Associative algebras arising from surfaces with orbifold points.

Based on joint works with
Jan Geuenich and Diego Velasco.

Cluster algebras: 20 years on
CIRM, Luminy

Surfaces with orbifold points.

$$(\Sigma, \mathbb{M}, \mathbb{O})$$

$\mathbb{O} \subseteq \Sigma \setminus (\mathbb{M} \cup \partial\Sigma)$, every element of $\mathbb{O}$ has an **order**
finite,
may be empty

$\phi \neq \mathbb{M} \subseteq \Sigma$, at least one point from each component of $\partial\Sigma$
finite

2-dimensional,
compact,
connected,
oriented,
topological surface

Typically: $\Sigma \setminus \mathbb{M} = \mathbb{H}^2/\Gamma$, $\Gamma \underset{\text{discrete}}{\leq} PSL_2(\mathbb{R})$

$\mathbb{M} \longleftrightarrow$ fixed points of parabolics

$\mathbb{O} \longleftrightarrow$ fixed points of elliptics

For $(\Sigma, M, \mathcal{O}) = (\Sigma, M)$, ie., when $\mathcal{O} = \emptyset$,

Fock–Goncharov
Fomin–Shapiro–Thurston
Fomin–Thurston
Penner

$\rightsquigarrow$ skew-symme**tric** cluster algebras

When all orbifold points have order 2,

Felikson–Shapiro–Tumarkin $\rightsquigarrow$ skew-symmetriz**able** cluster algebras

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When the orbifold points have arbitrary order,

Chekhov–Shapiro $\rightsquigarrow$ generalized cluster algebras

In this talk:

- $\mathbb{M} \subseteq \partial\Sigma \neq \emptyset$
- all orders of orbifold points $= 2$

  or

  all orders of orbifold points $= 3$

First construction:

Species with potential

for

surfaces with orbifold points of order 2

(jt. with Jan Geuenich)

Species?

The path algebra of a quiver Q is the tensor algebra $T_R(A)$, where

$$R = \bigtimes_{i \in Q_0} K \quad \text{Cartesian product}$$

$$A = \bigoplus_{a \in Q_1} Ka \quad R\text{-}R\text{-bimodule}$$

Thm. (Gabriel) $K = \overline{K} \Rightarrow$ every f.d. K-algebra is Morita-equivalent to an admissible quotient of **some** path algebra $T_R(A)$.

Species = suitable notion of (R, A) to deal with $K \neq \overline{K}$.

For future reference

$\mathbb{F}_q :=$ finite field with q elements, $\mathrm{char}(\mathbb{F}_q) \neq 2$

Write $\mathrm{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q) = \{\mathbb{1}, \theta\}$ (so, $\theta : \mathbb{F}_{q^2} \to \mathbb{F}_{q^2}$ is the Frobenius)

$\mathbb{F}_{q^2}$–$\mathbb{F}_{q^2}$-bimodules $\mathbb{F}_{q^2}^{\mathbb{1}}$ and $\mathbb{F}_{q^2}^{\theta}$:

$\mathbb{F}_{q^2}^{\mathbb{1}} := \mathbb{F}_{q^2}$ with usual left and right $\mathbb{F}_{q^2}$-actions.

$\mathbb{F}_{q^2}^{\theta} := \begin{cases} \mathbb{F}_{q^2} \text{ as abelian group,} \\ \text{with left and right } \mathbb{F}_{q^2}\text{-actions given by} \\ z \cdot m := zm \quad \text{(left action)} \\ m \cdot z := m\theta(z) \quad \text{(right action)} \end{cases}$

Notice: $\mathbb{F}_{q^2}^{\mathbb{1}} \not\cong \mathbb{F}_{q^2}^{\theta}$ as $\mathbb{F}_{q^2}$–$\mathbb{F}_{q^2}$-bimodules

$\underline{\Sigma} = (\Sigma, M, \mathbb{O}) =$ surface with orbifold points of order 2

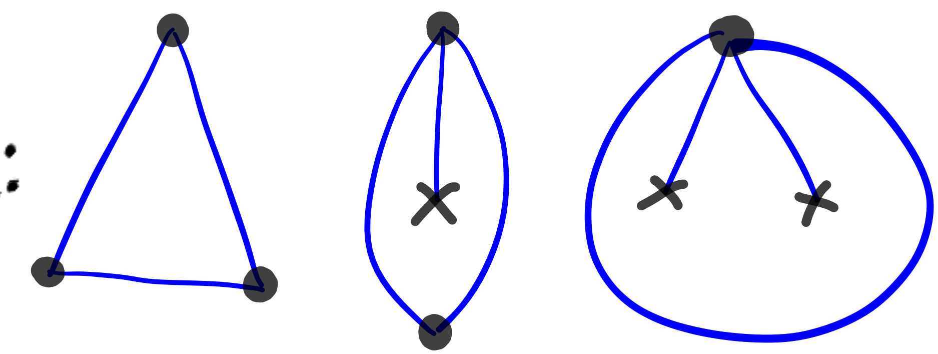
$\bullet$ = point in M

$\times$ = point in $\mathbb{O}$

We shall associate species and potentials to the triangulations of $\underline{\Sigma}$.

