Scaling limits of pattern-avoiding permutations

Mickaël Maazoun — UMPA, ENS de Lyon

Joint work with F. Bassino, M. Bouvel, V. Féray, L. Gerin and A. Pierrot

Journées ALEA 2018

May 15th, 2018

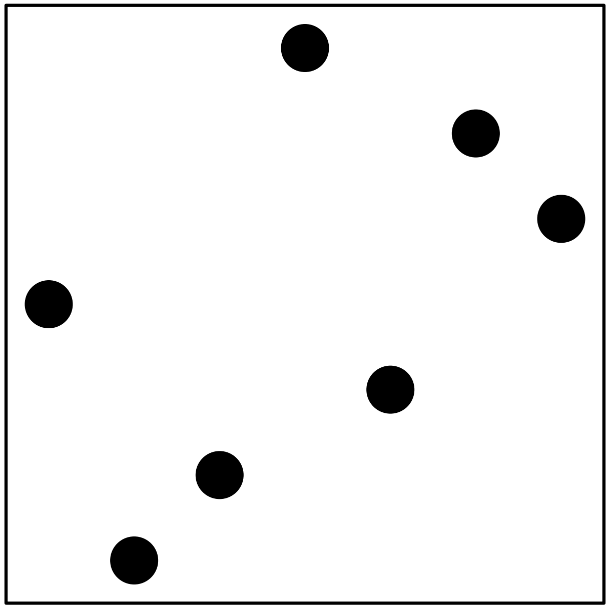
1 – The scaling limit of separable permutations

After Bassino, Bouvel, Féray, Gerin, Pierrot 2016

Permutations

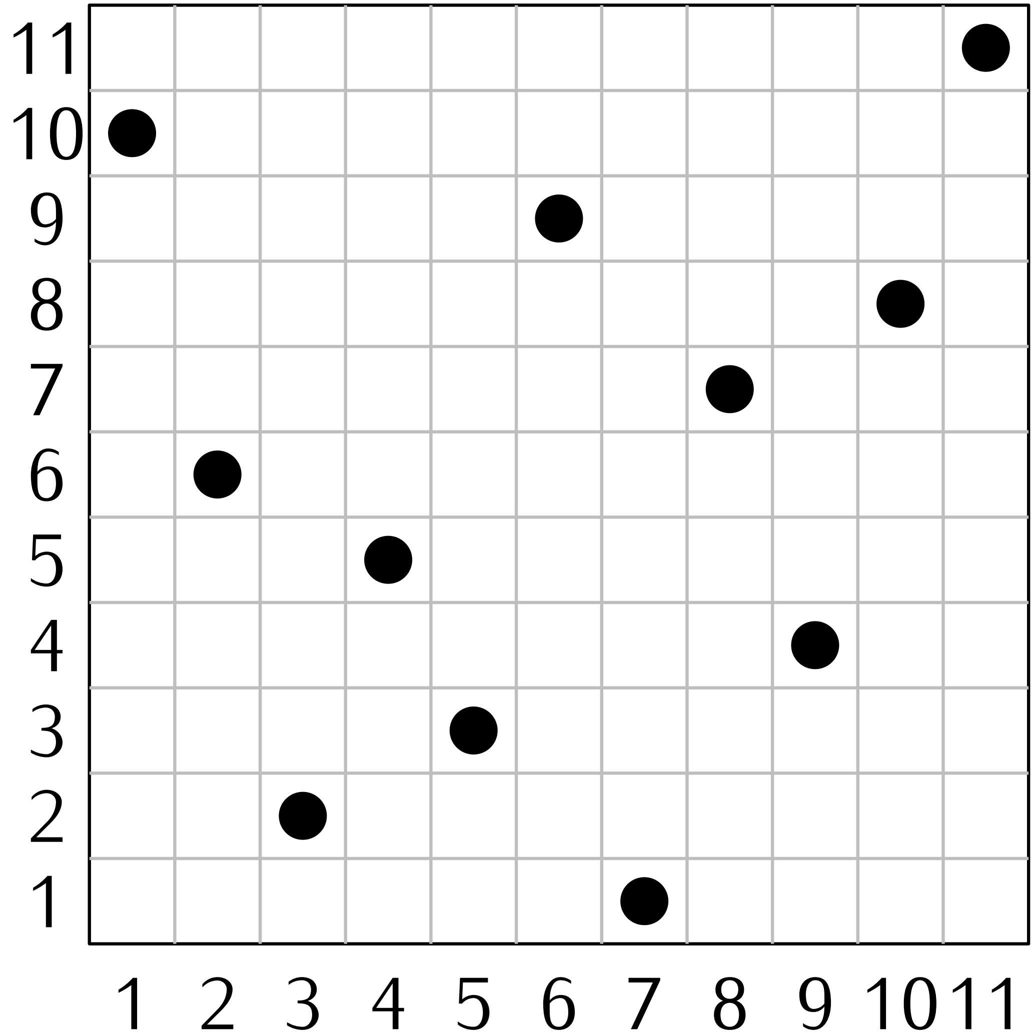
A permutation $\sigma \in \mathfrak{S}_n$ is a word $(\sigma(1), \ldots, \sigma(n))$ which contains every element of $\{1, \ldots, n\}$.

