Topics on general stochastic matching models

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(Based on common works with J. Mairesse, O. Perry and A. Bušić)

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Bipartite matching model

Classical skill-based queueing theory

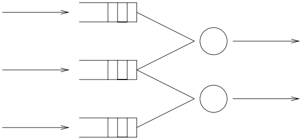


Figure: Queueing model of a call center.

Bipartite matching model: Complete symmetry between customers/servers. C/S arrive and depart simultaneously

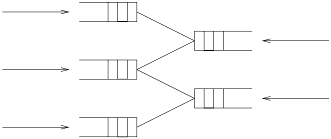


Figure: Bipartite matching model

Bipartite matching model

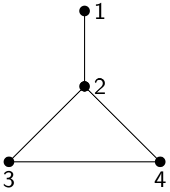
1. R. Caldentey, E.H. Kaplan, and G. Weiss. "FCFS Infinite bipartite matching of servers and customers". *Adv. Appl. Probab*, 41(3):695–730, 2009.
2. I. Adan and G. Weiss. "Exact FCFS matching rates for two infinite multi-type sequences". *Operations Research*, 60(2):475–489, 2012.
3. A. Bušić, V. Gupta, and J. Mairesse. "Stability of the bipartite matching model". *Adv. Appl. Probab.*, 45(2):351–378, 2013.

Applications

- **Healthcare systems**: Organ transplantation systems, Blood banks... (**bipartite** graphs);
- **Matching interfaces**: On-line dating, Job search, Public Housing allocations,... (**bipartite** graphs);
- **Collaborative economy**: Peer-to-peer sharing platforms, BlaBlaCar, Uberdrive, Bike-sharing...(**general** graphs);
- Assemble-to-order systems (**general** graphs and **hypergraphs**).
- ...

General stochastic matching model

Fix a simple connected graph $G = (\mathcal{V}, \mathcal{E})$,



- Items of the various classes in $\mathcal{V}$ arrive one by one; their class i is drawn following μ on $\mathcal{V}$.
- Any incoming item is matched, if possible, with a compatible item present in the system. Otherwise it is stored in a buffer;
- If several possible matches are possible, the incoming item follows a given matching policy ϕ.

General stochastic matching model

Usual types of matching policies:

- Priority type:
  - 2 choses 3 or 4 over 1,
  - 2 choses 1 over 3 or 4,
  - ...
- Class-uniform: visit the compatible classes in a uniformly random order, and pick an item of the first non-empty one.
- 'Match the Longest' (ML), 'Match the Shortest' (MS),...
- FCFM, LCFM, etc.

State space(s)

Let $\mathcal{V}^*$ be the free monoid associated to $\mathcal{V}$, and

$$\mathbb{W} = \left\{ w \in \mathcal{V}^* \ : \ \forall (i,j) \in \mathcal{E},\ |w|_i |w|_j = 0 \right\}.$$

Buffer detail

At any arrival time n,

$$W_n = w = w_1 w_2 ... w_q \in \mathcal{V}^*,$$

where $w_j =$ class of the i-th oldest item in line.

Class detail

At any n,

$$X_n = [W_n] := (|w|_i)_{i \in \mathcal{V}} \in \mathbb{N}^{|\mathcal{V}|},$$

i.e. the commutative image of the buffer detail at n.

Stochastic recursive representations

For any admissible matching policy ϕ,

if the original state is $Y \in \mathbb{W}$ we get that

$$\begin{cases} W_0^{\{Y\}} &= Y; \\ W_{n+1}^{\{Y\}} &= \left(W_n^{\{Y\}} \odot_\phi V_n\right), \; n \in \mathbb{N}, \end{cases} \qquad \text{a.s.}$$

For any *class-admissible* matching policy ϕ,

if the original state is $Y \in \mathbb{W}$,

$$\begin{cases} X_0^{\{[Y]\}} &= [Y]; \\ X_{n+1}^{\{[Y]\}} &= \left(X_n^{\{[Y]\}} \circledcirc_\phi V_n\right), \; n \in \mathbb{N}, \end{cases} \qquad \text{a.s.}$$

$\hookrightarrow$ Priorities, Match the Longest, class-uniform are class-admissible;

$\hookrightarrow$ LCFM, FCFM are not.