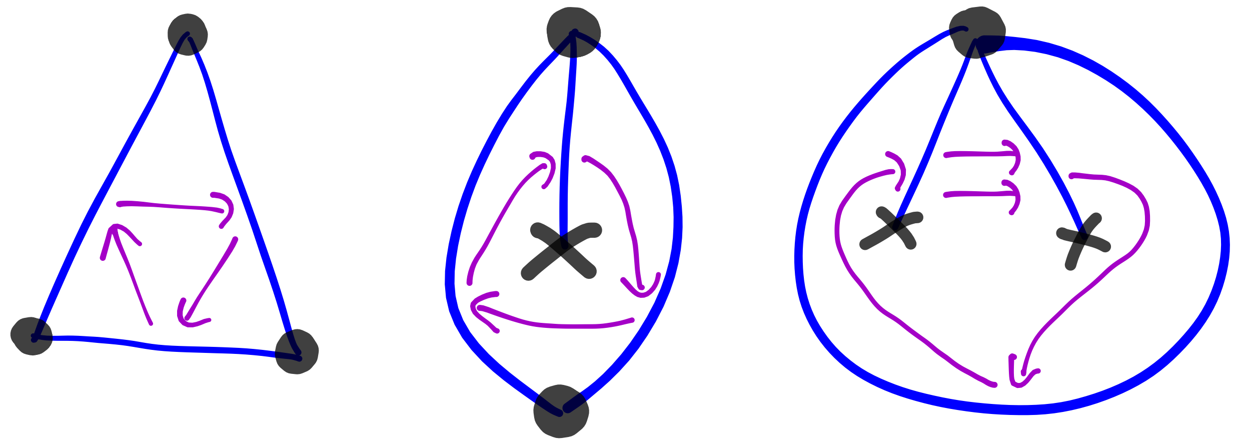
Every triangulation splits $\underline{\Sigma}$ into triangles.

Possible shapes of a triangle:



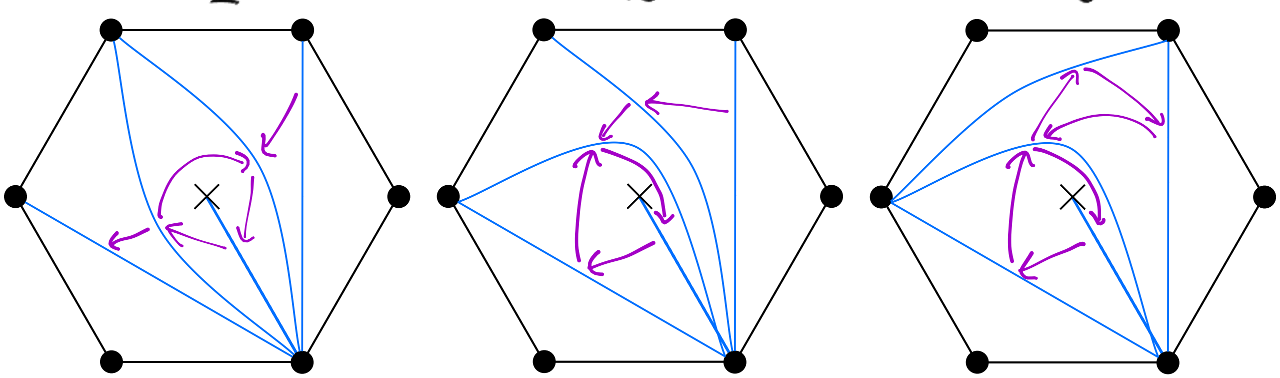
Quiver of a triangulation τ

$Q(\tau) := \begin{cases} \text{vertices} = \text{arcs in } \tau \\ \text{arrows: drawn on } \Sigma \text{ according to the rules} \end{cases}$

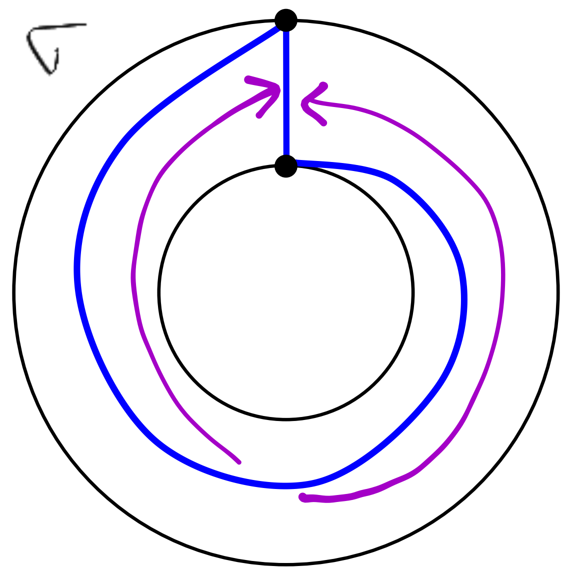


Examples:

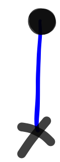
τ_1 τ_2 τ_3



σ



Chain complex of a triangulation

$X_0(\tau) := \tau \setminus \{\text{pending arcs}\}$ pending arc := 

$X_1(\tau) :=$ arrows of $Q(\tau)$ connecting elements of $X_0(\tau)$

$X_2(\tau) :=$ "obvious" 3-cycles on $X_1(\tau)$ (up to rotation)

$$C_n(\tau) := \mathbb{F}_2 X_n(\tau) \qquad \mathbb{F}_2 := \mathbb{Z}/2\mathbb{Z}$$

$$C_\bullet(\tau) : 0 \to C_2(\tau) \xrightarrow{d_2} C_1(\tau) \xrightarrow{d_1} C_0(\tau) \to 0$$

