Morsifications and mutations

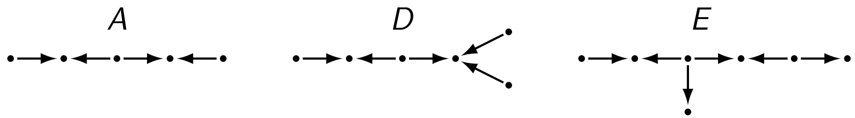
Sergey Fomin

University of Michigan

joint work with P. Pylyavskyy, E. Shustin, and D. Thurston

`arXiv:1711.10598`

The *ADE* enigma



The *ADE* nomenclature, in its version involving quivers, arises in two seemingly unrelated contexts:

- classification of simple plane curve singularities [V. Arnold, 1972];
- classification of cluster algebras of finite type [SF–A. Zelevinsky, 2003].

Is there a connection?

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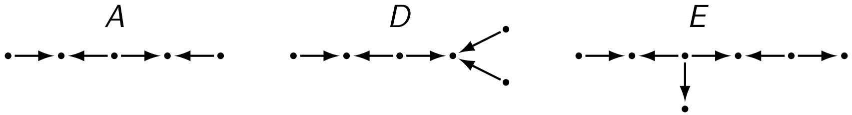

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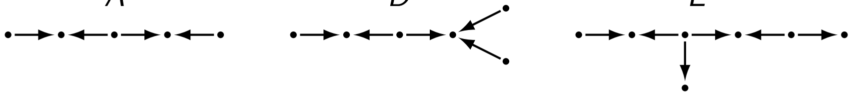

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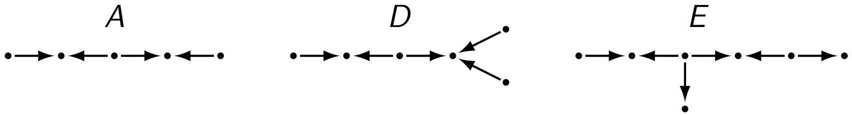

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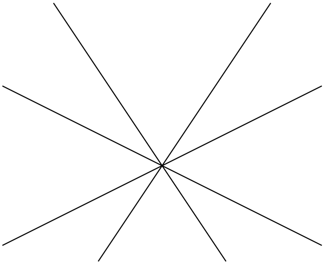
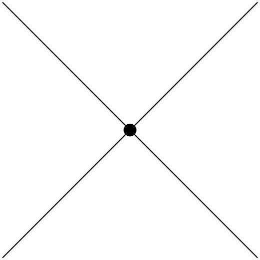
Overview

complex plane curve singularity

↓

real forms





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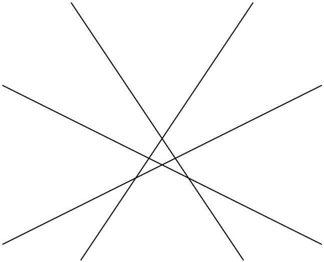

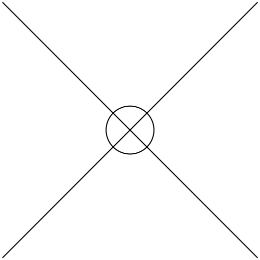

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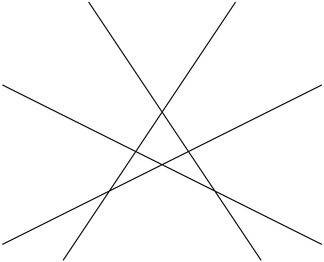

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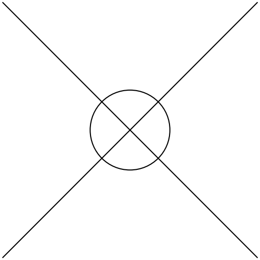

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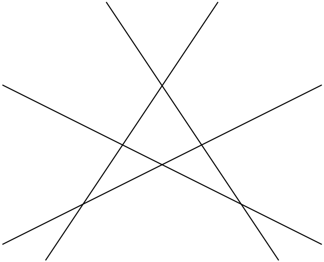

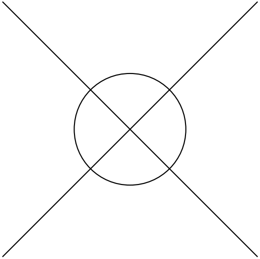

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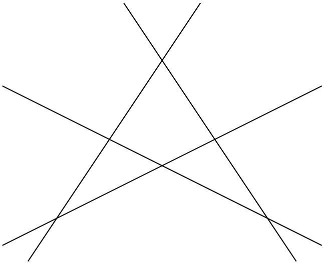

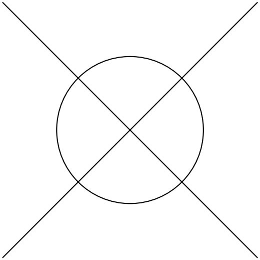

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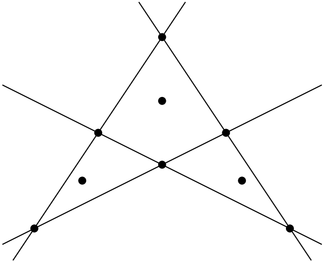