Outline

Stability problem

The **stability region** of (G, ϕ), denoted $\textsc{Stab}(G, \phi)$, is the set of probability measures μ on $\mathcal{V}$ such that the natural Markov chain of (G, μ, ϕ) is positive recurrent.

Natural necessary condition on μ

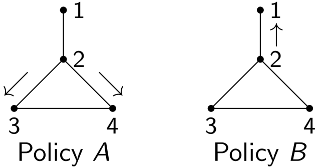
$\textsc{Stab}(G, \phi)$ is included in the set

$$\textsc{NCond}(G) := \left\{ \mu : \sum_{i \in \mathcal{I}} \mu(i) < \sum_{j \in \mathcal{E}(\mathcal{I})} \mu(j) \text{ for all independent sets } \mathcal{I} \right\}.$$

ϕ is said maximal if $\textsc{Stab}(G, \phi) = \textsc{NCond}(G)$.

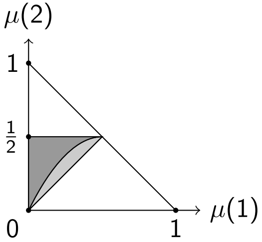
- $\textsc{NCond}$ generalizes the Complete resource pooling condition of [1], [2] and [3] for the bipartite model;
- Probabilistic analog to the necessary and sufficient condition of the Marriage Theorem.

Dependence on the matching policy: example of the 'Paw graph'



Stability regions

For $\mu(3) = \mu(4)$,

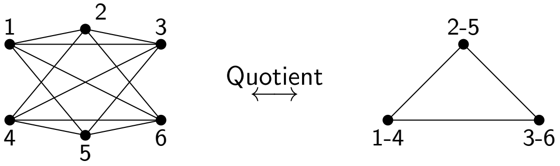


Anti-separable graphs

Definition

The graph G is said *anti-separable of order p* if there exists a partition of $\mathcal{V}$ into p independent sets $\mathcal{I}_1, \ldots, \mathcal{I}_p,\ p \geq 3$, such that

$$\forall i \neq j, \forall u \in \mathcal{I}_i, \forall v \in \mathcal{I}_j, \qquad u \text{ is a neighbor of } v\ .$$



Anti-separable $\simeq$ Complete

An anti-separable graph of order p is projected onto the complete graph of size p, if quotiented by the equivalence relation "not being neighbors".

Stability and the geometry of G

Definition

A connected graph G is said to be

- *matching-stable* if $\textsc{Ncond}(G)$ is non-empty and all admissible policies are maximal;
- *matching-unstable* if $\textsc{Stab}(G, \phi) = \emptyset$ for all admissible ϕ.

Theorem

For any connected graph G,

(i) G is matching-unstable if and only if is bipartite;

(ii) If G is non-bipartite, then $\textsc{ml}$ is maximal;

(iii) If G is anti-separable, then it is matching-stable.

- J. Mairesse and P. Moyal. "Stability of the stochastic matching model", *Journ. Appl. Probab.* **53**(4), 1064-1077, 2016.

Partial converse

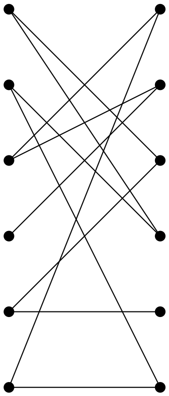
By a continuous-time declination of the model and fluid (in)stability arguments,

Theorem

Let $\mathscr{G}_7$ denote the set of connected graphs inducing an odd cycle of size 7 or more, but no 5-cycle and no Paw graph.
Then the *only* matching-stable graphs in $\mathscr{G}_7^c$ are anti-separable.

- P. Moyal and O. Perry. "On the instability of matching queues", *Annals Appl. Probab.* **27**(6), 3385-343, 2017.