$$abc \longmapsto a+b+c$$

$$a \longmapsto h(a) - t(a)$$

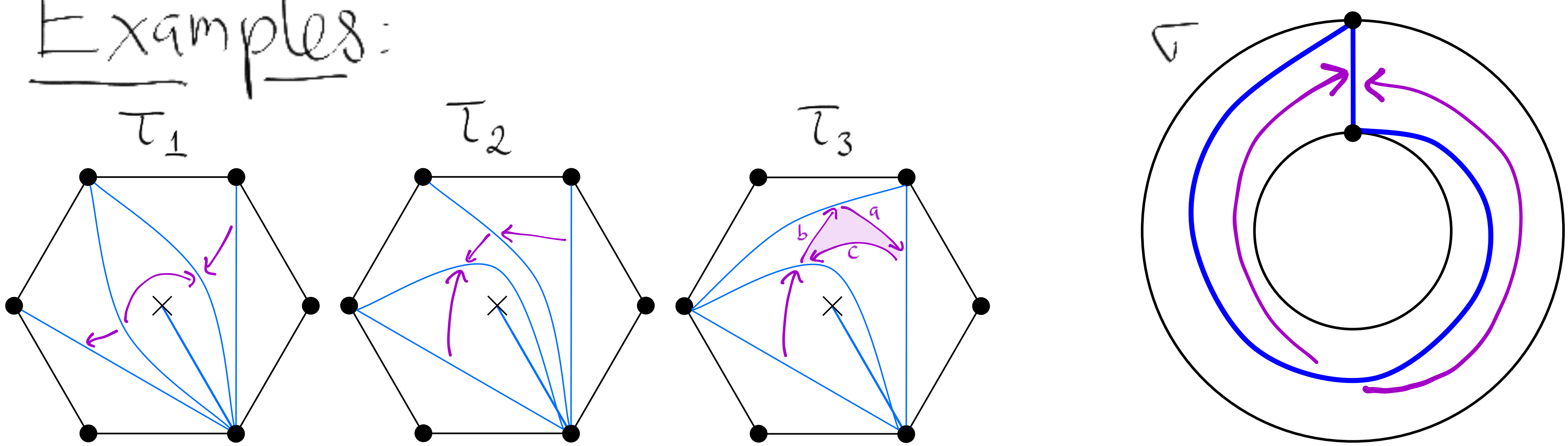
$$C^\bullet(\tau) := \mathrm{Hom}_{\mathbb{F}_2}(C_\bullet(\tau), \mathbb{F}_2)$$

Colored triangulations

Def. A *colored triangulation* is a pair (τ, ξ), where:

- τ is a triangulation of $\underline{\Sigma} = (\Sigma, M, \mathbb{O})$,
- ξ is a 1-cocycle in $C^{\bullet}(\tau)$ $\left(\begin{array}{l} \xi(a)+\xi(b)+\xi(c)=0 \\ \forall abc \in X_2(\tau) \end{array} \right)$

Examples:



$Z^1(\tau_1) = C^1(\tau_1)$ $\quad$ $Z^1(\tau_2) = C^1(\tau_2)$ $\quad$ $Z^1(\tau_3) = \{\xi \mid \xi(a)+\xi(b)+\xi(c)=0\}$ $\quad$ $Z^1(\sigma) = C^1(\sigma)$

The species of a colored triangulation

Suppose (τ, ξ) = colored triangulation

Set $R := \bigtimes_{j \in \tau} F_j$ where $F_j := \begin{cases} \mathbb{F}_{q^2} & j \text{ not pending} \\ \mathbb{F}_q & j \text{ pending} \end{cases}$

The definition of $A = A(\tau, \xi)$ is a bit trickier:

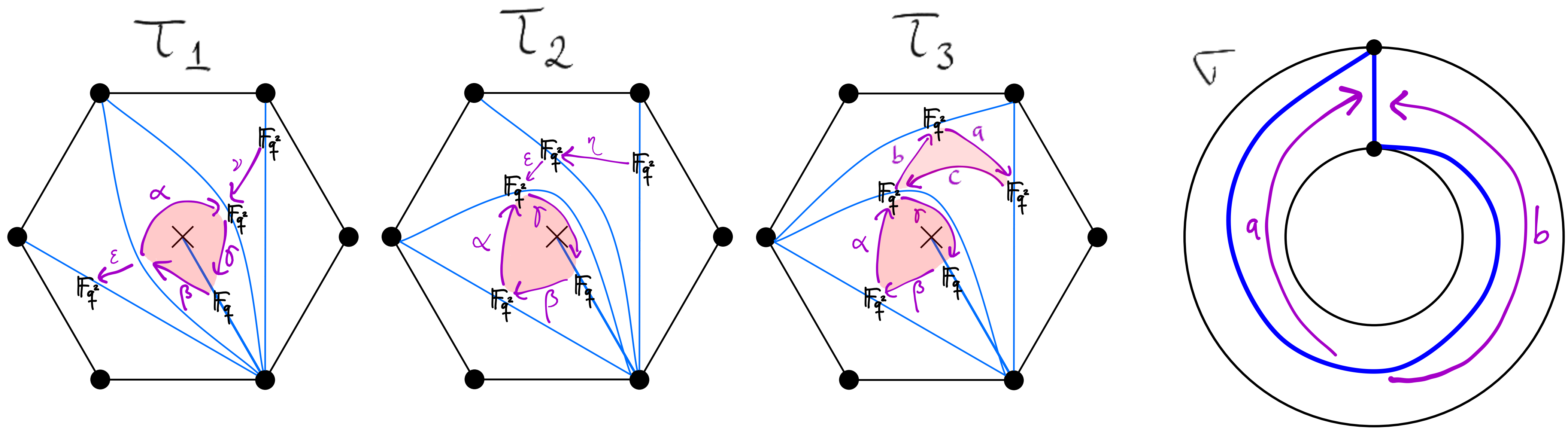
- For $a \in X_1(\tau)$, set $A_a := \mathbb{F}_{q^2}{}^{\theta^{\xi(a)}}$ $\left(\mathbb{F}_2 := \mathbb{Z}/\mathbb{Z}_2 \cong Gal(\mathbb{F}_{q^2}/\mathbb{F}_q)\right)$ $\xi(a) \longleftrightarrow \theta^{\xi(a)}$
- For $a \notin X_1(\tau)$, set $A_a := F_{h(a)} F_{t(a)}$ with usual bimodule structure.

$$A(\tau, \xi) := \bigoplus_{a \in Q_1(\tau)} A_a$$

The potential of a colored triangulation

With $(R, A(\tau,\xi))$ so defined, the (super) potential is

$$S(\tau,\xi) := \sum_{\Delta} (\text{obvious 3-cycles from } \Delta) \in \underbrace{T_R(A(\tau,\xi))}_{\text{tensor algebra}}$$



$S(\tau_1,\xi) = \alpha\beta\gamma$ $\quad$ $S(\tau_2,\xi) = \alpha\beta\gamma$ $\quad$ $S(\tau_3,\xi) = abc + \alpha\beta\gamma$ $\quad$ $S(\sigma,\xi) = 0$

Thm (Geuenich-LF).