Diagram of $(4128376) \in \mathfrak{S}_8$:



Permutation patterns

$\sigma = (10, 6, 2, 5, 3, 9, 1, 7, 4, 8, 11) \in \mathfrak{S}_{11}$



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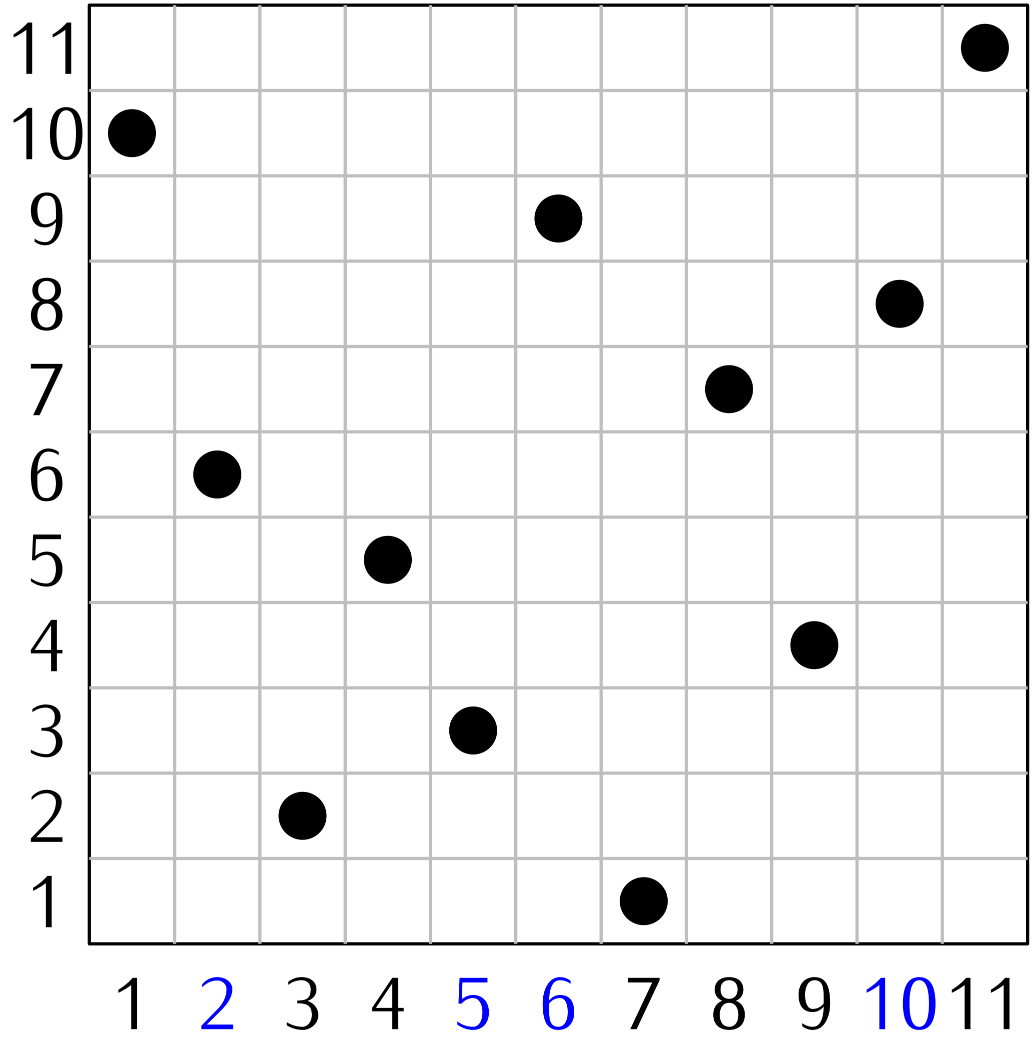

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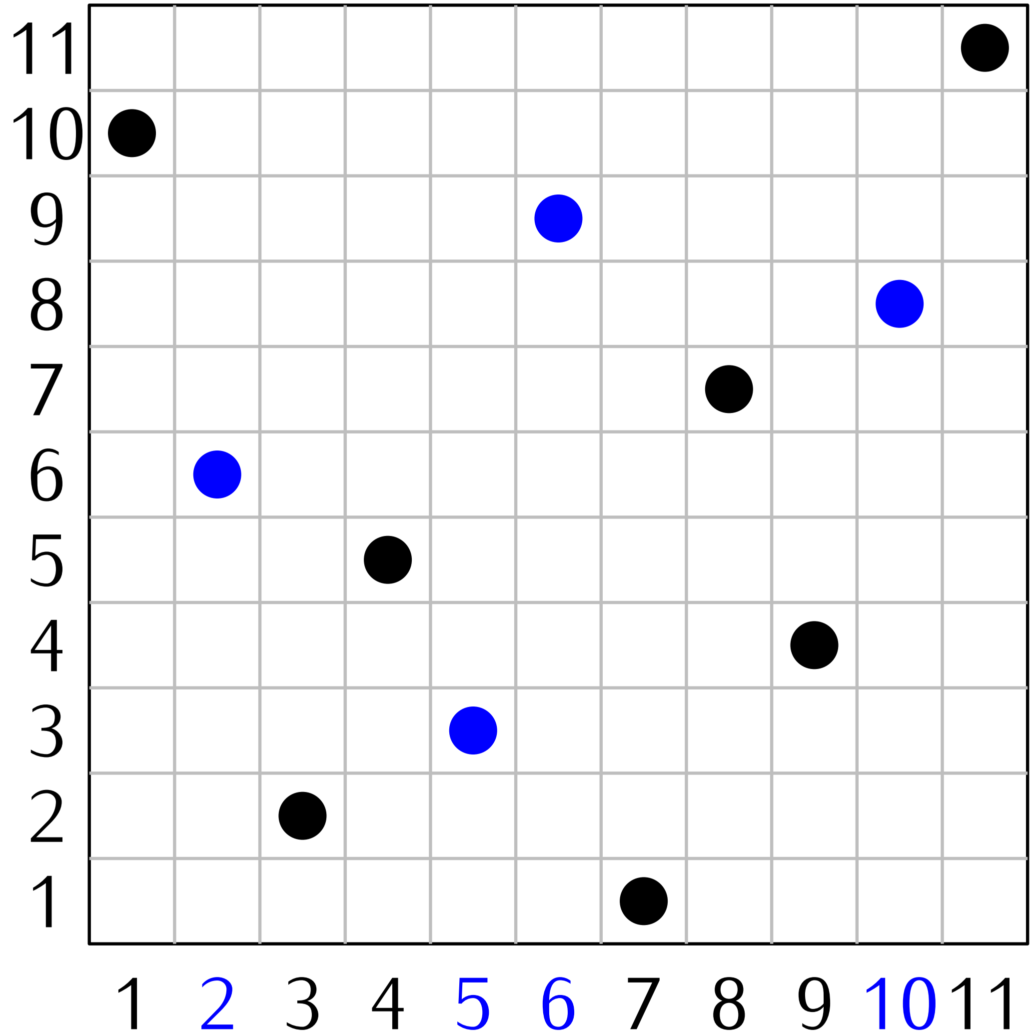

