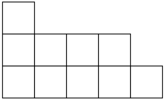
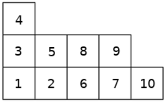
Corners of Young tableaux and periodic urns

C. Banderier, P. Marchal, M. Wallner

Filling at random a Young diagram

Consider $\mathcal{F}$ a Young (or Ferrers) diagram of size N.

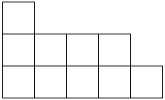


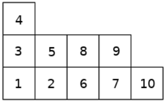


A Young tableau with shape $\mathcal{F}$ is a *standard filling* of $\mathcal{F}$: put an entry between 1 and N into each cell so as to be increasing along the rows and columns.

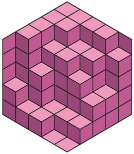
Filling at random a Young diagram

Consider $\mathcal{F}$ a Young (or Ferrers) diagram of size N.





A Young tableau with shape $\mathcal{F}$ is a *standard filling* of $\mathcal{F}$: put an entry between 1 and N into each cell so as to be increasing along the rows and columns.
If $\mathcal{F}$ if fixed, there are several possible standard fillings. Choose one at random, uniformly. Interpreting the entries as heights, how does the random surface look like ?
Analogy with height functions associated with random tilings.

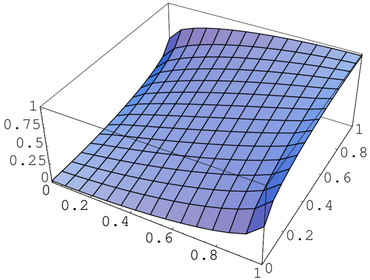


The rectangular case: scaling limit

For Young tableaux, the only case studied so far is the case of a rectangular tableau of size (m, n).
Associated surface: function $f : [0,1] \times [0,1] \to [0,1]$
If the cell (i, j) has entry k, put $f(i/m, j/n) = k/mn$.

The rectangular case: scaling limit

For Young tableaux, the only case studied so far is the case of a rectangular tableau of size (m, n).
Associated surface: function $f : [0,1] \times [0,1] \to [0,1]$
If the cell (i, j) has entry k, put $f(i/m, j/n) = k/mn$.
Pittel-Romik (2007): if $m, n \to \infty$, $m/n \to \ell$, existence of a deterministic limit function f, expressed as the solution of a variational problem.



The rectangular case: fluctuations along the edge

Two asymptotic regimes (M., 2016)

In the corner: Let $X_{m,n}$ be the entry in the southeast corner.

$$\frac{\sqrt{2}(1+\ell)\left(X_{m,n}-\mathbb{E}X_{m,n}\right)}{n^{3/2}} \overset{law}{\rightarrow} Gaussian$$

Along the edge: Suppose the tableau is an (n, n) square. Let $Y_{i,n}$ be the entry in the cell $(1, i)$. Fix $0 < t < 1$. Then for large n,

$$\frac{r(t)(Y_{1,\lfloor tn\rfloor}-\mathbb{E}Y_{1,\lfloor tn\rfloor})}{n^{4/3}} \overset{law}{\rightarrow} Tracy-Widom$$

Similar results for random surfaces (Johansson et al., Bouttier et al., ...) By analogy, the expected behaviour in the centre of the tableau is the gaussian free field (Kenyon et al.).

The rectangular case: fluctuations along the edge

Two asymptotic regimes (M., 2016)
In the corner: Let $X_{m,n}$ be the entry in the southeast corner.

$$\frac{\sqrt{2}(1+\ell)\left(X_{m,n} - \mathbb{E}X_{m,n}\right)}{n^{3/2}} \stackrel{law}{\rightarrow} Gaussian$$

Along the edge: Suppose the tableau is an (n, n) square. Let $Y_{i,n}$ be the entry in the cell $(1, i)$. Fix $0 < t < 1$. Then for large n,

$$\frac{r(t)(Y_{1,\lfloor tn \rfloor} - \mathbb{E}Y_{1,\lfloor tn \rfloor})}{n^{4/3}} \stackrel{law}{\rightarrow} Tracy - Widom$$

Similar results for random surfaces (Johansson et al., Bouttier et al., ...) By analogy, the expected behaviour in the centre of the tableau is the gaussian free field (Kenyon et al.).
Nothing else is known for non-rectangular diagrams.
Our goal: understanding what happens in the southeast corner of Young tableau with general fixed shape.

Linear extension of a tree

If T is a tree of size $N+1$, a linear extension is a function $f : T \to \{0, 1 \ldots N\}$ such that $f(child) > f(parent)$.

Linear extension of a tree

If T is a tree of size $N+1$, a linear extension is a function $f : T \to \{0, 1 \ldots N\}$ such that $f(child) > f(parent)$.
Number of linear extensions given by

$$\frac{(N+1)!}{\prod_{v \in T} h(v)}$$

where $h(v)$ is the size of the subtree below v.