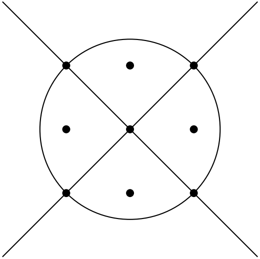

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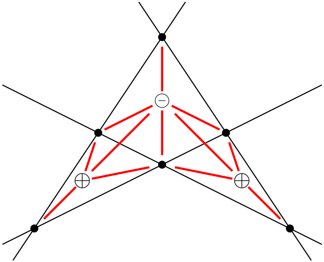

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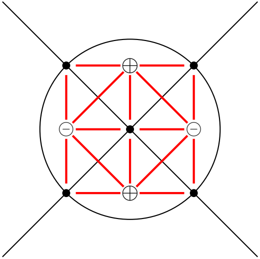
AΓ-diagrams



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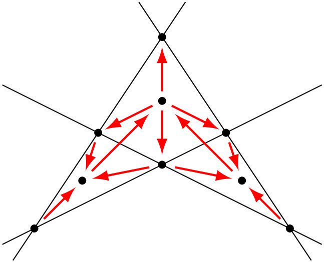

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$A\Gamma$-diagrams

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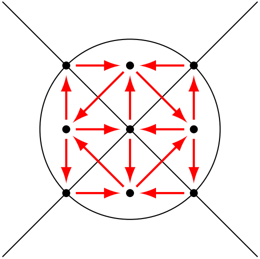
quivers



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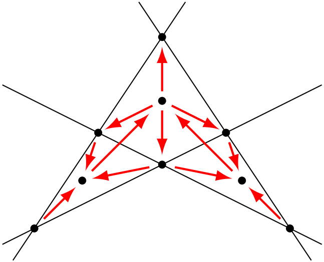

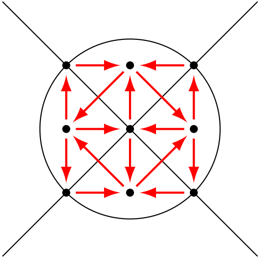

Main conjecture (proved modulo technical conditions):

topological type of the complex singularity $\longleftrightarrow$ mutation equivalence class of quivers

Singularities of plane curves

In this talk,

singularity = *isolated* singularity of a *plane* complex analytic curve

i.e., a germ (C, z) of a reduced analytic curve $C \subset \mathbb{C}^2$ at a singular point $z \in \mathbb{C}^2$ such that z is the only singular point of C inside some ball $\mathbf{B} \subset \mathbb{C}^2$ centered at z; without loss of generality, we assume $z = (0, 0)$.

We study singularities up to *topological equivalence*, i.e., up to homeomorphisms of a neighborhood of the singular point.

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Example: Quasihomogeneous singularities

A singularity is called *quasihomogeneous* of type (a, b) (here $a \geq b \geq 2$) if it can be given by an equation of the form $f(x, y) = \sum_{\substack{bi+aj=ab \\ i,j \geq 0}} c_{ij} x^i y^j = 0$.

A quasihomogeneous singularity of type (a, b) (here $a \geq b \geq 2$) is topologically equivalent to the singularity

$$x^a \pm y^b = 0.$$

$a = 2, b = 2$: a *node*: two locally smooth transverse branches
$a = 3, b = 2$: a *cusp*
$a = 4, b = 2$: a *tacnode*: two smooth branches with simple tangency
$a = 3, b = 3$: three smooth transverse branches

Real singularities

A *real singularity* (C, z) is an analytic curve $C \subset \mathbb{C}^2$ invariant under complex conjugation, with $z \in C$ its real singular point. Equivalently, C can be given by $f(x, y) = 0$ where all coefficients in the power series expansion of f at z are real. A singularity is called *totally real* if all its local branches are real.

A real singularity topologically equivalent to a complex one is called its *real form*.

Any complex plane curve singularity has at least one real form, including a totally real form (with all local branches real).

There may be many distinct real forms, up to conjugation-equivariant topological equivalence.

Example

A complex node has two distinct real forms:

- a *hyperbolic node* (equivalent over $\mathbb{R}$ to $x^2 - y^2 = 0$);
- an *elliptic node* (equivalent over $\mathbb{R}$ to $x^2 + y^2 = 0$).

Nodal deformations

A *nodal deformation* of a singularity (C, z) inside the Milnor ball $\mathbf{B}$ is an analytic family of curves $C_t \cap \mathbf{B}$ such that

- t varies in a small disk centered at $0 \in \mathbb{C}$;
- for $t = 0$, we recover the original curve: $C_0 = C$;
- each C_t is smooth along $\partial\mathbf{B}$, and intersects $\partial\mathbf{B}$ trasversally;
- for any $t \neq 0$, the curve C_t has only ordinary nodes inside $\mathbf{B}$;
- the number of these nodes does not depend on t.

A *real nodal deformation* of a real singularity (C, z) is obtained by taking a nodal deformation $(C_t \cap \mathbf{B})$ equivariant with respect to complex conjugation, and restricting t to a small interval $[0, \tau) \subset \mathbb{R}$.