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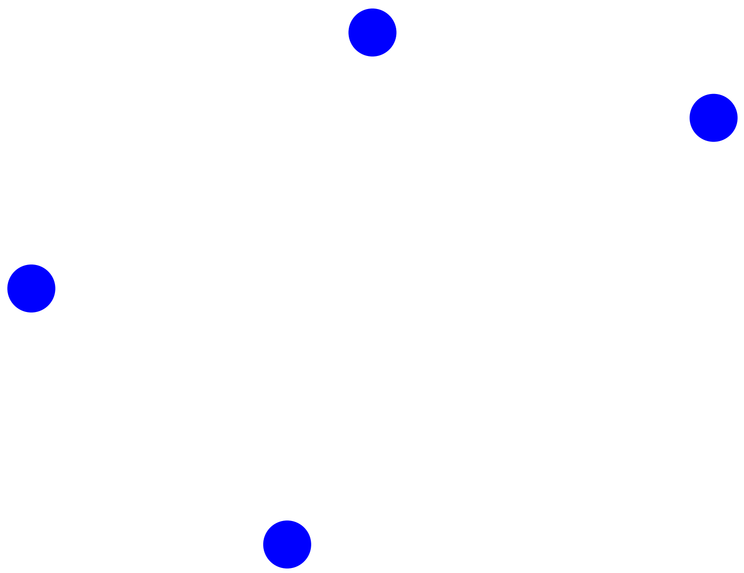

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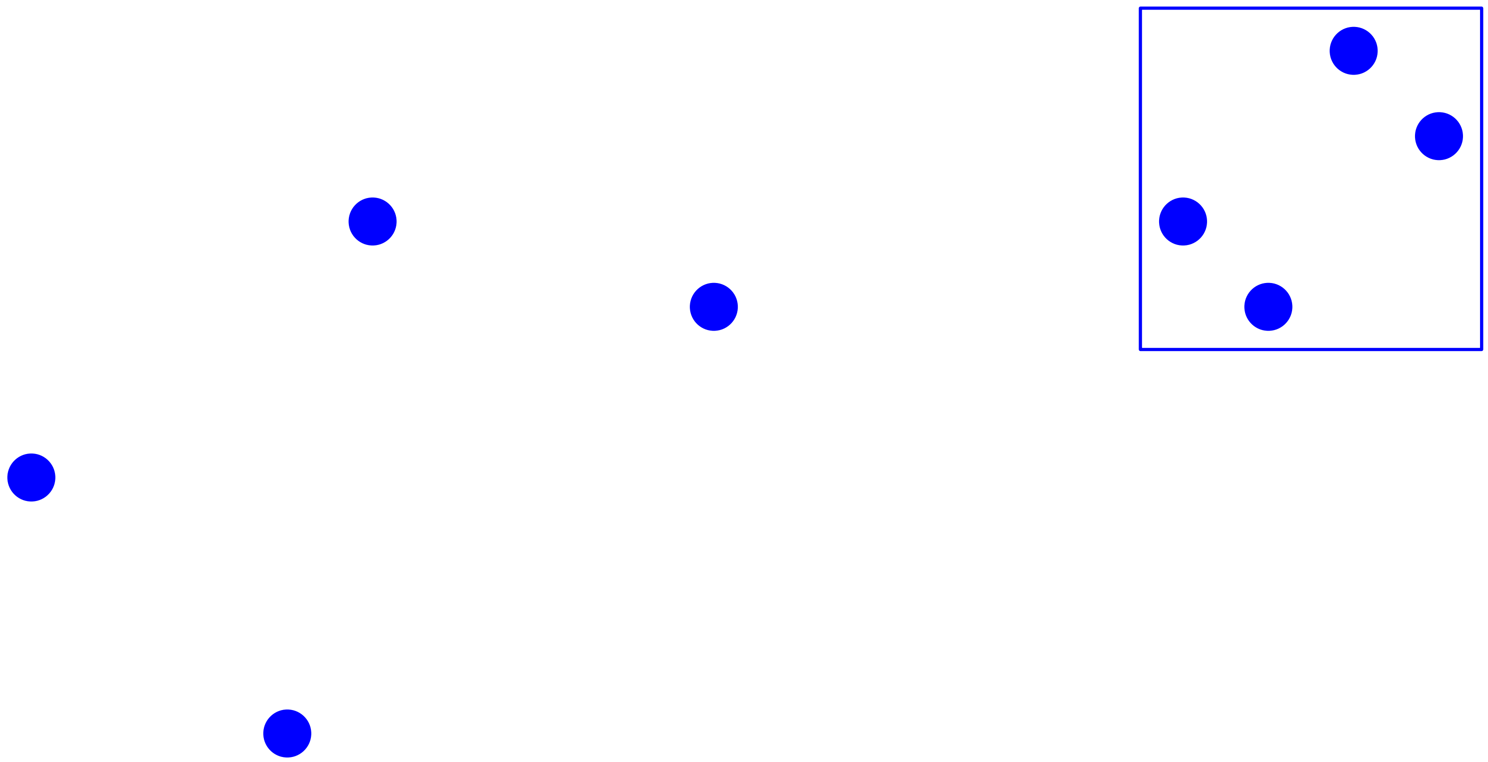

