Scaling limits of pattern-avoiding permutations

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Joint work with F. Bassino, M. Bouvel, V. Féray, L. Gerin and A. Pierrot

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1 – The scaling limit of separable permutations

After Bassino, Bouvel, Féray, Gerin, Pierrot 2016
A permutation $\sigma \in \mathcal{S}_n$ is a word $(\sigma(1), \ldots, \sigma(n))$ which contains every element of $\{1, \ldots, n\}$.

Diagram of $(4128376) \in \mathcal{S}_8$: 

![Diagram of (4128376) ∈ S8](image)
Permutation patterns

\( \sigma = (10, 6, 2, 5, 3, 9, 1, 7, 4, 8, 11) \in S_{11} \)
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\[ \text{pat}\{2,5,6,10\}(\sigma) = (2143) \]
Permutation patterns

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\[\text{pat}_{\{2, 5, 6, 10\}}(\sigma) = (2143)\]

\[\text{pat}_{\{8, 11\}}(\sigma) = (12)\]
Classes of permutation and pattern-avoidance

*Permutation class:* set of permutations closed under pattern extraction. Can always be written as $\text{Av}(B)$, the set of permutations that avoid patterns in some *basis* $B$. 
Permutation class: set of permutations closed under pattern extraction. Can always be written as \( \text{Av}(B) \), the set of permutations that avoid patterns in some basis \( B \).

Separable permutations: \( \text{Av}(3142, 2413) \)

(Avis-Newborn ’80, Bose-Buss-Lubiw ’93)
Separable permutations

Signed tree $\tau$
Separable permutations

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Alternating-signs Schröder tree

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perm(τ) = (1 2 10 7 6 5 8 9 4 3)

Counted by large Schröder numbers

1, 2, 6, 22, 90, 394, 1806, 8558, … ≈ (3 + \sqrt{8})^n n^{-3/2}
A large uniform separable permutation
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Permutons

A permuton is a probability measure on $[0, 1]^2$ with both marginals uniform.
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Permutations of all sizes are densely embedded in permutons.

We say that a sequence $(\sigma_n)$ converges to $\mu$ when $\mu_{\sigma_n} \xrightarrow{w} \mu$. 
Density of patterns

For $\sigma \in \mathcal{S}_n$, $\pi \in \mathcal{S}_k$, $\tilde{\text{occ}}(\pi, \sigma)$ is the proportion of $\pi$ among the $\binom{n}{k}$ subpermutations of $\sigma$ of size $k$. 
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For instance, $\tilde{\text{occ}}(21, \cdot)$ is the proportion of inversions: $\tilde{\text{occ}}(21, 13254) = 2/10$
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For instance, $\tilde{occ}(21, \cdot)$ is the proportion of inversions:
$\tilde{occ}(21, 13254) = 2/10$

Similarly, if $\mu$ is a permuton, the density of $\pi$ in $\mu$ is the probability that $k$ independent points drawn from $\mu$ are ordered like $\pi$.

$$
\tilde{occ}(\pi, \mu) = \int_{[0,1]^{2k}} \mu(dx_1 dy_1) \cdots \mu(dx_k dy_k) 1[(\vec{x}, \vec{y}) \sim \pi].
$$
Density of patterns

For \( \sigma \in \mathcal{S}_n, \pi \in \mathcal{S}_k \), \( \tilde{\mathrm{occ}}(\pi, \sigma) \) is the proportion of \( \pi \) among the \( \binom{n}{k} \) subpermutations of \( \sigma \) of size \( k \).

For instance, \( \tilde{\mathrm{occ}}(21, \cdot) \) is the proportion of inversions:
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\tilde{\mathrm{occ}}(21, 13254) = \frac{2}{10}
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Similarly, if \( \mu \) is a permuton, the density of \( \pi \) in \( \mu \) is the probability that \( k \) independent points drawn from \( \mu \) are ordered like \( \pi \).