The marriage problem on graphs



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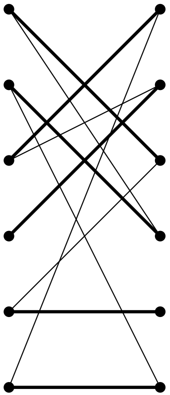
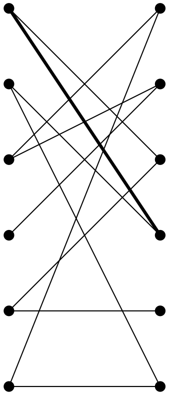


Hall's Marriage Theorem

Let $G = (\mathcal{V}, \mathcal{E})$ be a bipartite graph and for all subsets $A \subset \mathcal{V}$, $\mathcal{E}(A)$ denote the neighborhood of the nodes of A. Then there exists a perfect matching iff for all $A \subset \mathcal{V}$, $|A| \leq |\mathcal{E}(A)|$.

The matching problem on graphs



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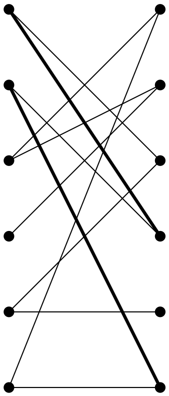

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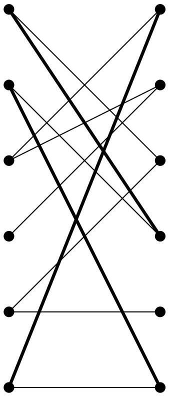

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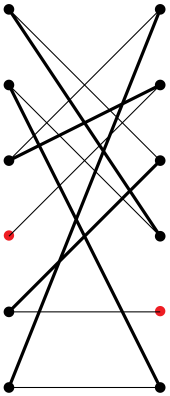


Online matching algorithms fail in general to construct a perfect matching.

Outline

First Come, First Matched matching model

First Come, First Matched matching model

- At this point, we do not know whether First Come, First Matched has a maximal stability region or not.
- This matching policy proves to have a maximal stability region for the Bipartite Matching model.

1. R. Caldentey, E.H. Kaplan, and G. Weiss. "FCFS Infinite bipartite matching of servers and customers". *Adv. Appl. Probab*, 41(3):695–730, 2009.
2. I. Adan, A. Bušić, J. Mairesse and G. Weiss. "Reversibility and further properties of the FCFM Bipartite matching model". ArXiv math.PR 1507.05939.

Auxiliary Markov chains

Let $\bar{\mathcal{V}}$ be a copy of $\mathcal{V}$ and $\mathbf{V} = \mathcal{V} \cup \bar{\mathcal{V}}$. We define at all n the following $\mathbf{V}^*$-valued chains B_n and F_n,

Backwards chain

For all n, let $i(n) \leq n$ is the index of the oldest item in line. For any $\ell \in [\![1, n - i(n) + 1]\!]$, we set

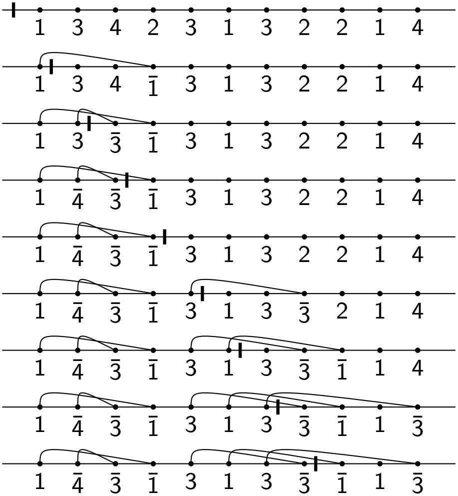
$$B_n(\ell) = \begin{cases} V_{i(n)+\ell-1} & \text{if } V_{i(n)+\ell-1} \text{ has not been matched up to time } n; \\ \overline{V_k} & \text{if } V_{i(n)+\ell-1} \text{ is matched with } V_k, \text{ with } k \leq n. \end{cases}$$