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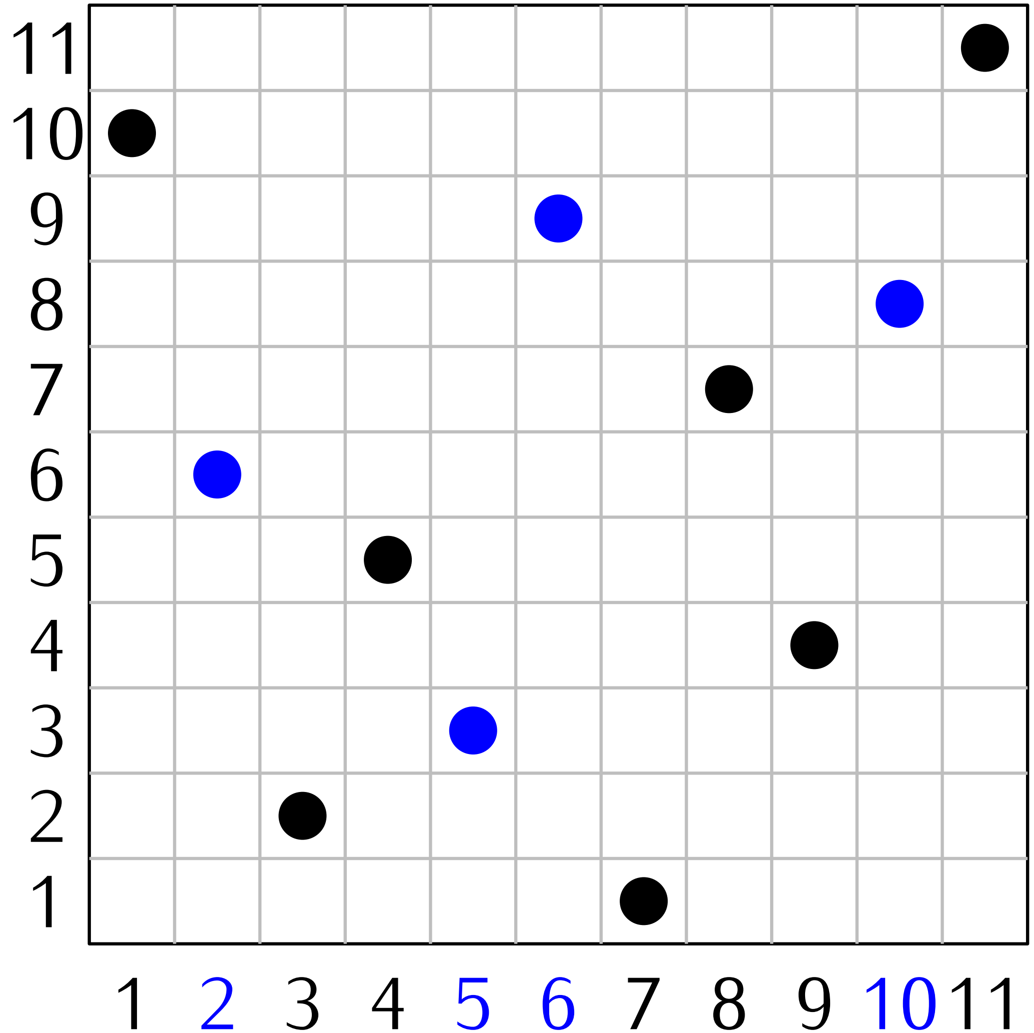