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**Theorem (Hoppen et. al., 2013)**
The sequence \( (\sigma_n) \) converges to \( \mu \) iff for every \( \pi \in \mathcal{S} \),
\[
\tilde{\mathrm{occ}}(\pi, \sigma_n) \rightarrow \tilde{\mathrm{occ}}(\pi, \mu).
\]
Sequences of random permutations

If $\sigma_n$ is a sequence of random permutations, we can consider the convergence in distribution of the random permutons $\mu_{\sigma_n}$. Let $\sigma_n = \text{uniform of size } n$ in some class $\mathcal{C}$. 
Sequences of random permutations

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\mathcal{C} = \mathcal{S} : \sigma_n \xrightarrow{\mathbb{P}} \text{Leb}_{[0,1]^2}.
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$$\mathcal{C} = \text{Av}(231) \text{ or Av}(321) : \sigma_n \xrightarrow{\mathbb{P}} (\text{id}, \text{id})_* \text{Leb}_{[0,1]}.$$
Sequences of random permutations

If $\sigma_n$ is a sequence of random permutations, we can consider the convergence in distribution of the random permutons $\mu_{\sigma_n}$. Let $\sigma_n = \text{uniform of size } n$ in some class $C$.

$C = \text{Av}(2413, 3142) = \{\text{separables}\}$:

**Theorem** (Bassino, Bouvel, Féray, Gerin, Pierrot 2016) $\sigma_n$ converges in distribution to some random permuton $\mu$, called the Brownian separable permuton.
A portmanteau theorem for random permutons

Theorem (Bassino, Bouvel, Feray, Gerin, M., Pierrot. 2017)
The following are equivalent:
1. The random measure $\mu_{\sigma_n}$ converges in distribution to some random permuton $\mu$.
2. The random variables $\tilde{\text{occ}}(\pi, \sigma_n)$ converge in distribution, jointly in $\pi \in S$.
3. $\mathbb{E}[\tilde{\text{occ}}(\pi, \sigma_n)]$ converges to some $\Delta_{\pi}$ for every $\pi \in S$.

Moreover, the law of $\mu$ is characterized by
$\mathbb{E}[\tilde{\text{occ}}(\pi, \mu)] = \Delta_{\pi}, \pi \in S$
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The following are equivalent:
1. The random measure $\mu_{\sigma_n}$ converges in distribution to some random permuton $\mu$.
2. The random variables $\tilde{\text{occ}}(\pi, \sigma_n/n)$ converge in distribution, jointly in $\pi \in \mathcal{S}$.
3. $\mathbb{E}[\tilde{\text{occ}}(\pi, \sigma_n)]$ converges to some $\Delta_\pi$ for every $\pi \in \mathcal{S}$.

Moreover, the law of $\mu$ is characterized by $\mathbb{E}[\tilde{\text{occ}}(\pi, \mu)] = \Delta_\pi$, $\pi \in \mathcal{S}$

Remark: $\mathbb{E}[\tilde{\text{occ}}(\pi, \sigma_n)] = \mathbb{P}[\text{pat}_{l_n^k}(\sigma_n) = \pi]$, where $l_n^k$ is a uniform $k$-subset of $[n]$, independent of $\sigma_n$.  

A portmanteau theorem for random permutons

It suffices to show that $\forall k$, $\text{pat}_{I_n^k}(\sigma_n)$ converges, where $I_n^k$ is a uniform $k$-subset of $[n]$, independent of $\sigma_n$. 
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When $\sigma_n$ uniform in $C_n$, this is only a matter of enumeration: how many of the $\binom{n}{k}|C_n|$ pairs $(l, \sigma) \in [n] \times C_n$ are such that $\text{pat}_l(\sigma) = \pi$?
Idea of proof

Use the bijection with signed Schröder trees: $\sigma_n = \text{perm}(t_n)$, where $t_n$ is a uniform signed Schröder tree with $n$ leaves.
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Fix \( k(= 3) \). Then \( \text{pat}_{I_n}(\sigma_n) = \text{perm}(t_n|_{I_n}^k) \), where \( t_n|_{I_n}^k \) is the reduced subtree of \( t_n \) induced by the leaves with indexes in \( I_n^k \).
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![Diagram of signed Schröder trees and pat]
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What does it look like as $n \to \infty$?
Analytic combinatorics for leaf-counted trees