Forwards chain

For all n, let $j(n) > n$ be the largest index of an item that is matched with an item entered up to n. For any $\ell \in [\![1, j(n) - n]\!]$, we let

$$F_n(\ell) = \begin{cases} V_{n+\ell} & \text{if } V_{n+\ell} \text{ is not matched with an item arrived up to } n; \\ \overline{V_k} & \text{if } V_{n+\ell} \text{ is matched with } V_k \ (k \leq n). \end{cases}$$

Example on the Paw graph



$$W_0 = \emptyset,\ B_0 = \emptyset,\ F_0 = \emptyset$$

$$W_1 = 1,\ B_1 = 1,\ F_1 = 34\bar{1}$$

$$W_2 = 13,\ B_2 = 13,\ F_2 = \bar{3}\bar{1}$$

$$W_3 = 1,\quad B_3 = 1\bar{4}\bar{3},\quad F_3 = \bar{1}$$

$$W_4 = \emptyset,\ B_4 = \emptyset,\ F_4 = \emptyset$$

$$W_5 = 3,\ B_5 = 3,\ F_5 = 13\bar{3}$$

$$W_6 = 31,\ B_6 = 31,\ F_6 = 3\bar{3}\bar{1}$$

$$W_7 = 313,\ B_7 = 313,\ F_7 = \bar{3}\bar{1}1\bar{3}$$

$$W_8 = 13,\ B_8 = 13\bar{3},\quad F_8 = 11\bar{3}$$

Reversibility

Let for any $w \in \mathbf{V}^*$ and any $i \in \mathcal{V}$, $|w|_i$ be the number of occurrences of letter i or $\bar{i}$ in the word w.

Proposition

Suppose that $\mu \in \textsc{Ncond}(G)$. Then the Backwards chain $\{B_n\}$ and the Forwards chain $\{F_n\}$ both admit the following unique stationary distribution on $\mathbf{V}^*$:

$$\Pi_B(\mathbf{w}) = \alpha \prod_{i=1}^{p} \mu(i)^{|\mathbf{w}|_i + |\overline{\mathbf{w}}|_i}.$$

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$\hookrightarrow$ Consequence of Kelly's Lemma together with

Proposition

For any two admissible states $\mathbf{w}, \mathbf{w}' \in \mathbf{V}^*$ for $\{B_n\}$, the states $\overleftarrow{\overline{\mathbf{w}}}$ and $\overleftarrow{\overline{\mathbf{w}'}}$ are admissible for $\{F_n\}$ and we have that

$$\Pi_B(\mathbf{w})\mathbf{P}\left[B_{n+1} = \mathbf{w}' | B_n = \mathbf{w}\right] = \Pi_B\left(\overleftarrow{\overline{\mathbf{w}'}}\right) \mathbf{P}\left[F_{n+1} = \overleftarrow{\overline{\mathbf{w}}} | F_n = \overleftarrow{\overline{\mathbf{w}'}}\right].$$

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Theorem

Consider a matching model $(G, \mu, \textsc{fcfm})$, where G is non-bipartite. Then the model is stable if and only if $\mu \in \textsc{Ncond}(G)$, and in that case the only stationary probability of the Markov chain $\{W_n\}$ is given by

$$\Pi_W(w) = \alpha \prod_{\ell=1}^{q} \frac{\mu(w_\ell)}{\mu\Big(\mathcal{E}\left(\{w_1, ..., w_\ell\}\right)\Big)}, \quad \text{for any } w = w_1 ... w_q \in \mathcal{V}^*.$$

Outline

Stochastic recursive representation

- Under stationary ergodic assumptions, we aim at an explicit construction of a (possibly unique) stationary version of the system, using coupling from the past;
- As the model is 2-periodic we work on the Palm space of the input tracked by batches of two;
- For an initial state $Y \in \mathbb{W}_2 = \{w \in \mathbb{W} : |w| \text{ is even }\}$ we study the stochastic recursion

$$\begin{cases} W_0^{\{Y\}} & = Y; \\ W_{2(n+1)}^{\{Y\}} & = \left(W_{2n}^{\{Y\}} \odot_\phi V_{2n}\right) \odot_\phi V_{2n+1} \circ \theta^n,\ n \in \mathbb{N}, \end{cases} \qquad \bar{\mathbf{P}} - \text{ a.s..}$$