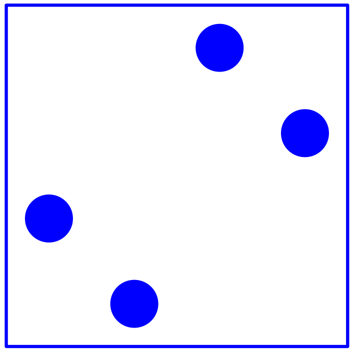

$\mathrm{pat}_{\{2,5,6,10\}}(\sigma) = (2143)$

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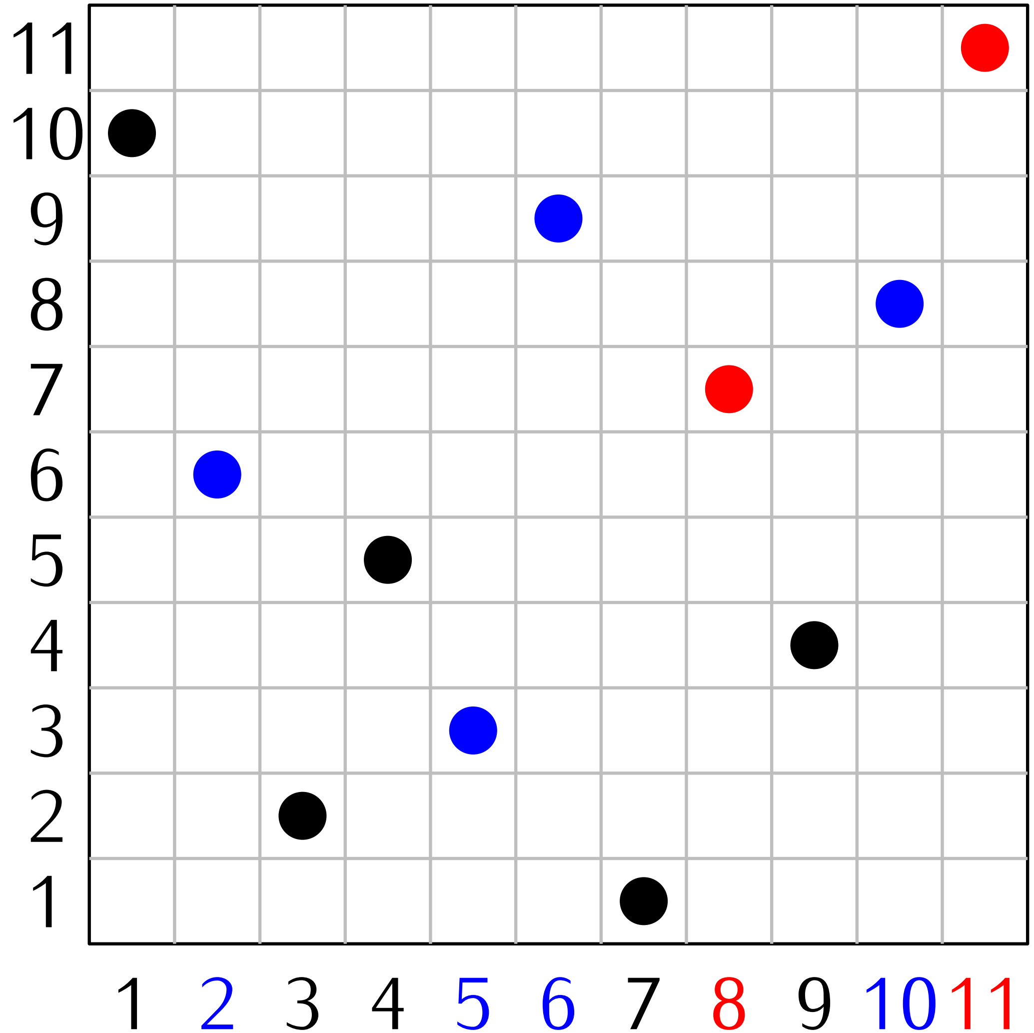

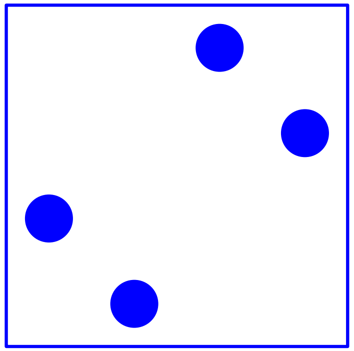

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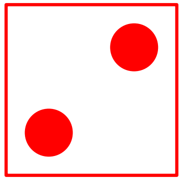

$\mathrm{pat}_{\{8,11\}}(\sigma) = (12)$

Classes of permutation and pattern-avoidance

Permutation class: set of permutations closed under pattern extraction. Can always be written as $\mathrm{Av}(B)$, the set of permutations that avoid patterns in some *basis* B.

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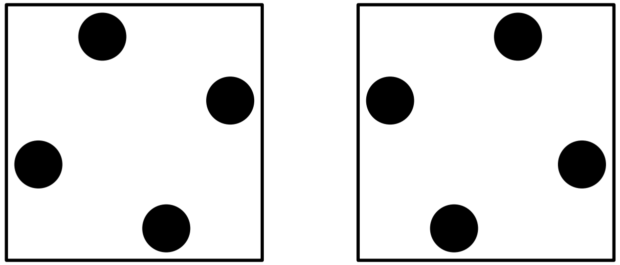
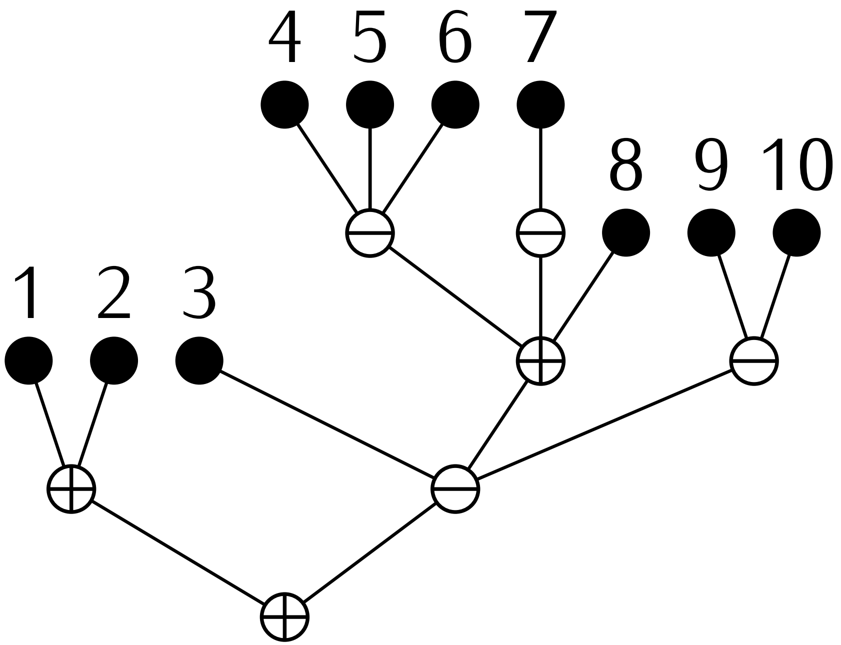
Separable permutations: $\mathrm{Av}(3142, 2413)$