(1) $\exists$ natural notion of "colored flip" of colored triangulations $\left(\tau \xleftrightarrow{f_j} \sigma \rightsquigarrow Z^{-1}(\tau) \xleftrightarrow[1:1]{\text{colored } f_j} Z^{-1}(\sigma)\right)$

(2) $\exists$ natural notion of mutation μ_j of species with potential,

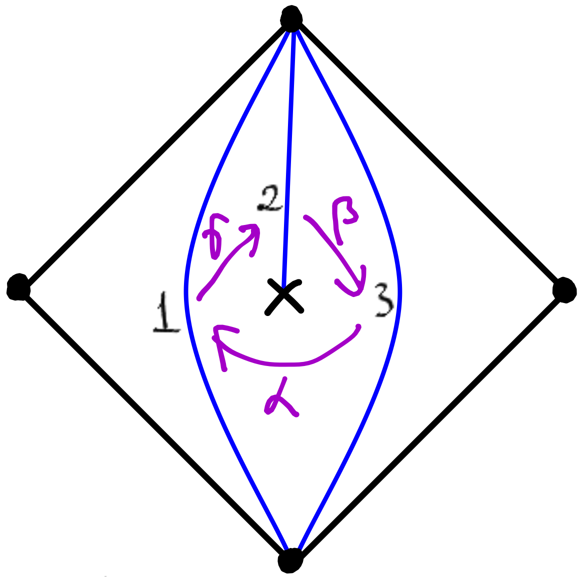
(3) $(\tau, \xi) \xleftrightarrow{f_j} (\sigma, \lambda) \Longrightarrow (A(\tau,\xi), S(\tau,\xi)) \xleftrightarrow{\mu_j} (A(\sigma,\lambda), S(\sigma,\lambda))$

(4) for τ fixed, $[\xi_1] = [\xi_2]$ if and only if

$$\mathcal{P}(A(\tau,\xi_1), S(\tau,\xi_1)) \cong \mathcal{P}(A(\tau,\xi_2), S(\tau,\xi_2))$$

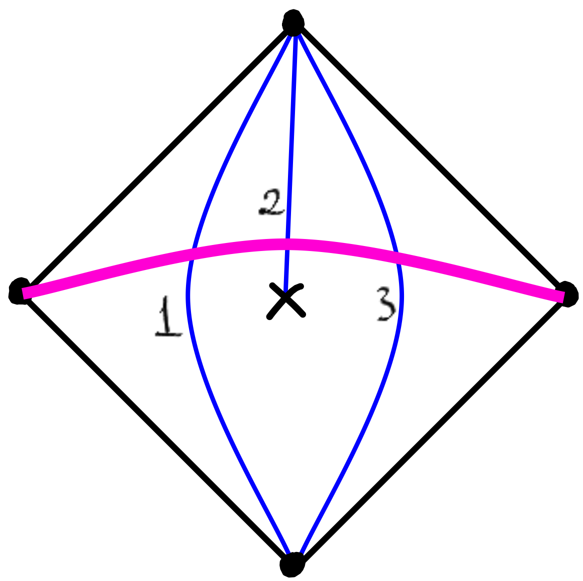
Example

$(\tau, 0)$ $\quad\quad\quad$ $A(\tau, 0)$ $\quad\quad\quad$ $S(\tau, 0) = \alpha\beta\gamma$



(with zero cocycle)

$$
\begin{tikzcd}
 & \mathbb{F}_q \arrow[dr, "\beta", "\mathbb{F}_{q^2} \otimes_{\mathbb{F}_q} \mathbb{F}_q"'] & \\
\mathbb{F}_{q^2} \arrow[ur, "\mathbb{F}_q \otimes_{\mathbb{F}_q} \mathbb{F}_{q^2}", "\gamma"'] & & \mathbb{F}_{q^2} \arrow[ll, "\alpha"', "\mathbb{F}_{q^2} \otimes_{\mathbb{F}_{q^2}} \mathbb{F}_{q^2}"]
\end{tikzcd}
$$



$$
\begin{tikzcd}
 & \mathbb{F}_q^2 \arrow[dr, "\beta = \mathbb{1}"] & \\
\mathbb{F}_{q^2} \arrow[ur, "\gamma = 0"] & & \mathbb{F}_{q^2} \arrow[ll, "\alpha = 0"]
\end{tikzcd}
$$

$$
\left.\begin{array}{l}
F_M(q) = \underline{1} + u_3 + \dfrac{q^2-1}{q-1} \cdot u_2 u_3 + u_2^2 u_3 + u_1 u_2^2 u_3 \\
g_M = (0, 0, -1, 0, 0)
\end{array}\right\} \rightsquigarrow
$$

correct Laurent expansion of cluster variable

Second construction:

Gentle algebras

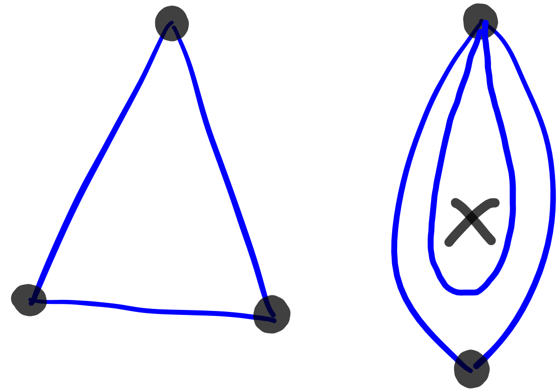
for

polygons with one orbifold point of order 3

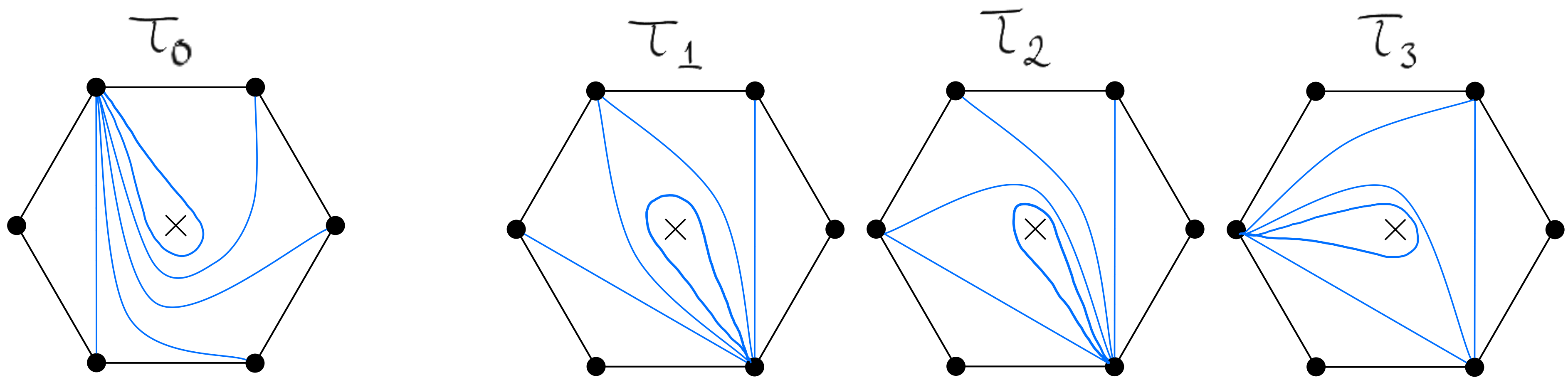
(jt. with Diego Velasco)

$\underline{\Sigma} = (\Sigma, M, \mathbb{O})$ = polygon with one orbifold point of order 3.

Two types of triangles:

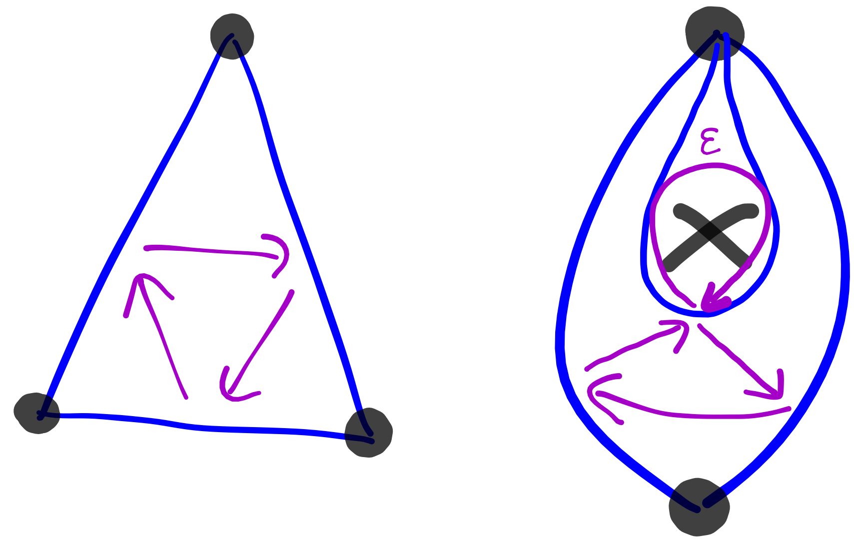


Examples of triangulations:

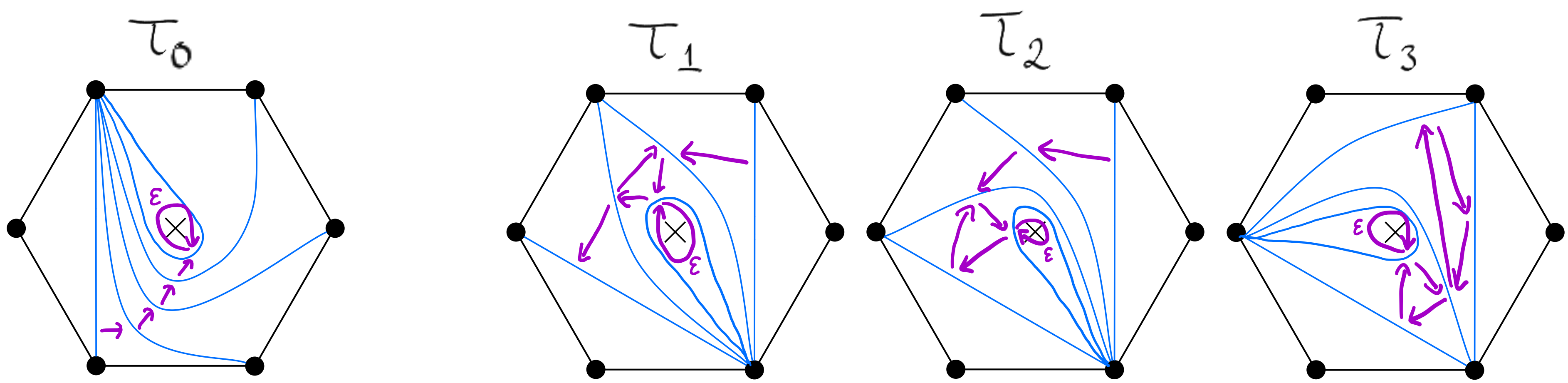


Quiver of a triangulation τ

$$Q(\tau) := \begin{cases} \text{vertices} = \text{arcs in } \tau \\ \text{arrows: drawn on } \Sigma \text{ according to the rules} \end{cases}$$



Examples:

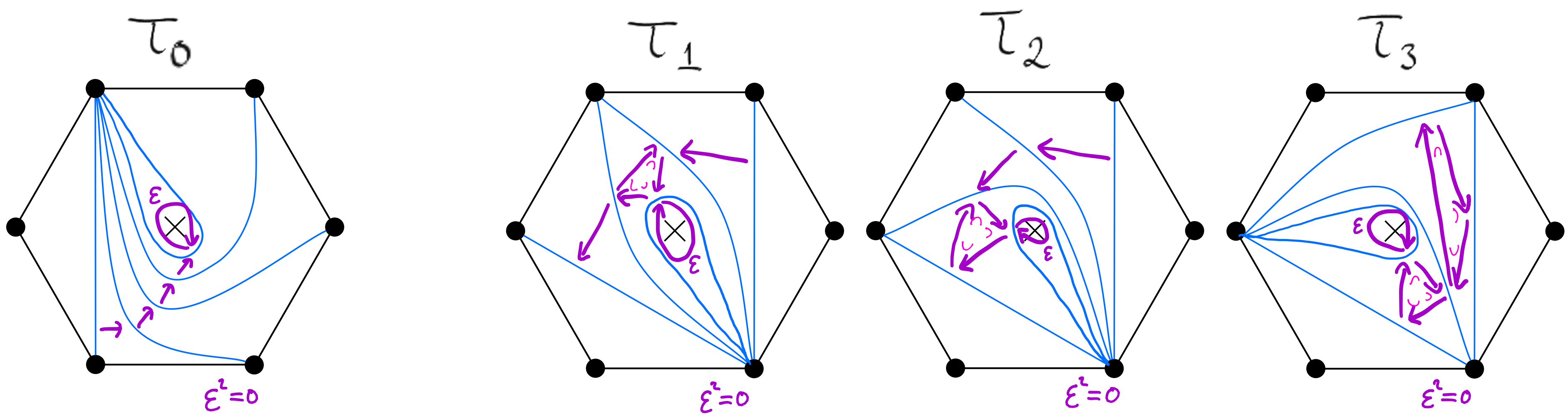


Potential $S(\tau)$

$$S(\tau) = \varepsilon^3 + \sum_{\Delta} \triangle \in \mathbb{C}\langle Q(\tau)\rangle$$

$\partial_\varepsilon(S(\tau)) = 3\varepsilon^2$, so $\varepsilon^2 = 0$ in the quotient

$$\Lambda(\tau) := \mathbb{C}\langle Q(\tau)\rangle \big/ J(S(\tau))$$



Thm (LF-Velasco).

For every triangulation τ of Σ:

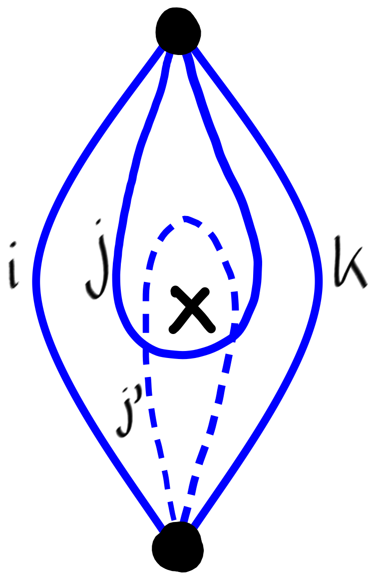
(1) $\left\{\begin{matrix}\text{arcs} \\ \text{on } \Sigma\end{matrix}\right\} \xleftrightarrow{1:1} \left\{\begin{matrix}\text{indecomposable E-rigid} \\ \text{(decorated) representations of } \Lambda(\tau)\end{matrix}\right\}$

(2) $\left\{\begin{matrix}\text{triangulations} \\ \text{of } \Sigma\end{matrix}\right\} \xleftrightarrow{1:1} \left\{\begin{matrix}\text{maximal collections of pairwise} \\ \text{E-orthogonal representations}\end{matrix}\right\}$

(3) $\forall$ arc $j \in \tau$:

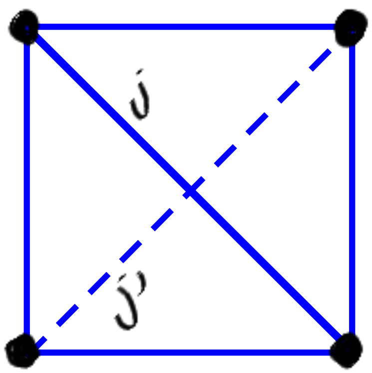
$CC_{\Lambda(\tau)}(M(j)) \cdot CC_{\Lambda(\tau)}(M(j')) =$ polynomial of Chekhov-Shapiro evaluated in $(CC_{\Lambda(\tau)}(M(k)) \mid k \in \tau \setminus \{j\})$

A bit more precisely:



$$\rightsquigarrow CC_{\Lambda(\tau)}(M(j)) \cdot CC_{\Lambda(\tau)}(M(j')) =$$

$$= CC_{\Lambda(\tau)}(M(i))^2 + CC_{\Lambda(\tau)}(M(i))\, CC_{\Lambda(\tau)}(M(k)) + CC_{\Lambda(\tau)}(M(k))^2$$



$$\rightsquigarrow CC_{\Lambda(\tau)}(M(j)) \cdot CC_{\Lambda(\tau)}(M(j')) =$$

$=$ usual Ptolemy binomial.

Corollary (LF-Velasco).

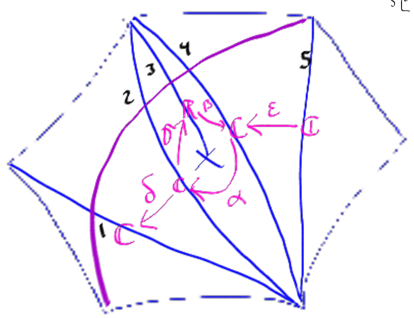
For every triangulation τ of Σ,

$\mathcal{C}_{\Lambda(\tau)} \cong$ generalized cluster algebra of Chekhov-Shapiro.