Morsifications

A *real morsification* of a real singularity (C, z) is a real nodal deformation $C_t = \{f_t(x, y) = 0\}$ such that

- all critical points of f_t are real, with nondegenerate Hessian;
- all saddle points of f_t are at the zero level (i.e., lie on C_t).

A real morsification attains the natural upper bound on the number of real hyperbolic nodes in a real nodal deformation of a given real singularity. See [P. Leviant–E. Shustin, arXiv:1703.05510].

singularity   not a morsification   morsification

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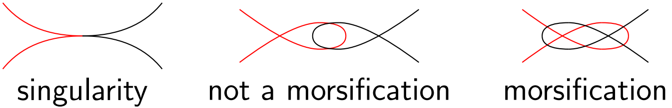

Existence of morsifications

Theorem (N. A'Campo–S. Guseĭn-Zade, 1974)

Any totally real plane curve singularity possesses a real morsification.

These morsifications have been successfully used to compute the monodromies and the intersection forms of plane curves singularities.

Extending this theorem, P. Leviant–E. Shustin [2017] established the existence of morsifications for a wide class of real singularities.

Conjecture

Any real plane curve singularity possesses a real morsification.

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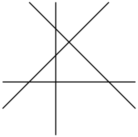
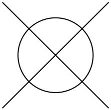
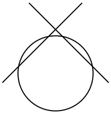
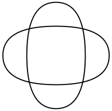
Example: Morsifying different real forms

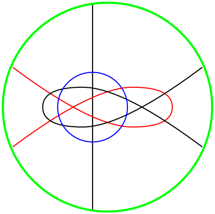
$x^3y - xy^3 = 0$	$xy(x-y+t)(x+y-2t)=0$	
$x^4 - y^4 = 0$	$(x^2-y^2)(x^2+y^2-t^2)=0$	
	$(x^2-(y-1.2t)^2)(x^2+y^2-t^2)=0$	
$(x^2+4y^2)(4x^2+y^2)=0$	$(x^2+4y^2-t^2)(4x^2+y^2-t^2)=0$	

Divides (after N. A'Campo)

A *divide* D is the image of a generic relative immersion of a collection of intervals and circles into a disk $\mathbf{D} \subset \mathbb{R}^2$. Their images (the *branches* of D) must satisfy:

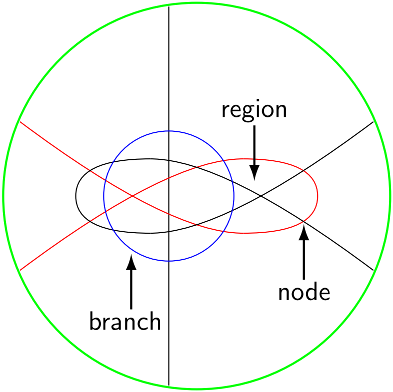
- the immersed circles do not intersect the boundary $\partial\mathbf{D}$;
- the immersed intervals have pairwise distinct endpoints which lie on $\partial\mathbf{D}$;
- the immersed intervals intersect $\partial\mathbf{D}$ transversally;
- all intersections and self-intersections of the branches are transversal;
- no three branches intersect at a point;

plus a couple of additional technical requirements. We do not distinguish between divides related by a homeomorphism between their respective ambient disks.



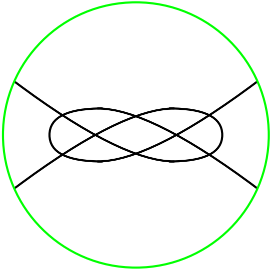
Regions and nodes of a divide

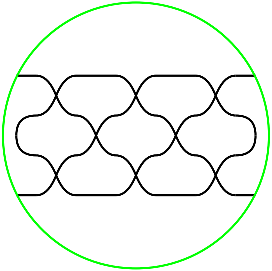
The connected components of the complement $\mathbf{D} \setminus D$ which are disjont from the boundary $\partial\mathbf{D}$ are the *regions* of D. The singular points of D are its *nodes*.



Divides of morsifications

Any real morsification defines a divide.





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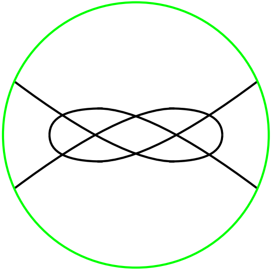
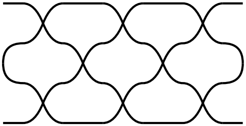
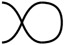
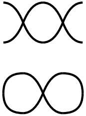
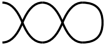
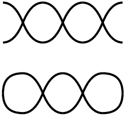
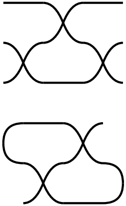
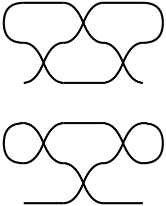
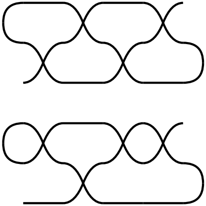
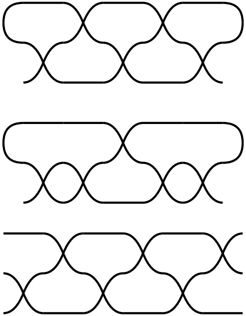