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Recursive trees counted by number of leaves.

$$T(z) = z + F(T(z))$$ (Schröder: $F(t) = \sum_{k \geq 2} t^k$).
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This is the case for Schröder ($F$ rational)
Uniform $k$-subtree in large unsigned trees

$T$ has square-root singularity at $\rho$ and $F$ analytic at $T(\rho)$. Then, the $g.f$ of trees with $k$ marked leaves that induce the $k$-tree $\tau$ is

$$z^k T'(z) \prod_{\nu \text{ internal node of } \tau} T'(z)^{\deg(\nu)} \frac{1}{\deg(\nu)!} F^{(\deg(\nu))}(T(z))$$
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$$\sim \rho \ C_\tau (1 - \frac{z}{\rho})^{-\#\{\text{nodes in } \tau\}/2}.$$ 

Dominates when $\tau$ binary. (Then $C_\tau$ doesn’t depend on $\tau$). Transfer: $t_n|I_n^k$ converges in distribution to a uniform binary tree.
Uniform $k$-subtree in large signed trees

Counting signed trees that induce a given signed tree $\tau$: adding parity constraints on the height of the marked leaf in the marked trees.
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Replace instances of $T'$ by $T'_0$ (even height) or $T'_1$ (odd height). $T'_0 + T'_1 = T'$ and $T'_1 = F'(T)T'_0$, so $T'_0 \sim T'_1 \sim \frac{1}{2}T'$. 
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g.f. of Trees with $k$ marked leaves that induce the signed $k$–tree $\tau$:

$$z^k (T'_0 + T'_1) T'_0^a T'_1^b T' \prod_{\text{internal node of } \tau} \frac{1}{\deg(v)!} F^{(\deg(v))}(T(z))$$

where $a$ (resp. $b$) is the number of edges of $\tau$ incident to two nodes of the same (resp. different) signs.
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g.f. of Trees with \( k \) marked leaves that induce the signed \( k \)-tree \( \tau \):

\[
z^k(T'_0 + T'_1) T'_0^a T'_1^b T'^k \prod_{v \text{ internal node of } \tau} \frac{1}{\deg(v)!} F^{(\deg(v))}(T(z))
\]

where \( a \) (resp. \( b \)) is the number of edges of \( \tau \) incident to two nodes of the same (resp. different) signs

Hence all signed binary trees have the same asymptotic probability. Hence convergence in distribution to a permuton.
$t_n$ uniform of size $n$ in $T = z + F(T)$ is Galton-Watson conditioned on the number of leaves. $F$ "nice" = critical GW with exponential moments (Boltzmann sampling)
Link with scaling limits of trees

t_n uniform of size \( n \) in \( T = z + F(T) \) is Galton-Watson conditioned on the number of leaves. \( F \) "nice" = critical GW with exponential moments (Boltzmann sampling)

In this case, if \( C_n \) is the contour exploration of \( t_n \), then for some constant \( c > 0 \), \( cn^{-1/2} C_n \) converges in distribution to the normalized Brownian excursion. (Kortchemski ’12, Pitman-Rizzolo ’15)
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$t_n$ uniform of size $n$ in $T = z + F(T)$ is Galton-Watson conditioned on the number of leaves. $F$ "nice" = critical GW with exponential moments (Boltzmann sampling)

So uniform extracted subtrees from $C_n$ converge to uniform extracted subtrees from the Brownian excursion, which are uniform binary trees (Aldous ’93, Le Gall ’93)
2 – Universality of permuton limits in substitution-closed classes.
Joint work with F. Bassino, M. Bouvel, V. Féray, L. Gerin and A. Pierrot
[arXiv:1706.08333]
Substitution decomposition

Generalizing $\oplus$ and $\ominus$?
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For $\sigma \in \mathcal{S}_k$, $\rho_1, \ldots, \rho_k \in \mathcal{S}$, define $\sigma[\rho_1, \ldots, \rho_k]$ by replacing the $i$-th dot in $\sigma$ by $\pi_i$.