Caldero-Chapoton algebra of $\Lambda(\tau)$

Thank you for your attention!

$$X=\begin{array}{c|c} & \begin{array}{ccccc}1&2&3&4&5\end{array}\\ \begin{array}{c}1\\2\\3\\4\\5\end{array} & \begin{bmatrix}0&1&0&0&0\\-1&0&-1&1&0\\0&2&0&-2&0\\0&-1&1&0&1\\0&0&0&-1&0\end{bmatrix}\end{array}$$



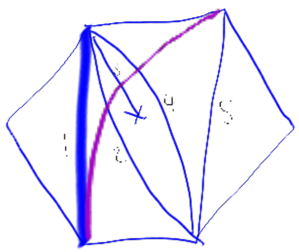
$$\begin{tikzcd} & \mathbb{R}^2 \arrow[dr, "\mathbb{1}"] & \\ \mathbb{C} \arrow[ur, "\nabla"] & & \mathbb{C} \arrow[ll, bend left, "\alpha=0"] & 0 \arrow[l, "\varepsilon=0"'] \end{tikzcd}$$

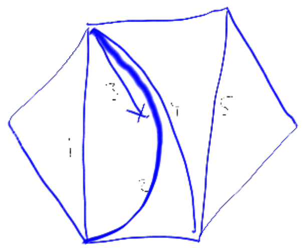
$$\chi(\mathbb{RP}^1)=0$$

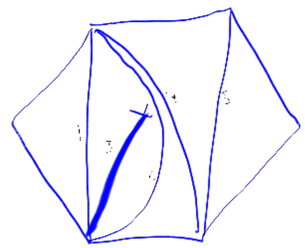
$$\begin{tikzcd} & \mathbb{C}^2 \arrow[dr, "{[1\ i]}"] & \\ 0 \arrow[r] & \mathbb{C} \arrow[u, "{\begin{bmatrix}1\\ i\end{bmatrix}}"] & \mathbb{C} \arrow[l] & 0 \arrow[l] \end{tikzcd}$$

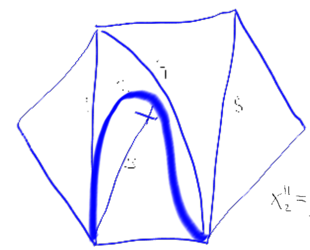
$$[-i\ \ 1]$$

$$\cancel{F=1+u_1+u_4+u_1u_4+\boxed{}\,u_3u_4+u_3^2u_4+\boxed{}\,u_1u_3u_4+u_1u_3^2u_4+u_1u_2u_3^2u_4}$$









$$B=\begin{bmatrix}0&-1&0&0&0\\1&0&-1&1&0\\0&2&0&-2&0\\0&-1&1&0&1\\0&0&0&-1&0\end{bmatrix}$$

$$(x_1, x_2, x_3, x_4, x_5)$$

$$B_1=\begin{bmatrix}0&1&-1&0&0\\-1&0&1&-1&0\\2&-2&0&0&0\\0&1&0&0&1\\0&0&0&-1&0\end{bmatrix}$$

$$\left(x_1,\ x_2'=\frac{x_1x_4+x_3^2}{x_2},\ x_3,\ x_4,\ x_5\right)$$

$$x_3'=\frac{x_1+x_2'}{x_3}=\frac{x_1+\left(\frac{x_1x_4+x_3^2}{x_2}\right)}{x_3}=\frac{x_1x_2+x_1x_4+x_3^2}{x_2x_3}$$

$$(x_1,\ x_2',\ x_3',\ x_4,\ x_5)$$

$$B_2=\begin{bmatrix}0&-1&1&0&0\\1&0&-1&-1&0\\-2&2&0&0&0\\0&1&0&0&1\\0&0&0&-1&0\end{bmatrix}$$

$$B_3=\begin{bmatrix}0&1&0&-1&0\\-1&0&1&1&0\\0&-2&0&0&0\\1&-1&0&0&1\\0&0&0&-1&0\end{bmatrix}$$

$$x_2''=\frac{x_1+x_3'^2x_4}{x_2'}=\frac{\left(\frac{x_1x_2+x_1x_4+x_3^2}{x_2x_3}\right)^2x_4}{\left(\frac{x_1x_4+x_3^2}{x_2}\right)}=$$

$$=\frac{x_2\cdot\left(x_1x_2^2x_3^2+(x_1x_2+x_1x_4+x_3^2)^2x_4\right)}{x_2^2x_3^2\cdot(x_1x_4+x_3^2)}$$

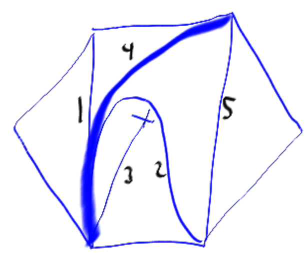
$$=\frac{x_1x_2^2x_3^2+\left(x_1^2x_2^2+2x_1x_2(x_1x_4+x_3^2)+(x_1x_4+x_3^2)^2\right)x_4}{x_2x_3^2\cdot(x_1x_4+x_3^2)}$$

$$=\frac{x_1x_2^2x_3^2+x_1^2x_2^2x_4+2x_1x_2x_4(x_1x_4+x_3^2)+(x_1x_4+x_3^2)^2x_4}{x_2x_3^2\cdot(x_1x_4+x_3^2)}$$

$$=\frac{x_1x_2^2(x_3^2+x_1x_4)+2x_1x_2x_4(x_1x_4+x_3^2)+(x_1x_4+x_3^2)^2x_4}{x_2x_3^2(x_1x_4+x_3^2)}$$

$$=\frac{x_1x_2^2+2x_1x_2x_4+x_1x_4^2+x_3^2x_4}{x_2x_3^2}$$

$$(x_1,\ x_2'',\ x_3',\ x_4,\ x_5)$$



$$B_4=\begin{bmatrix}0&0&0&1&0\\0&0&1&-1&1\\0&-2&0&0&0\\-1&1&0&0&-1\\0&-1&0&1&0\end{bmatrix}$$

$$x_4'=\frac{x_1x_5+x_2''}{x_4}=\frac{x_1x_5+\left(\frac{x_1x_2^2+2x_1x_2x_4+x_1x_4^2+x_3^2x_4}{x_2x_3^2}\right)}{x_4}$$

$$=\frac{x_1x_2x_3^2x_5+\boxed{x_1x_2^2}+2x_1x_2x_4+\boxed{x_1x_4^2}+x_3^2x_4}{x_2x_3^2x_4}$$

$$F_M=1+1_{\cdot}+\boxed{}\,u_3u_4+u_3^2u_4+u_2u_3^2u_4$$

$$0\to M\to I_4\to I_1\oplus I_5 \qquad g_M=\begin{bmatrix}1\\0\\0\\-1\\1\end{bmatrix}$$

$$z=x_1x_4^{-1}x_5\left(1+x_2x_3^{-2}x_5^{-1}+\boxed{}\,x_3^{-2}x_4x_5^{-1}+x_2^{-1}x_3^{-2}x_4^2x_5^{-1}+x_1^{-1}x_2^{-1}x_4x_5^{-1}\right)$$