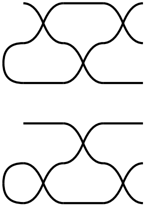
Divides for quasihomogeneous singularities ($b=2$)

$a=2$	$a=3$	$a=4$	$a=5$	$a=6$
				
A_1	A_2	A_3	A_4	A_5
node	cusp	tacnode		

Divides for quasihomogeneous singularities ($b = 3$)

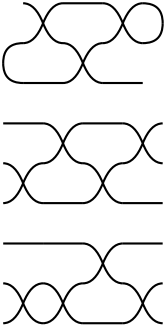
$a=3$	$a=4$	$a=5$	$a=6$
			
D_4	E_6	E_8	$E_8^{(1,1)}$

Divides for singularities of types D_5, D_6, E_7



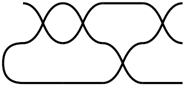
D_5

cusp
$+$
transversal line



D_6

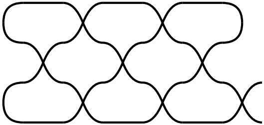
tacnode
$+$
transversal line



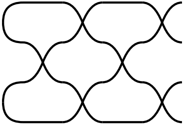
E_7

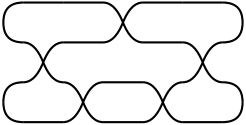
cusp
$+$
cuspidal tangent

Divides for non-quasihomogeneous singularities



$$y = x^{3/2} + x^{7/4}$$





two transversal cusps (either real or complex conjugate)

Algebraic divides

Divides arising from morsifications are called *algebraic*.

It is difficult to detect whether a given divide is algebraic or not.

Divides coming from different morsifications of the same real singularity share some basic features.

The branches of a divide are obtained by deforming the local branches of the original real singularity. Each real local branch deforms into an immersed interval with endpoints on $\partial\mathbf{D}$. Each pair of distinct complex conjugate local branches deforms into an immersed circle inside $\mathbf{D}$. The numbers of intersections (resp., self-intersections) of the individual branches of the divide do not depend on the choice of morsification/divide, nor does the total number of regions in a divide.

Algebraic divides

Divides arising from morsifications are called *algebraic*.

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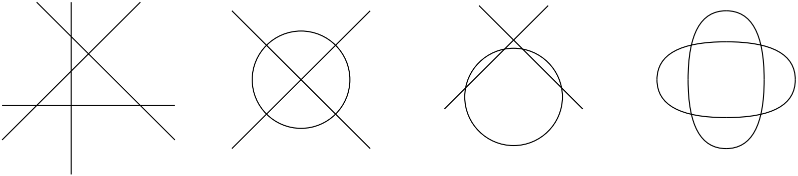
The Milnor number

The combined number of nodes and regions of an algebraic divide is equal to the *Milnor number* of the associated singularity.



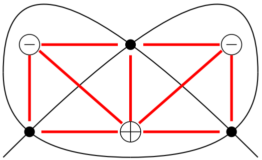
$6+3=9$ $\qquad$ $5+4=9$ $\qquad$ $5+4=9$ $\qquad$ $4+5=9$

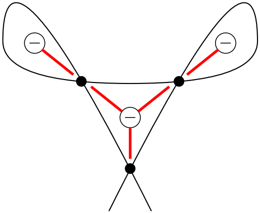
Thus this number does not depend on the choice of the real form of a complex singularity, nor on the choice of its morsification.

A'Campo-Guseĭn-Zade diagrams

The $\mathrm{A}\Gamma$-*diagram* of a divide D is a vertex-colored graph constructed as follows:

- place a vertex at each node of D, and color it black;
- place one vertex into each region of D;
- color these vertices $\oplus$ or $\ominus$ so that adjacent regions receive different colors;
- draw an edge across each segment separating two adjacent regions;
- connect the nodes on the boundary of each region to the vertex inside it.



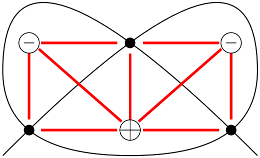


The $A\Gamma$-diagram of a morsification

For a divide coming from a real morsification, the vertices of the $A\Gamma$-diagram correspond to the critical points of the morsified curve:

$$\begin{array}{ccl} \oplus & \longleftrightarrow & \text{local maxima} \\ \ominus & \longleftrightarrow & \text{local minima} \\ \bullet & \longleftrightarrow & \text{saddle points} \end{array}$$

The number of vertices in the $A\Gamma$-diagram is equal to the Milnor number of the singularity.



Recovering a singularity from a $A\Gamma$-diagram

Theorem (P. Leviant–E. Shustin, 2017)

The $A\Gamma$-diagram of a real morsification of a real isolated plane curve singularity determines the complex topological type of the singularity.

For totally real singularities, a version of this result was obtained by L. Balke–R. Kaenders [1996].

Problem

Given $A\Gamma$-diagrams of two morsifications of real singularities, determine whether these two singularities have the same complex topological type.

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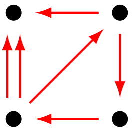
A *quiver* is a finite directed graph.

No oriented cycles of length 1 or 2.

