of the strict transform $C^{(1)}$.

In the previous example, set $z = y^2 - x^3$.
Notice that the strict transform of z by π defines one of the local coordinates y_1. Then:

(r,s)	$(0,0)$	$(0,1)$	$(0,2)$	$(1,1)$	$(1,0)$
$M_{r,s}$	x^9	x^6z	x^3z^2	x^5yz	x^8y

and

$$F_{t_1} = a_{0,0}M_{0,0}t_1^6 + a_{1,0}M_{0,1}t_1^4 + a_{0,2}M_{0,2}t_1^2 + a_{1,1}M_{1,1}t_1 + a_{0,1}M_{1,0}t_1^3.$$

for certain $a_{i,j} \in \mathbf{R}$.

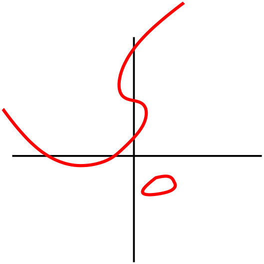
Let C be a real branch with two characteristic pairs.

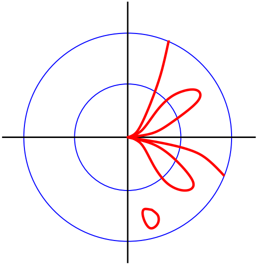
Definition. We say that C_{t_0,t_1} is a **multi-Harnack smoothing** of C if

- C_{t_0,t_1} is a Harnack smoothing of C with respect to the coordinates axes (x, y) for $0 < t_0 \ll t_1 \ll 1$.
- $C_{t_1}^{(1)} := C_{t_0=0,t_1}^{(1)}$ is a Harnack smoothing of the strict transform $C^{(1)}$ with respect to the coordinate axes (x_1, y_1).
- The charts of C_{t_0,t_1} and C_{t_1} have **regular intersection**.

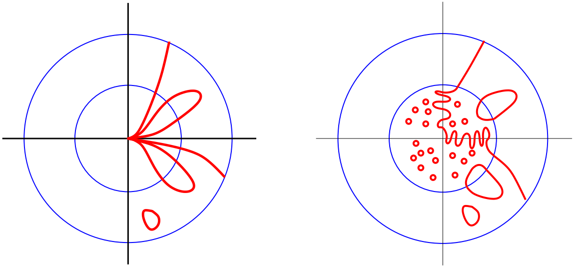
If C_{t_0,t_1} is a **multi-Harnack smoothing** of C then:

- $C_{t_1}^{(1)}$ is a Harnack smoothing of $C^{(1)}$ w.r.t. (x_1, y_1).
- C_{t_1} is Newton **non-degenerate** with peripheral roots of the same sign.





- Regular intersection means that the charts of C_{t_1} and C_{t_0,t_1} glue up providing the maximal number of ovals.



Theorem 2.

If $C_{\underline{t}}$ is a multi-Harnack smoothing then the topological type of the triples

$$(\mathbf{R}\bar{B}, \mathbf{R}C_{\underline{t}}, B \cap (\mathbf{R}^*)^2),$$

is determined by the embedded topological type of the branch $(C, 0) \subset (\mathbf{C}^2, 0)$.

Theorem 3.

Consider a msqh-smoothing C_{t_0,t_1} of a real branch $(C,0)$, with two characteristic pairs. Assume that:

- $C_{t_1}^{(1)}$ *is a sqh* M*-smoothing of* $C^{(1)}$.
- C_{t_0,t_1} *is a sqh* M*-smoothing of* C_{t_1}.
- $C_{t_1}^{(1)}$ *is in maximal position w.r.t. the exceptional divisor* $x_1 = 0$.
- *The charts of* C_{t_0,t_1} *and* C_{t_1} *have regular intersection.*

Then, C_{t_0,t_1} *defines a* M*-smoothing of* $(C,0)$*, for* $0 < t_0 \ll t_1 \ll 1$.

Remarks.

- Theorem 2 and 3 can be stated for arbitrary real plane branches.

- Unicity of the topological type in Theorem 2 is deduced from Mikhalkin theorem, combined with explicit form of the toric maps appearing in the process.

- We build the deformations in terms of monomials in polynomials $x, y = y_0, y_1, \dots, y_{g-1} \in \mathbf{R}[x, y]$ such that

$$(x, f)_0, (y_0, f)_0, \dots, (y_{g-1}, f)_0$$

define the minimal sequence of generators of the semigroup of the branch C.

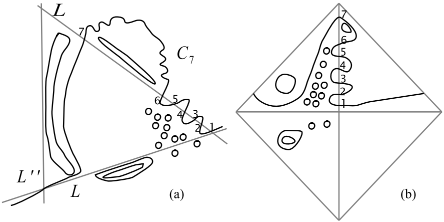
- The asymptotic size of the ovals in msqh-M-smoothings provides also the generators of this semigroup.

Remark. We can build with this method Harnack M-msqh smoothings which are not multi-Harnack.

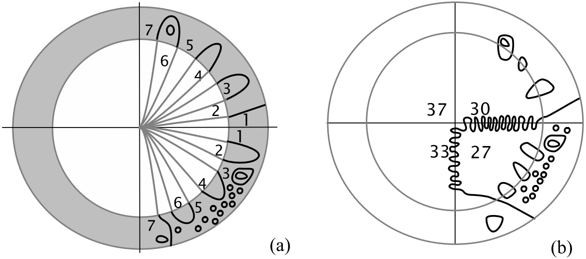
Example: Let $f = (y^2 - x^3)^7 - x^{24}$.

We can construct geometrically a degree seven M-curve with Newton polygon with vertices $(0, 7)$, $(0, 6)$ and $(0, 0)$, whose real part is represented in the figure.

We build from it a sqh-smoothing of $C^{(1)}$.



We obtain the following after blowing down and taking a Harnack smoothing.



(a)
(b)

Thanks!