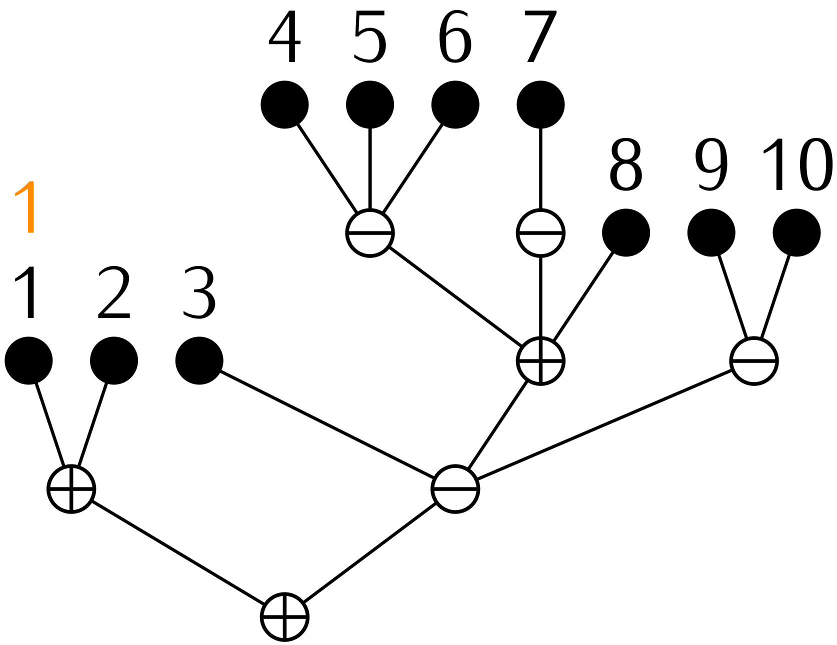
(Avis-Newborn '80, Bose-Buss-Lubiw '93)

Separable permutations



Signed tree τ

Separable permutations



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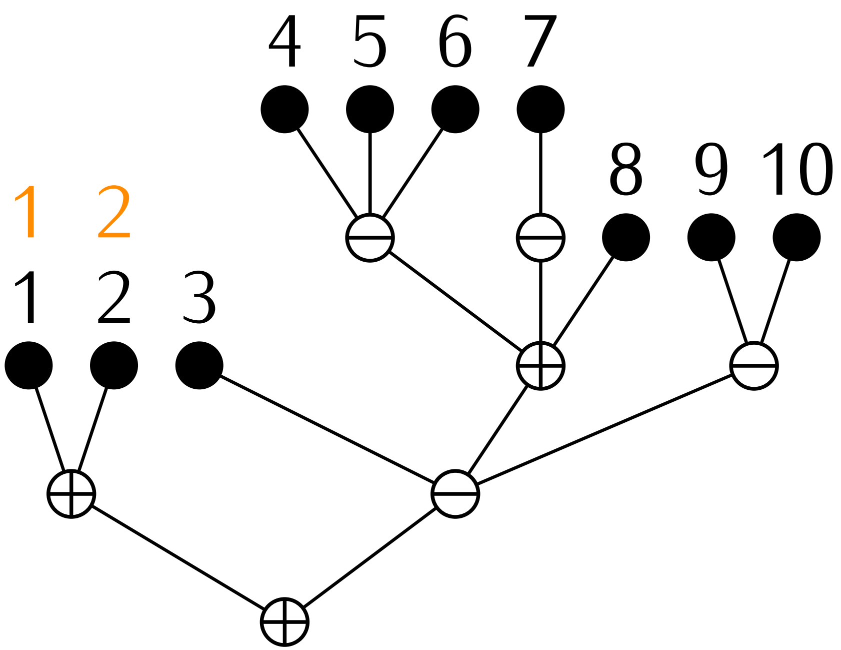

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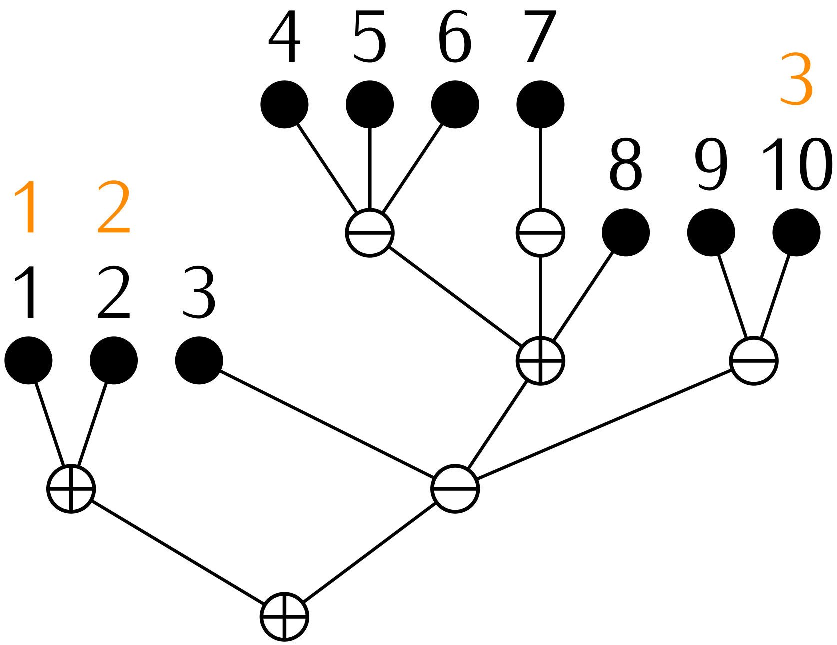