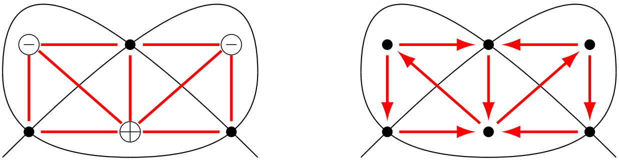


We do not distinguish between quivers (on the same vertex set) which differ by simultaneous reversal of all the arrows.

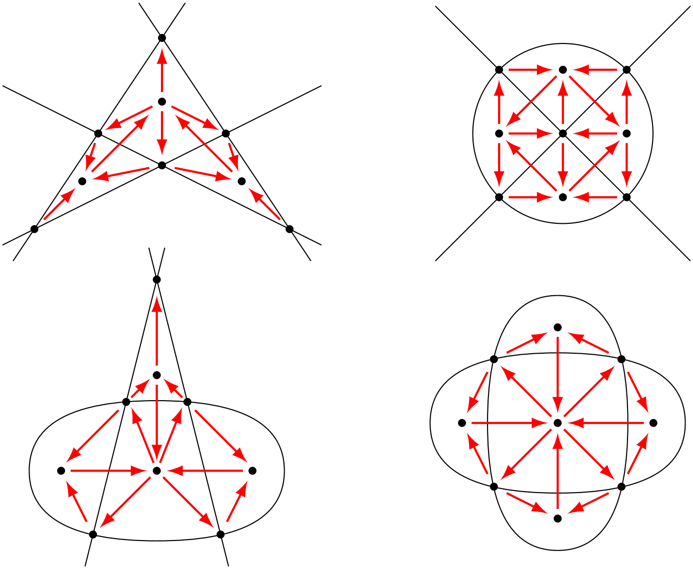
Quiver of a divide

Any divide D gives rise to a quiver $Q(D)$, constructed by orienting the edges of the $\mathrm{A\Gamma}$-diagram using the rule

$$\bullet \color{red}{\rightarrow} \oplus \color{red}{\rightarrow} \ominus \color{red}{\rightarrow} \bullet$$



What do these quivers have in common?



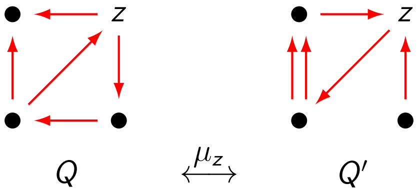
Quiver mutations

Given a vertex z in a quiver Q, the *quiver mutation* at z transforms Q into a new quiver $Q' = \mu_z(Q)$ constructed in three steps:

1. For each 2-edge path $x \to z \to y$, introduce a new edge $x \to y$.

2. Reverse the direction of all edges incident to z.

3. Remove oriented 2-cycles.



Quiver mutation is involutive: $\mu_z(Q') = Q$.

Mutation equivalence

Quivers related via iterated mutations are called *mutation equivalent.*

A mutation equivalence class defines a *cluster algebra*.

Many important rings arise in this way.

Main conjecture

Conjecture

Given two real morsifications of real isolated plane curve singularities, the following are equivalent:

- *the two singularities have the same complex topological type;*
- *the quivers associated with the two morsifications are mutation equivalent to each other.*

To rephrase, isolated plane curve singularities are topologically classified by the mutation classes of quivers coming from their morsifications (equivalently, by the corresponding cluster algebras).

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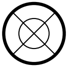
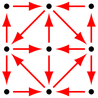
Unpacking the main conjecture

general concept	illustration
complex singularity	4 transversal smooth branches
real singularity	$x^4 - y^4 = 0$
morsification	$(x^2 - y^2)(x^2 + y^2 - t^2) = 0$
divide	
quiver	
mutation class	$E_7^{(1,1)}$

Main results (preview)

We prove our main conjecture modulo some technical assumptions, each of which we optimistically expect to be redundant.

These assumptions are satisfied for all morsifications constructed using known general methods.

Example: transversal overlays of Lissajous divides.

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Singularities and cluster algebras

The link between morsifications and quiver mutations is suggestive of a deep intrinsic relationship between singularities and cluster algebras.

Example

A quasihomogeneous singularity $x^a + y^b = 0$ is described by the same quivers as the standard cluster structure on the homogeneous coordinate ring of the Grassmannian $\mathrm{Gr}_{a,a+b}(\mathbb{C})$.

An algebraic connection between quasihomogeneous singularities and Grassmannians (using additive categorification) was proposed in [B. T. Jensen, A. King, and X. Su, 2016].

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Link of an isolated singularity (after J. Milnor)

The link $L(C,z)$ of an isolated complex plane curve singularity (C,z) is defined by intersecting C with a small sphere centered at z.

The links arising in this way are called *algebraic links*.

The link $L(C,z)$ completely determines—and is determined by—the local topology of a singular complex plane curve (C,z).

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A'Campo links of divides

Let D be a divide in the unit disk $\mathbf{D} = \{(x, y) \mid x^2 + y^2 \leq 1\} \subset \mathbb{R}^2$.

