Extremal Graph Theory

March, 13th 2018

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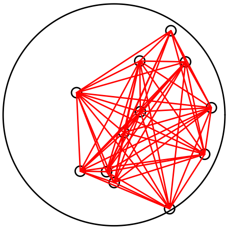
63 pairs out of 66



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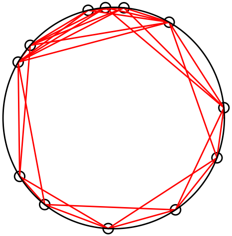

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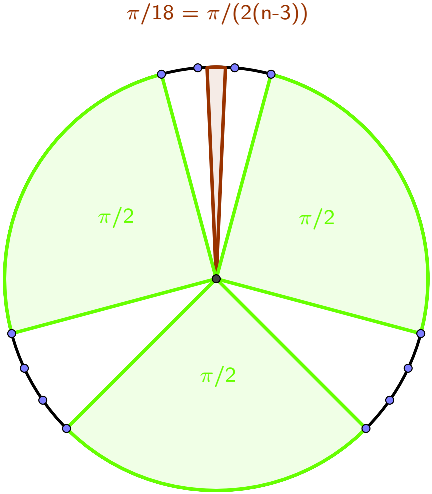

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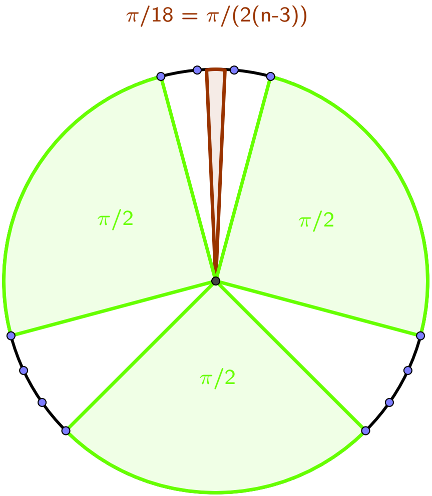

$\Rightarrow \frac{n^2}{6} - \frac{n}{2}$ pairs at distance at most $\sqrt{2}$.

Is it a good bound?

Parallel Processing

- How many processors are needed to find the maximum of n numbers in k rounds?

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  - Split the numbers into $n^{2/3}$ groups of $n^{1/3}$ numbers each.

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- Exercise! k rounds allowed $\Rightarrow O\left(n^{1+\frac{1}{2^k-1}}\right)$ processors are sufficient.

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Question

Are these bounds good?

Turán's theorem – Independent sets

Theorem (Turán, 1941)

G graph with n vertices and m edges. Then

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- The sum is minimised when G is $2m/n$-regular.

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- $\alpha(G_S) \leq 3$.
- So

$$3 \geq \frac{n^2}{2m + n},$$

i.e.

$$m \geq \frac{n^2}{6} - \frac{n}{2}.$$

Parallel Processing

Order is tight: induction on k. Wlog $p \geq n$.

- Statement true if $k = 1$. Suppose that finding the max. of n numbers in k rounds requires $\Omega\left(n^{1+\frac{1}{2^k-1}}\right)$ processors.

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- By induction, $p = \Omega\left(|I|^{1+\frac{1}{2^k-1}}\right)$.

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As $p \geq n$, it implies that

$$p = \Omega\left(\left(\frac{n^2}{p}\right)^{1+\frac{1}{2^k-1}}\right)$$

that is

$$p^{\frac{2^{k+1}-1}{2^k-1}} = \Omega\left(n^{\frac{2^k+1}{2^k-1}}\right)$$

$$\Rightarrow p = \Omega\left(n^{\frac{2^k}{2^{k+1}-1}}\right) = \Omega\left(n^{1+\frac{1}{2^{k+1}-1}}\right).$$

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Is this a good definition?

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  - Thus

$$\begin{aligned}\mathbf{E}(\mathbf{m}') &= \theta\binom{n}{r}\binom{n-r}{n'-r}\binom{n}{n'}^{-1} \\ &= \theta\binom{n'}{r}.\end{aligned}$$

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- Consequently, $\left(\binom{n}{r}^{-1} \mathrm{ex}(n, F)\right)_n$ is a decreasing sequence in $[0, 1]$.

When are we good at finding $\pi(F)$?

Theorem (Turán, 1941)

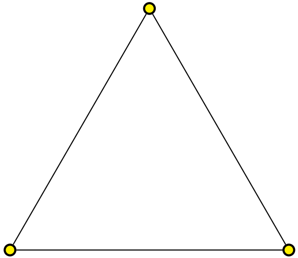
Fix $n \geq t \geq 2$.

$$\mathrm{ex}(n, K_{t+1}) = \frac{1}{2}\left(1 - 1/t\right)\left(n^2 - k^2\right) + \binom{k}{2}$$

where $k = n \pmod{t}$. Further, it is attained only by the complete multipartite graph on n vertices with balanced part sizes.

Turán Graphs and Blow-ups

The Turán graph (complete balanced multi-partite graph) is the balanced blow-up of the complete graph (part sizes are $\lfloor n/t \rfloor$ or $\lceil n/t \rceil$).