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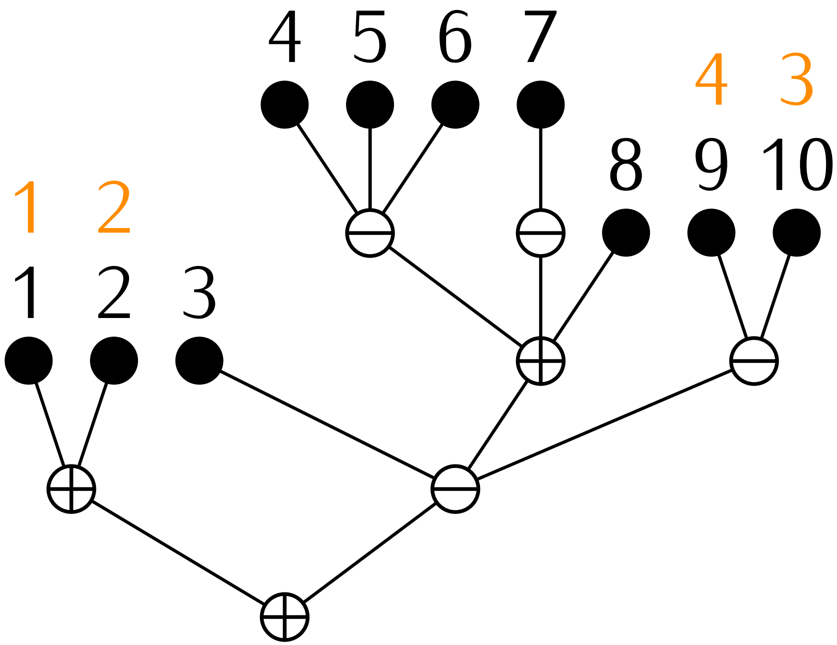

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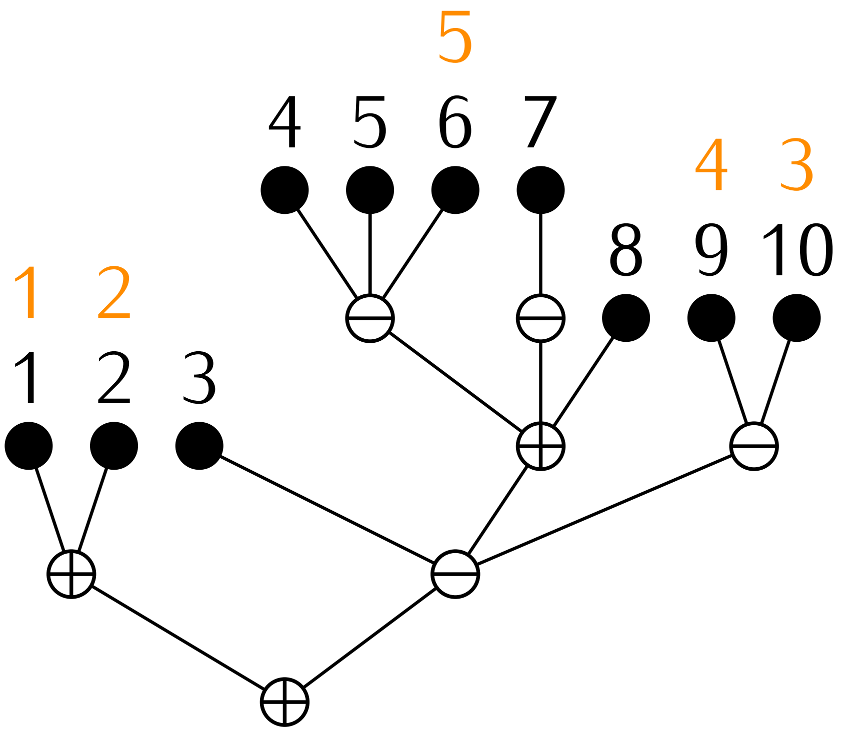

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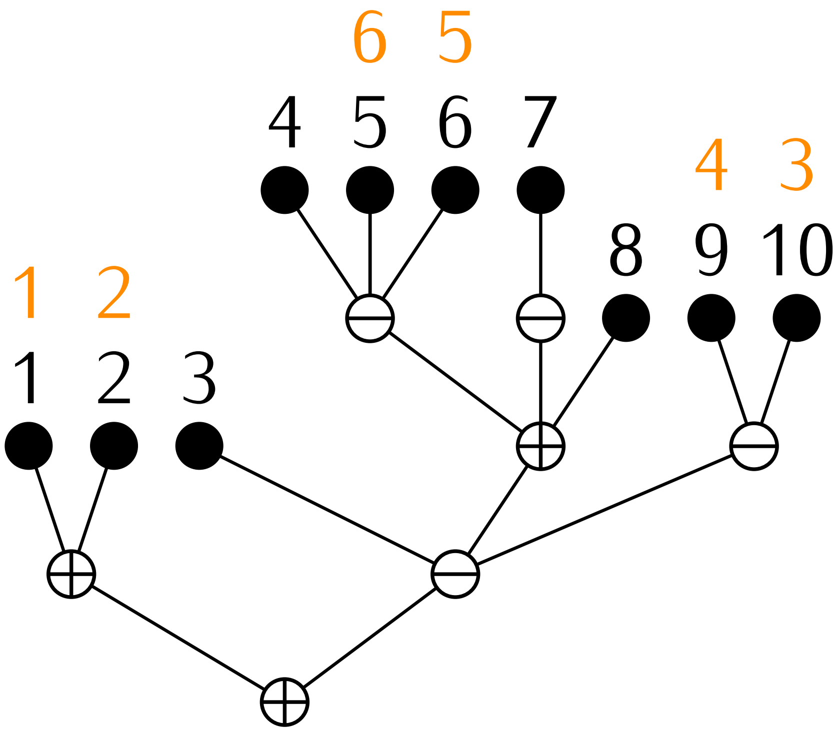