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$\oplus$ (resp. $\ominus$) is just the substitution into $(12 \cdots r)$ (resp. $(r \cdots 21)$).

Given $\sigma$, either:
Substitution decomposition

Generalizing $\Theta$ and $\Theta$?
For $\sigma \in S_k$, $\rho_1, \ldots, \rho_k \in S$, define $\sigma[\rho_1, \ldots, \rho_k]$ by replacing
the $i$-th dot in $\sigma$ by $\pi_i$.


$\Theta$ (resp. $\Theta$) is just the substitution into
$(12 \cdots r)$ (resp. $(r \cdots 21)$).

Given $\sigma$, either:

- We can find a proper interval mapped to an interval, and
then $\sigma$ can be written as a substitution of smaller permutations
Substitution decomposition

Generalizing $\oplus$ and $\ominus$?

For $\sigma \in S_k$, $\rho_1, \ldots, \rho_k \in S$, define $\sigma[\rho_1, \ldots, \rho_k]$ by replacing the $i$-th dot in $\sigma$ by $\pi_i$.

Example: $132[21, 312, 2413] = 219784635$. $\ominus$ (resp. $\Theta$) is just the substitution into $(12 \cdots r)$ (resp. $(r \cdots 21)$).

Given $\sigma$, either:

- We can find a proper interval mapped to an interval, and then $\sigma$ can be written as a substitution of smaller permutations
- Or $\sigma$ can’t be decomposed by a nontrivial substitution: $\sigma$ is a simple permutation. Ex: $1, 12, 21, 2413, 3142, 31524, \ldots \sim \frac{n!}{e^2}$. 
Substitution decomposition

\[(8, 10, 9, 2, 11, 1, 4, 7, 3, 6, 5)\]
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\(\oplus\)

\(\ominus\)

\(42513\)

\(2413\)
Theorem (Albert, Atkinson 2005): Any permutation can be decomposed into a substitution tree with $\oplus$, $\ominus$ nodes, and simple nodes of length $\geq 4$, unique as long as adjacent $\oplus$ and $\ominus$ are merged.
Substitution-closed families

$S \subset \{ \text{simple permutations of length } \geq 4 \}.$
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\( \langle S \rangle = \{\text{permutations built by substituting } \bullet, \oplus, \ominus, \text{ and } \alpha \in S\} \).
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Examples:

$\langle \emptyset \rangle = \{\text{separables}\} = \text{Av}(3142, 2413)$.

$\langle 3142 \rangle = \text{Av}(2413, 41352, 415263, 531642)$. 
Substitution-closed families

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Examples: \(<\emptyset> = \{ \text{separables} \} = \text{Av}(3142, 2413)\).
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Let \( \sigma_n \) be a uniform permutation of size \( n \) in \(<S>\).

\( S(z) = \sum_{\alpha \in S} z^{|\alpha|} \) generating function of the simples, radius \( R \). Set
\( a = S'(R) - 2/(1 + R)^2 + 1 \) and \( b = S''(R) \)
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**Theorem** (Bassino, Bouvel, Féray, Gerin, M., Pierrot 2017)

The limit in distribution of $\sigma_n$ is

- a **biased** Brownian separable permuton if $a > 0$ or $a = 0, b < \infty$,
- the same limit $\nu$ as an uniform simple permutation in $S$ if $a < 0$,
- a **stable permuton** if $a = 0, b = \infty$.

When $a \leq 0$ additional hypotheses are needed.
Biased Brownian separable permuton

Regime where the decomposition tree converges to a Brownian CRT.

Picture by I. Kortchemski
Biased Brownian separable permuton

Regime where the decomposition tree converges to a Brownian CRT.

The signs in a uniform subtree are biased: $\mathbb{P}(\oplus) = p$, and $p$ depends explicitly on $S$. Here $p = 0.2$.