The *A'Campo link* $L(D)$ is defined by

$$L(D) = \left\{ (x, y, u, v) \;\middle|\; \begin{array}{c} (x, y) \in D \\ (u, v) \in T_{(x,y)} D \\ x^2 + y^2 + u^2 + v^2 = 1 \end{array} \right\} \subset \mathbb{S}^3 \subset \mathbb{R}^4 \cong \mathbb{C}^2$$

(Special cases: (x, y) is a node; or $(x, y) \in \partial \mathbf{D}$.)

Theorem (N. A'Campo)

For an algebraic divide D arising from a morsification of a singular curve (C, z), the links $L(D)$ and $L(C, z)$ are isotopic to each other.

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Link equivalence of divides

Two divides are *link-equivalent* if their A'Campo links are isotopic.

Corollary

In the case of algebraic divides, link equivalence coincides with the topological equivalence of the corresponding singularities.

We can now restate our main conjecture as follows:

Conjecture

Algebraic divides are link-equivalent if and only if the corresponding quivers are mutation equivalent.

Problem

Identify a class of divides—as broad as possible—within which our main conjecture holds. In particular, does it hold for all divides?

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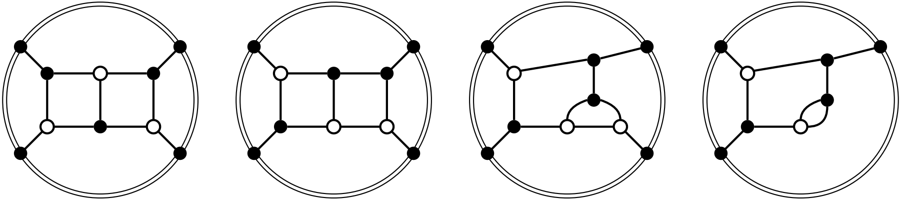
Plabic graphs

Plabic (planar bicolored) graphs were introduced by A. Postnikov to study parametrizations of cells in totally nonnegative Grassmannians.

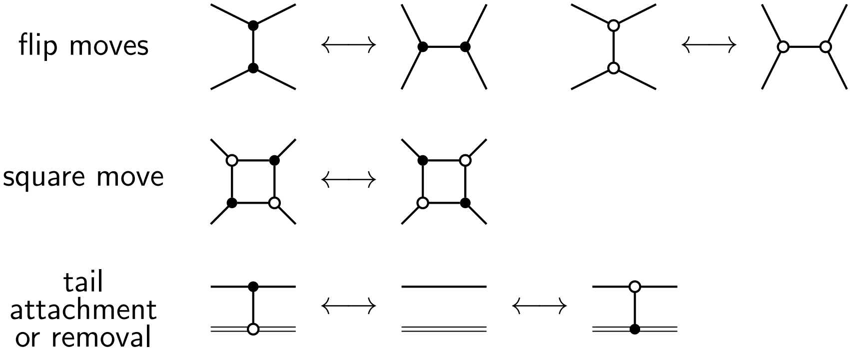


We view plabic graphs up to isotopy, and up to simultaneously changing the colors of all vertices.

We use a slightly modified version of the definition: our plabic graphs are *trivalent-univalent*, and we color both the interior and the boundary vertices.

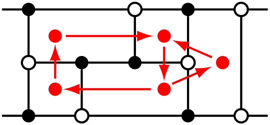
Local moves on plabic graphs

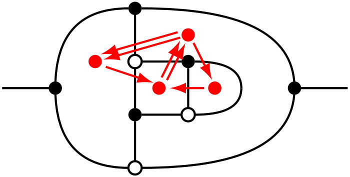
Two plabic graphs are called *move equivalent* if they can be obtained from each other via repeated application of local *moves* shown below.



Plabic graphs and quivers

Any plabic graph defines a quiver:





Local moves on plabic graphs translate into quiver mutations:

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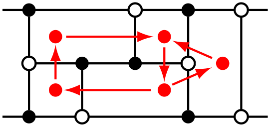

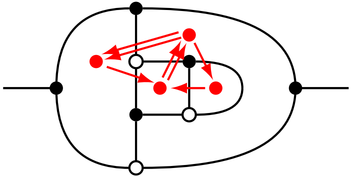

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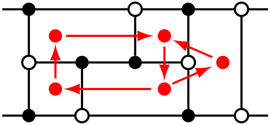
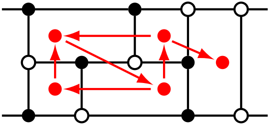
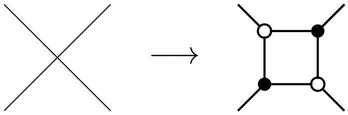
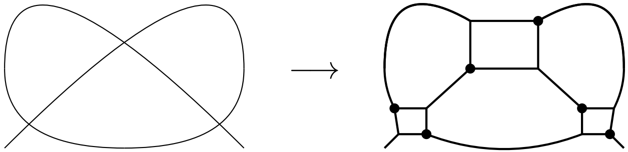