On the canonical space of arrivals, a stationary version of the system solves the functional equation

$$U \circ \theta = \left(U \odot_\phi V^0\right) \odot_\phi V^1, \text{ a.s..}$$

Backwards scheme

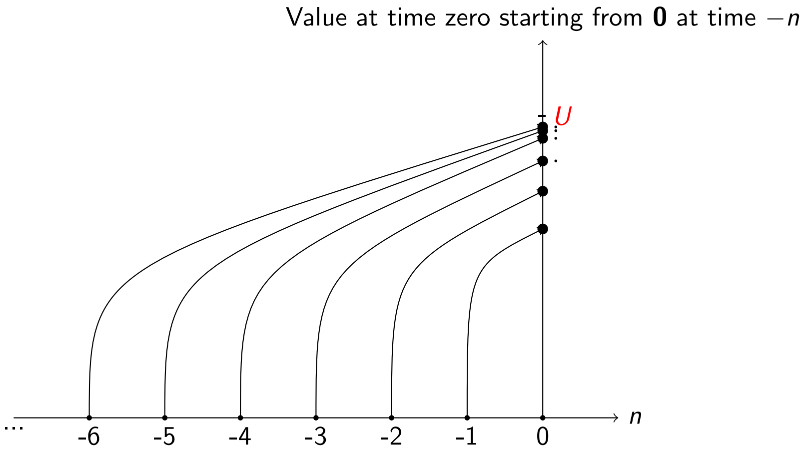

Figure: Loynes backwards scheme on $\mathbb{R}+$

'Block-wise' sub-additivity

Let for all ϕ and all $z \in \mathcal{V}^*$,

$$Q_\phi = \text{word of unmatched letters of } z \text{ by } \phi \text{ in arrival order.}$$

Definition (Sub-additivity)

An admissible matching policy ϕ is said to be *sub-additive* if, for all $z', z'' \in \mathcal{V}^*$ we have that

$$|Q_\phi(z'z'')| \leq |Q_\phi(z')| + |Q_\phi(z'')|\,.$$

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Proposition

The matching policies FCFM, LCFM, Priorities, class-uniform and ML are sub-additive.

Proof.

- Any '1-Lipschitz' policy ϕ (true for ML, Priorities or class-uniform) is sub-additive: $\|[w'] \circledcirc_\phi v - [w] \circledcirc_\phi v\| \leq \|[w'] - [w]\|$,
- Direct proof for FCFM, LCFM.

Sub-additivity (Cd)

Consider the 'Paw graph' and the following arrival scenario:

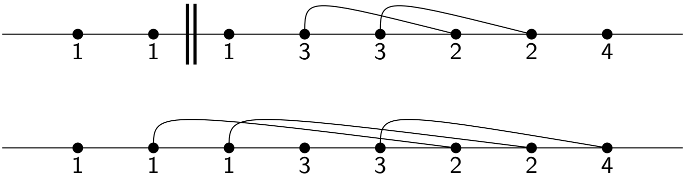

Figure: 'Match the Longest' is sub-additive...

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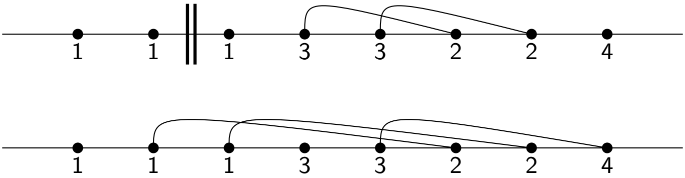

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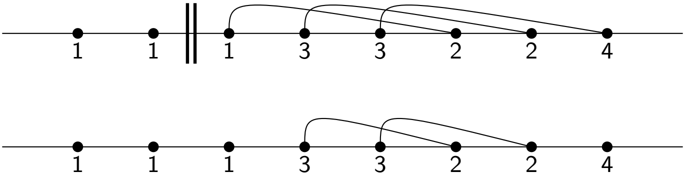


Figure: 'Match the Shortest' is not.