Linear extension of a tree

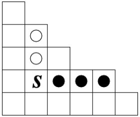
If T is a tree of size $N+1$, a linear extension is a function $f: T \to \{0, 1 \dots N\}$ such that $f(child) > f(parent)$.
Number of linear extensions given by

$$\frac{(N+1)!}{\prod_{v \in T} h(v)}$$

where $h(v)$ is the size of the subtree below v.
Analogue of the hook length formula for the number of standard fillings of a diagram $\mathcal{F}$:

$$\frac{N!}{\prod_{e \in \mathcal{F}} h(e)}$$

where $h(e)$ is the hook length of the cell e.



The tree associated with a diagram

If $\mathcal{F}$ is a Young diagram, associate a planar rooted tree T with the rule:

- The hook lengths along the first row of $\mathcal{F}$ are the same as the hook lengths along the left-most branch of T.
- All the non-leftmost vertices are leaves.
- T has a distinguished vertex v which is either the father of the left-most leaf if this leaf has brothers, or the left-most leaf.

The main result

Let $\mathcal{F}$ be a Young diagram, associate a planar rooted tree T with a distinguished vertex v. Enlarge T to obtain $\overline{T}$ by adding a father R to the root of T and adding children to R so that the size of $\overline{T}$ is $N+1$.

The main result

Let $\mathcal{F}$ be a Young diagram, associate a planar rooted tree T with a distinguished vertex v. Enlarge T to obtain $\overline{T}$ by adding a father R to the root of T and adding children to R so that the size of $\overline{T}$ is $N+1$.

Theorem

Let X be the entry of the southeast corner of $\mathcal{F}$ when one picks a random, uniform standard filling of $\mathcal{F}$.
Let $Y = \ell(v)$ where ℓ is a random, uniform linear extension of $\overline{T}$.
Then X and Y have the same law.

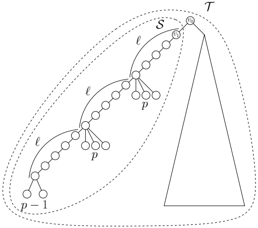
Triangular tableaux, periodic trees, periodic urns

If $\mathcal{F}$ is triangular, then the associated tree T will have a periodic pattern.

43	56	58	61	83										
30	44	45	57	73										
22	31	33	52	71										
21	25	32	42	60	62	74	75	85	90					
18	19	26	41	53	59	63	70	82	87					
12	14	23	29	46	54	55	69	76	84					
6	8	10	24	34	40	47	51	68	72	77	78	80	86	89
3	5	7	13	16	17	28	37	38	50	64	65	66	81	88
1	2	4	9	11	15	20	27	35	36	39	48	49	67	79

p p p

ℓ ℓ ℓ



Moreover, the law of $\ell(v)$ will be given by a model of *periodic urn*.

Periodic Pólya urns

Definition

A *periodic Pólya urn* of period p is a Pólya urn with replacement matrices
$M_1, M_2, \dots, M_p$, such that at step $np + k$ the replacement matrix M_k is used.

Particular case:

Definition

The *Airy Pólya urn* is a Pólya urn of period 2 with replacement matrix $M_1 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ for every odd step,
and replacement matrix $M_2 := \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ for every even step.

Caveat: not *exactly* modelizable as a classical Pólya urn (even with multidrawing).

The Airy case

The generalized Gamma distribution $GG(d, p)$ is the distribution on $\mathbb{R}_+$ with density $c(d, p)x^{d-1}e^{-x^p}$. When $p = 1$, this a usual Gamma distribution.

Computing the moments, we find:

Theorem

The of the number of black balls in an Airy Pólya urn converges in law to a generalized Gamma distribution:

$$\frac{2^{2/3}}{3} \frac{B_n}{n^{2/3}} \xrightarrow{\mathcal{L}} GG\,(1, 3)$$

As a corollary, let X_n be the entry of the southeast corner of the associated triangular Young tableau with total size $N_n = n(n+1)$.

$$\frac{2^{2/3}}{3} \frac{N_n - X_n}{n^{5/3}} \xrightarrow{\mathcal{L}} GG\,(1, 3)$$

The more general triangular case

For a more general triangular tableau, there exists a critical exponent α such that $(N_n - X_n)/n^{\alpha}$ has a limit law X. One can show that α only depends on the slope and that X is a product of independent generalized Gamma distributions. These distributions, which are related to Mittag-Leffler distributions (same growth of the moments), also appear in papers by Janson (2016) and Peköz-Röllin-Ross (2016).

The more general triangular case

For a more general triangular tableau, there exists a critical exponent α such that $(N_n - X_n)/n^\alpha$ has a limit law X. One can show that α only depends on the slope and that X is a product of independent generalized Gamma distributions. These distributions, which are related to Mittag-Leffler distributions (same growth of the moments), also appear in papers by Janson (2016) and Peköz-Röllin-Ross (2016).

Duality

There is a duality between the limit law X in the southeast corner and the limit law Y in the northwest corner. If $X' \sim X$, $Y' \sim Y$ and X', Y' are independent, then $X'Y'$ has Gamma distribution. Such factorizations often appear in the study of exponential functionals of Lévy processes (Bertoin-Yor).