Plabic graphs attached to divides

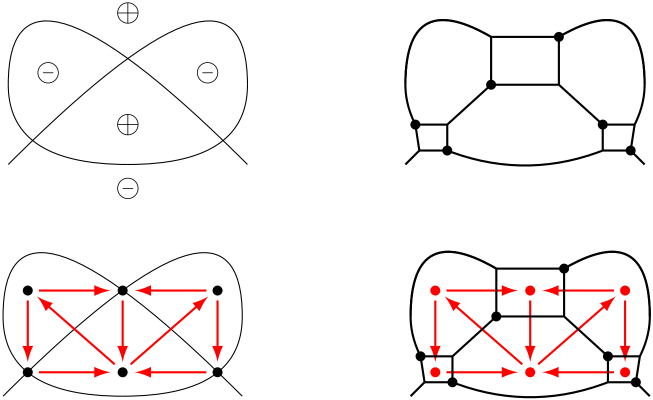
From a divide, one constructs a plabic graph:





Divide → plabic graph → quiver

The quiver obtained from the $\mathrm{A}\Gamma$-diagram of a divide D coincides with the quiver associated with a plabic graph attached to D:



M. Shapiro's conjecture

Conjecture (M. Shapiro)

Two plabic graphs are move equivalent (up to changing the colors of boundary vertices) if and only if their quivers are mutation equivalent.

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We establish one direction of this conjecture in a stronger version, without the assumption of algebraicity:

Theorem

If the plabic graphs associated to two divides are move equivalent, then these divides are link-equivalent.

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The proof uses T. Kawamura's construction of links of *graph divides*.

This construction associates a canonical link to any plabic graph.

These links are invariant under Postnikov's local moves.

The link associated with the plabic graph of an algebraic divide coincides with its A'Campo link.

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Divides and braids

Proved: move equivalence implies link equivalence.

Going in the opposite direction: need an analogue of Reidemeister's theorem for links of graph divides.

The links of graph divides are *quasi-positive* [T. Kawamura, 2004].

Our approach relies on Markov's theorem (link equivalence of braids).

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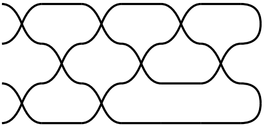
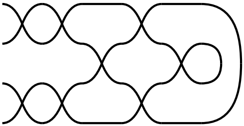
Scannable divides

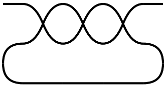
A *scannable divide* is a divide drawn inside a rectangle $[a_1, a_2] \times [b_1, b_2] \subset \mathbb{R}^2$ so that, moving along each branch, the x-coordinate makes all of its U-turns at the locations $x = a_1$ and $x = a_2$, approaching them from the right, resp. from the left. Every isolated plane curve singularity has a scannable real morsification.



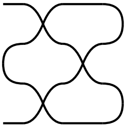
scannable



not scannable



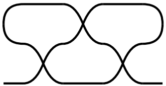
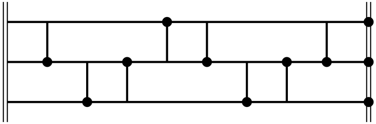
scannable



scannable

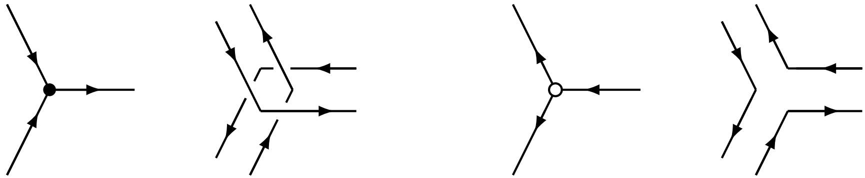
Plabic fences

For a scannable divide D, there is a natural choice of a plabic graph $\Phi = \Phi(D)$ attached to D which we call a *plabic fence*.

scannable divide	
plabic fence	

Links of oriented plabic graphs

Plabic fences have orientations with nice properties, which lead to an alternative construction of the associated links.



Local move:

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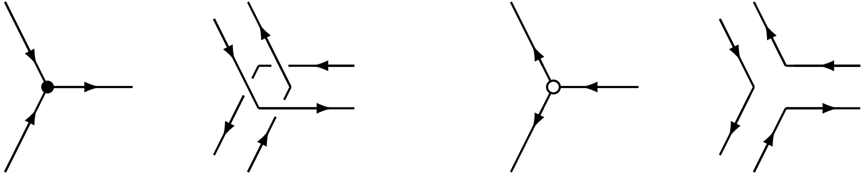

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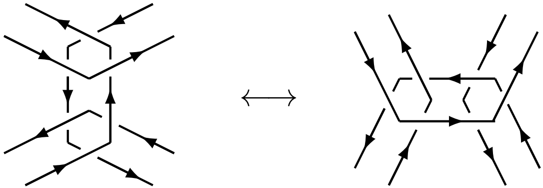
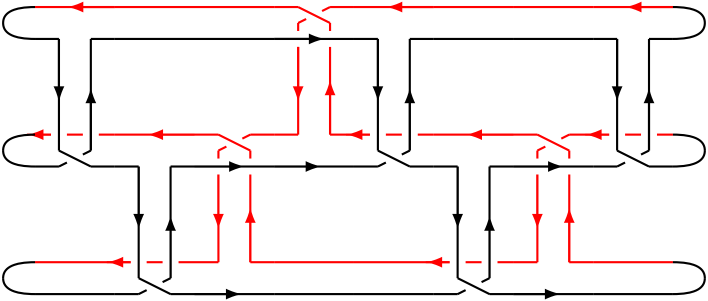