Coupling result

Definition

Let G a connected graph and ϕ be an admissible matching policy. Let $u \in \mathbb{W}_2$. We say that the word $z \in \mathcal{V}^*$ is an *erasing word* of u for (G, ϕ) if $|z|$ is even and $Q_\phi(z) = \emptyset$ and $Q_\phi(uz) = \emptyset$.

Theorem

Borovkov and Foss's Renovation Theorem applies in particular in particular if:

1. ϕ is sub-additive;
2. For any $w \in \mathbb{W}_2$ the r.v. $\tau(w) := \inf \left\{ n > 0, U_n^{\{w\}} = \emptyset \right\}$ is integrable (true in particular if $\mu \in \textsc{Ncond}(G)$ and if $\phi = \textsc{fcfm}$ or $\textsc{ml}$, or if G is anti-separable);
3. Erasing words occur often enough (true in particular if the input is iid).

Thus a unique solution exists, to which all sequences $(U_n^w)_{n\in\mathbb{N}}$, $w \in \mathbb{W}_2$, couple strongly from the past.

Existence of erasing words

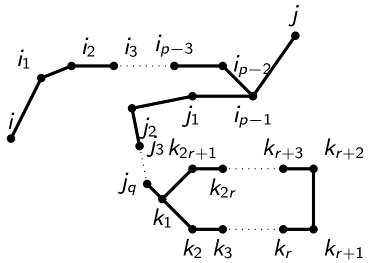
Proposition

If G is non-bipartite and ϕ is sub-additive, then any word $u \in \mathbb{W}_2$ admits an erasing word z for (G, ϕ).

Proof.

By sub-additivity is it enough to address the case $u = ij$ for $i \neq j$, and consider the minimal path $i - i_1 - \ldots - i_{p-1} - j$ connecting i to j. Then set:

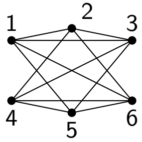
- if p odd, $z = i_1 i_2 \ldots i_{p-1}$;
- if p even, $z = i_1 i_2 \ldots i_{p-1} i_{p-1} j_1 j_1 j_2 j_2 \ldots j_q j_q k_1 k_1 k_2 k_3 \ldots k_{2r} k_{2r+1}$, where

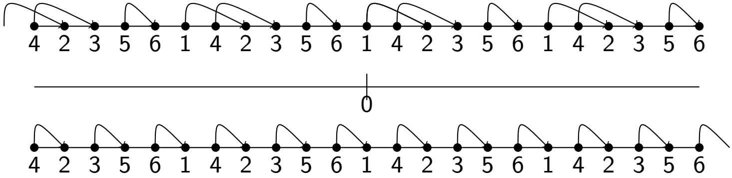


Constructing perfect bi-infinite ϕ-matchings

Corollary

Under the assumptions of the above Theorem, there exist exactly **two** perfect ϕ-matchings on $\mathbb{Z}$.

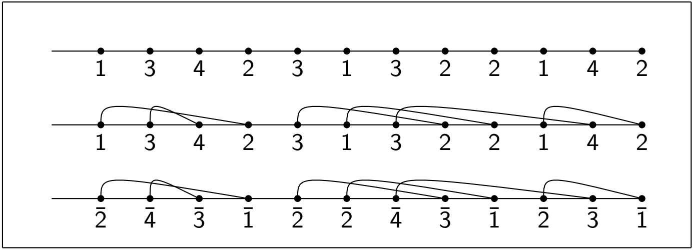




Back to FCFM

Corollary

If all matchings and 'exchanges' are completed by a perfect bi-infinite FCFM matching, then the matching obtained in reversed time on the copies of arrived items, is also a perfect bi-infinite FCFM matching.



- A. Bušić, J. Mairesse and P. Moyal. "A product form and a sub-additive theorem for the stochastic matching". ArXiv math.PR/1711.02620.