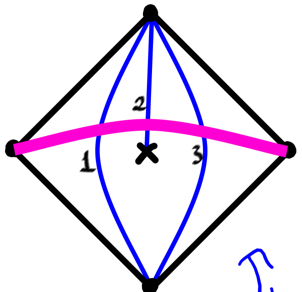
$$=\frac{x_1x_5+\boxed{x_1x_2^2}+\boxed{}\,x_1x_4+\boxed{x_1x_4^2}+x_2x_3^2}{x_2x_4x_3^2}$$

$$F_1=1+u_3+u_2u_3 \qquad g_1=\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}$$

$$F_2=1+u_4+u_3u_4 \qquad g_2=\begin{bmatrix}0\\0\\-1\\1\end{bmatrix}$$

$$F_1\cdot F_2=1+u_4+u_3u_4+u_3+u_3u_4+u_3^2u_4+u_2u_3+u_2u_3u_4+u_2u_3^2u_4$$

$$=1+u_4+2u_3u_4+u_3+u_3^2u_4+u_2u_3+u_2u_3u_4+u_2u_3^2u_4$$



$$B_0=\begin{bmatrix}0&-1&1\\2&0&-2\\-1&1&0\end{bmatrix}\quad (x_1,x_2,x_3)$$

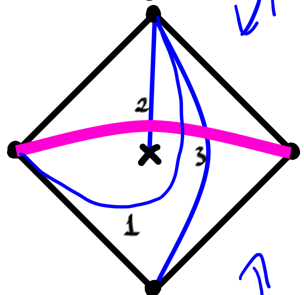
$$F_M(q)=1+u_3+\frac{q^2-1}{q-1}\cdot u_2u_3+u_2^2u_3+u_1u_2^2u_3$$

$$g_M=(0,0,-1,0,0)$$

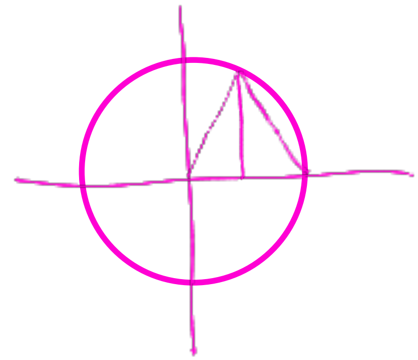
$$\frac{1}{x_3}\left(1+x_1x_2^{-2}+2x_2^{-1}x_3+x_1^{-1}x_2^{-2}x_3^2+x_1^{-1}x_3\right)=$$

$$=\frac{x_1x_2^2+x_1^2+2x_1x_3+x_3^2+x_2^2x_3}{x_3x_1x_2^2}$$

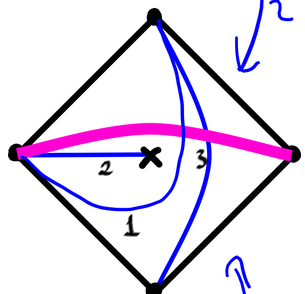
1



$$B_1=\begin{bmatrix}0&1&-1\\-2&0&0\\1&0&0\end{bmatrix}\quad \left(x_1'=\frac{x_2^2+x_3}{x_1},\ x_2,\ x_3\right)$$



2

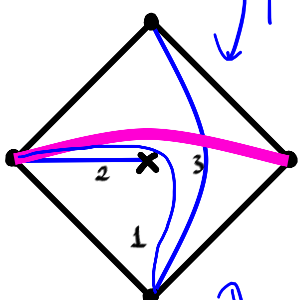


$$B_2=\begin{bmatrix}0&-1&-1\\2&0&0\\1&0&0\end{bmatrix}\quad \left(x_1',\ x_2'=\frac{x_1'+1}{x_2},\ x_3\right)$$

$$\parallel$$

$$\frac{\left(\frac{x_2^2+x_3}{x_1}\right)+1}{x_2}=\frac{x_2^2+x_3+x_1}{x_1x_2}$$

1



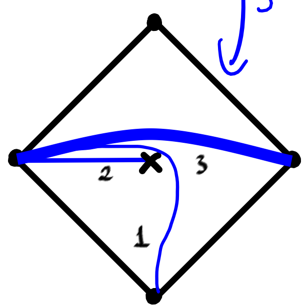
$$B_3=\begin{bmatrix}0&1&1\\-2&0&0\\-1&0&0\end{bmatrix}\quad \left(x_1''=\frac{x_2'^2x_3+1}{x_1'},\ x_2',\ x_3\right)$$

$$\parallel$$

$$\frac{\left(\frac{x_2^2+x_3+x_1}{x_1x_2}\right)^2x_3+1}{\left(\frac{x_2^2+x_3}{x_1}\right)}=\frac{x_1\left((x_2^2+x_3+x_1)^2x_3+x_1^2x_2^2\right)}{x_1^2x_2^2\cdot(x_2^2+x_3)}=\frac{x_3(x_2^2+x_3)^2+2x_3(x_2^2+x_3)x_1+x_3x_1^2+x_1^2x_2^2}{x_1x_2^2(x_2^2+x_3)}$$

$$=\frac{x_3x_2^2+x_3^2+2x_1x_3+x_1^2}{x_1x_2^2}$$

3



$$B_4=\begin{bmatrix}0&1&-1\\-2&0&0\\1&0&0\end{bmatrix}\quad x_3'=\frac{x_1''+1}{x_3}=\frac{x_2^2x_3+x_3^2+2x_1x_3+x_1^2+x_2^2x_1}{x_1x_2^2x_3}$$

$+1$