The link of a scannable divide

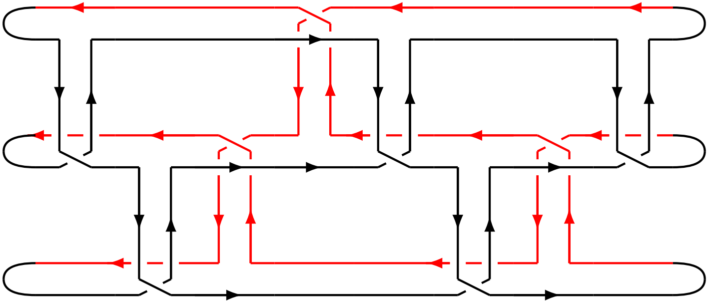
For a scannable divide, this construction recovers the A'Campo link [O. Couture–B. Perron, 2000].



The link of a scannable divide is the closure of a *positive braid*.

The link of a scannable divide

For a scannable divide, this construction recovers the A'Campo link [O. Couture–B. Perron, 2000].



The link of a scannable divide is the closure of a *positive braid*.

Positive braid isotopy

Positive isotopies are isotopies of *closed positive braids* via

- Artin's braid relations;
- cyclic shifts;
- positive Markov moves and their inverses.

Conjecture

Positive braids associated with link equivalent scannable algebraic divides are positive-isotopic.

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Scannable divides whose braids are positive-isotopic have move equivalent plabic fences (hence mutation equivalent quivers).

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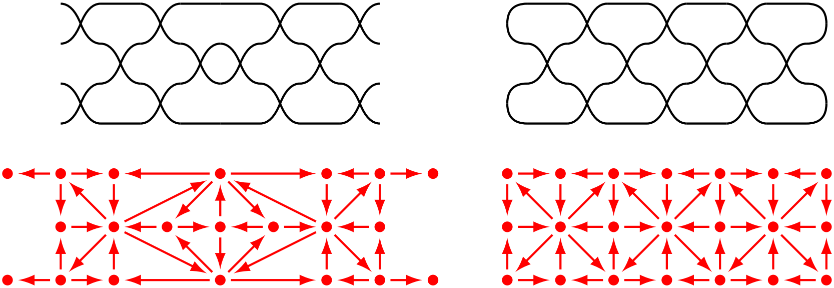
Example: quasihomogeneous singularity $x^8+y^4=0$

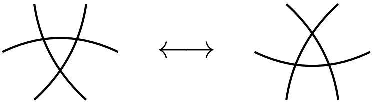


The braids associated with these divides are both equal to Δ^4.

The corresponding quivers are mutation equivalent.

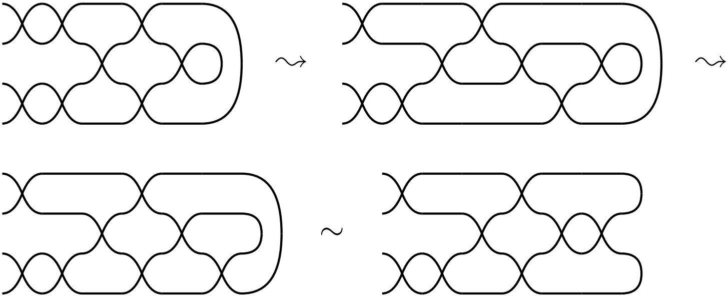
Triangle moves

Two divides are called $\bigtriangledown$-*equivalent* if they can be obtained from each other via a sequence of *triangle moves*:



Malleable divides

A divide is *malleable* if it is $\bigtriangledown$-equivalent to a scannable divide.



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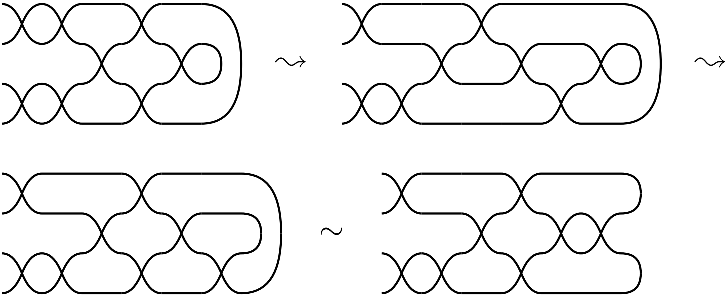

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Main conjecture for malleable divides

Triangle moves preserve both the isotopy class of the link of a divide and the mutation class of its quiver.

Consequently the last theorem extends to all malleable divides.

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Let D and D' be malleable divides $\bigtriangledown$-equivalent to scannable divides whose respective braids are positive-isotopic to each other. Then the quivers $Q(D)$ and $Q(D')$ are mutation equivalent.

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Simple singularities

Definition

A singularity is *simple* if any two curves in its topological equivalence class are locally diffeomorphic.

Simple singularities are classified by *ADE* Dynkin diagrams [V. Arnold, 1972].

Theorem

A plane curve singularity is simple if and only if some (equivalently, any) real morsification thereof defines a quiver of finite type. The type of a simple singularity matches the type of its quivers.

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