Picture by I. Kortchemski
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**Biased Brownian separable permuton**

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The regime $\alpha > 0$ covers most known substitution-closed classes: $S$ finite or subexponential, $S$ rational,...
Degenerate case $a<0$

Regime where the decomposition tree exhibits a condensation phenomenon. Roughly, $\sigma_n$ looks like a large uniform simple permutation in $S$ and converges to the same limit $\nu$.

Example: $\text{Av}(2413)$. We still need to understand the permuton limit of large simples in this class (+ technical hypotheses) to apply our theorem.
Stable permutons

Regime where the decomposition tree converges to a $\alpha$-stable tree, $\alpha$ explicit.
Stable permutons

Regime where the decomposition tree converges to a $\alpha$-stable tree, $\alpha$ explicit.

$\alpha = 1.5$  

$\alpha = 1.1$

Pictures by I. Kortchemski
Stable permutons

Regime where the decomposition tree converges to a $\alpha$-stable tree, $\alpha$ explicit.

$\alpha = 1.5$

$\alpha = 1.1$

Branches from each infinite-degree point are reordered according to an independent copy of $\nu$ (the limit of large simples in the class)
3 – Construction of the Brownian Permuton

[arXiv:1711.08986]
The (signed) Brownian excursion and CRT
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\[ d_e(x, y) = e(x) + e(y) - 2 \min_{[x,y]} e \]
The (signed) Brownian excursion and CRT

\[ d_e(x, y) = e(x) + e(y) - 2 \min_{[x,y]} e \]

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The (signed) Brownian excursion and CRT

\[ d_e(x, y) = e(x) + e(y) - 2 \min_{[x,y]} e \]

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Constructing the Brownian permuton

e(x)
Constructing the Brownian permuton
Define a shuffled pseudo-order on $[0,1]$: $x \triangleleft^S \epsilon y$ if and only if $
abla$ Brownian excursion, $S$ i.i.d. signs indexed by the local minima of $e$.
Theorem (M. 2017)
Almost surely there exists a Lebesgue-preserving, \( \triangleleft^S_{e} \)-increasing function (unique up to a.e. equality). The random measure \((id, \varphi) \ast \text{Leb}\) has the law of the Brownian separable permuton.

Define a shuffled pseudo-order on \([0,1]\): \( x \triangleleft^S_{e} y \) if and only if

\[
\begin{align*}
\text{or } & \quad \begin{cases} 
\oplus & \text{if } e(x) < 0, \\
\ominus & \text{if } e(x) > 0.
\end{cases}
\end{align*}
\]
Theorem (M. 2017)
Almost surely there exists a Lebesgue-preserving, \(\triangleleft^S_e\)-increasing function (unique up to a.e. equality). The random measure \((\text{id}, \phi) \ast \text{Leb}\) has the law of the Brownian separable permuton.

\(\phi\) is continuous at every leaf (point which is not a one-sided local minimum) of \(e\) (full Lebesgue measure).

\(\rightsquigarrow\) The support of \(\mu\) is of Hausdorff dimension 1.
Theorem (M. 2017)
Amost surely there exists a Lebesgue-preserving, $\triangleleft_S^e$-increasing function (unique up to a.e. equality). The random measure $(\text{id}, \varphi)_\ast \text{Leb}$ has the law of the Brownian separable permuton.

Discontinuities at every strict local minima of $e$ (dense) $\leadsto$ The support of $\mu$ is totally disconnected.
There exists a Brownian excursion $f$ defined on the same probability space such that $f \circ \varphi = e$. a.s., $\mathcal{I}_f$ is isometric to $\mathcal{I}_e$. 

\[
\begin{align*}
\varphi(x) \\
\downarrow \\
f(\varphi(x)) \\
\downarrow \\
e(x) \\
\downarrow \\
x
\end{align*}
\]
Self-similarity

The Brownian permuton can be obtained by cut-and-pasting three independent copies in distribution of itself. The first copy $\mu_0$ is cut according to a sample $(X_0, Y_0) \sim \mu_0$. The scaling is an independent Dirichlet$(1/2, 1/2, 1/2)$ vector. The relative position of $\mu_1$ and $\mu_2$ is chosen independently and uniformly between $\oplus$ and $\ominus$. 
Thank you!