K_3

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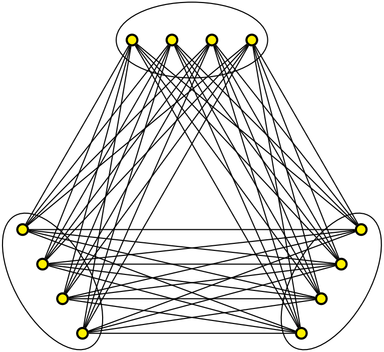

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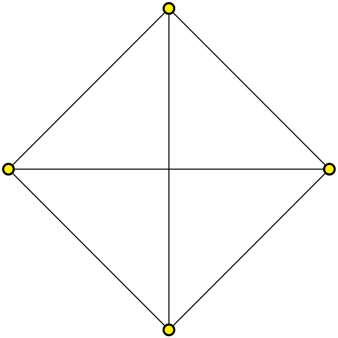

K_4

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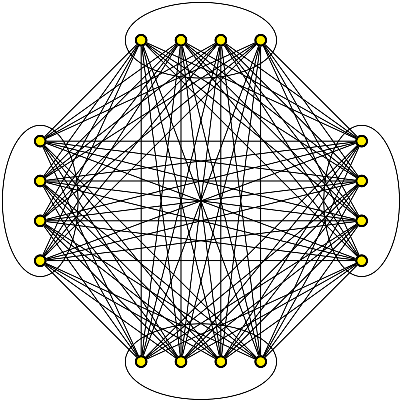

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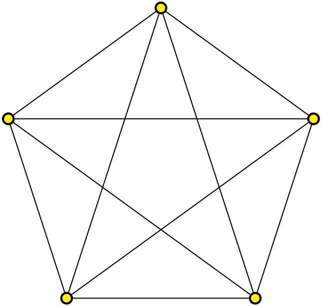

K_5

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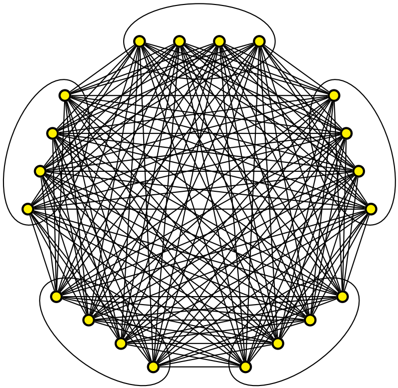

$K_5(4)$

When are we good at finding $\pi(F)$?

Theorem (Erdős-Stone-Simonovits, 1946)

Let F be a graph with chromatic number ≥ 3. Then

$$\pi(F) = 1 - \frac{1}{\chi(F) - 1}.$$

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Conjecture (Turán, 1940)

$\pi(K_4^3) = \frac{5}{9}$. *More specifically,*

$$\mathrm{ex}(n, K_4^3) = \begin{cases} \frac{m^2(5m-3)}{2} & \text{if } n = 3m \\ \frac{m(5m^2+2m-1)}{2} & \text{if } n = 3m+1 \\ \frac{m(m+1)(5m+2)}{2} & \text{if } n = 3m+2 \end{cases}$$

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- (V_0, V_1, V_2) balanced partition of V. Hyperedges either:
  - have two vertices in V_i and one in V_{i+1} (indices modulo 3); or
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- known: $\mathrm{ex}(n, K_4^3) \leq 0.561\binom{n}{3}$.

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- Suppose that F is a bipartite graph. It is proved that $\pi(F) = 0$, but what is the order of $\mathrm{ex}(n, F)$? Partial answers only.
- Why?

Super-saturation

Fix

- an r-graph F; and
- a real number $a > 0$.

There exists

- an integer n_0; and
- a real number $b > 0$

such that every r-graph G with $n_G > n_0$ vertices and $m > (\pi(F) + a)\binom{n}{r}$ contains at least $b\binom{n}{n_F}$ copies of F.

Proof

Consider an r-graph G with $m > (\pi(F) + a)\binom{n}{r}$.

- By the defintion, $\exists k$ such that $\text{ex}(k, F) \leq (\pi(F) + a/2)\binom{k}{r} =: x$.

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- By the defintion, $\exists k$ such that $\mathrm{ex}(k, F) \le (\pi(F) + a/2)\binom{k}{r} =: x$.
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  - On the other hand,

$$X = \binom{n-r}{k-r} m_G > \binom{n-r}{k-r}(\pi(F) + a)\binom{n}{r},$$

which contradicts the previous.

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- Number of k-subsets of $V(G)$ inducing a hypergraph with at least x hyperedges: at least $\frac{a}{2}\binom{n}{k}$.
- Each of those sets contains a copy of F, so G contains at least

$$\frac{a}{2}\binom{n}{k}\binom{n-n_F}{k-n_F}^{-1} = \overbrace{\frac{a}{2} \cdot \binom{k}{k-n_F}^{-1}}^{b} \binom{n}{n_F} = b \cdot \binom{n}{n_F}$$

copies of F.

Blow-up

Definition

The s-blow-up of an r-graph F is the r-graph $F(s)$ obtained from F by replacing:

- every vertex x by s vertices $x^1, \ldots, x^s$; and
- every hyperedge $x_1 \ldots x_r$ by a complete r-partite r-graph on copies, that is, all edges $x_1^{a_1} \ldots x_r^{a_r}$ with $1 \leq a_1, \ldots, a_r \leq s$.

Turán densities of blow-ups: $\pi(F(s)) = \pi(F)$

Two facts: $\mathrm{ex}(n, K_r^r) = 0$ and $\pi(K_r^r(s)) = 0$ (Erdős).

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Fix s. We show that $\pi(F(s)) = \pi(F)$, that is, $\forall \varepsilon > 0, \exists n_0$ such that every r-graph with $n > n_0$ and $m > (\pi(F) + \varepsilon)\binom{n}{r}$ contains $F(s)$.

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- Color the hyperedges of K with $n_F!$ colours, depending on the mapping from $V(F)$ to the parts of K.
- Ramsey tells us that there is a monochromatic copy of $K_{n_F}^{n_F}(s)$ in K (provided S is large enough *vs* s): this monochromatic copy yields a copy of $F(s)$ in G.

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- So $\pi(F) \leq \pi(K_t(s)) = \pi(K_t) = 1 - \frac{1}{t-1}$.