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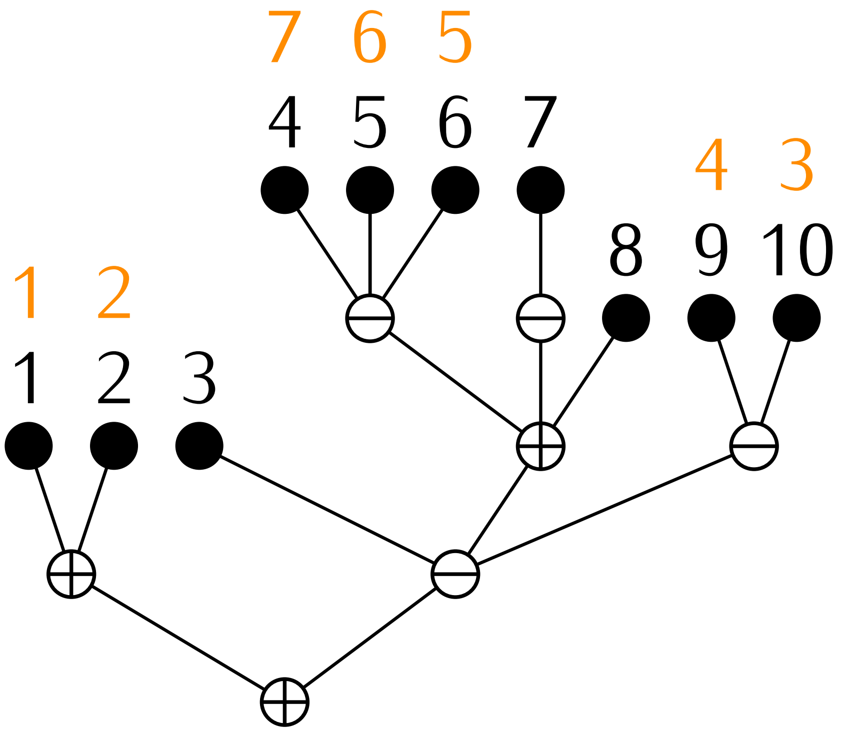

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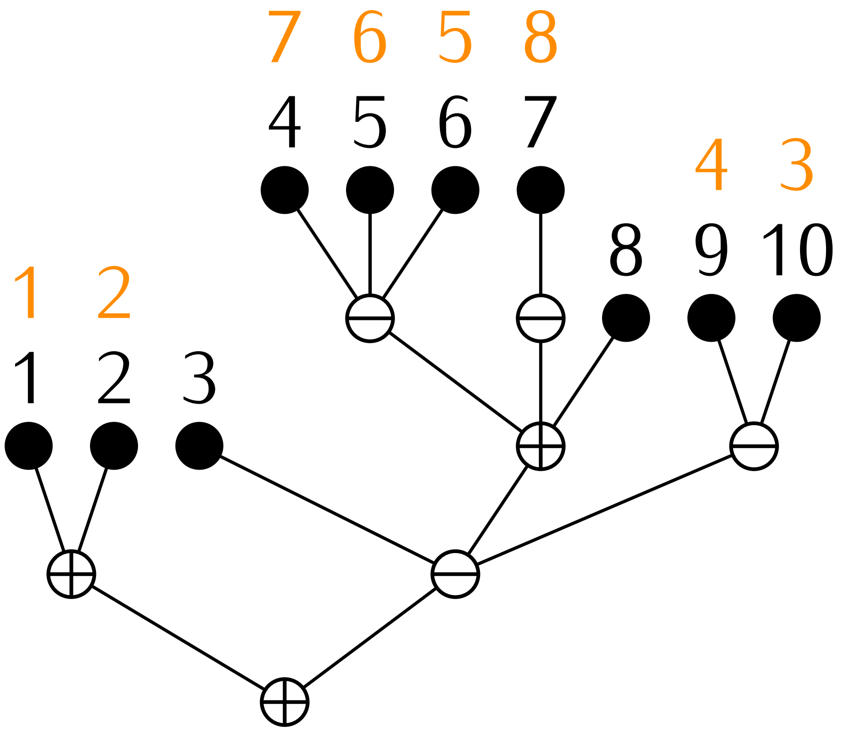

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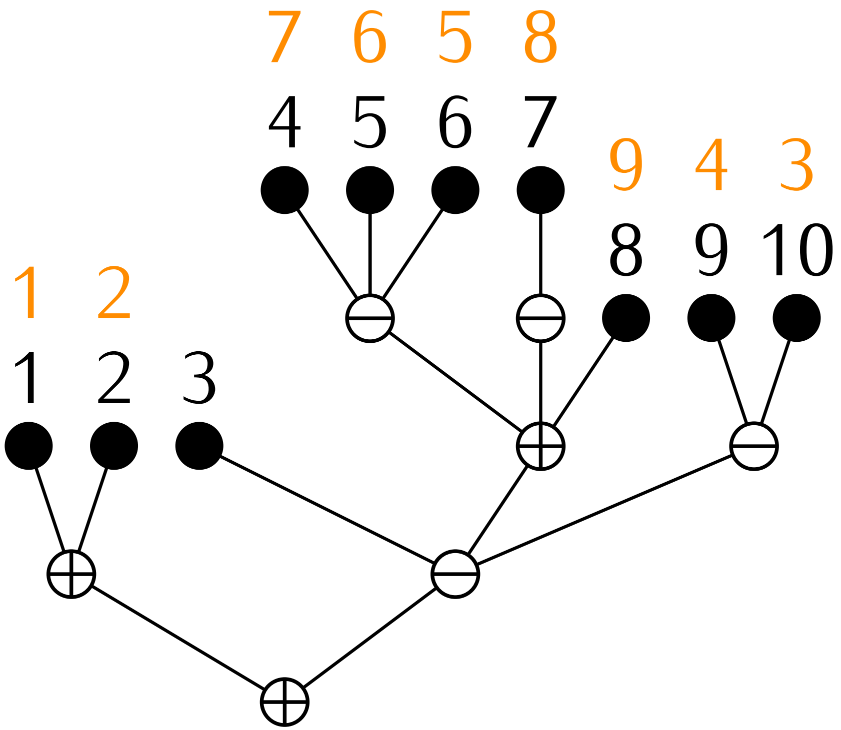

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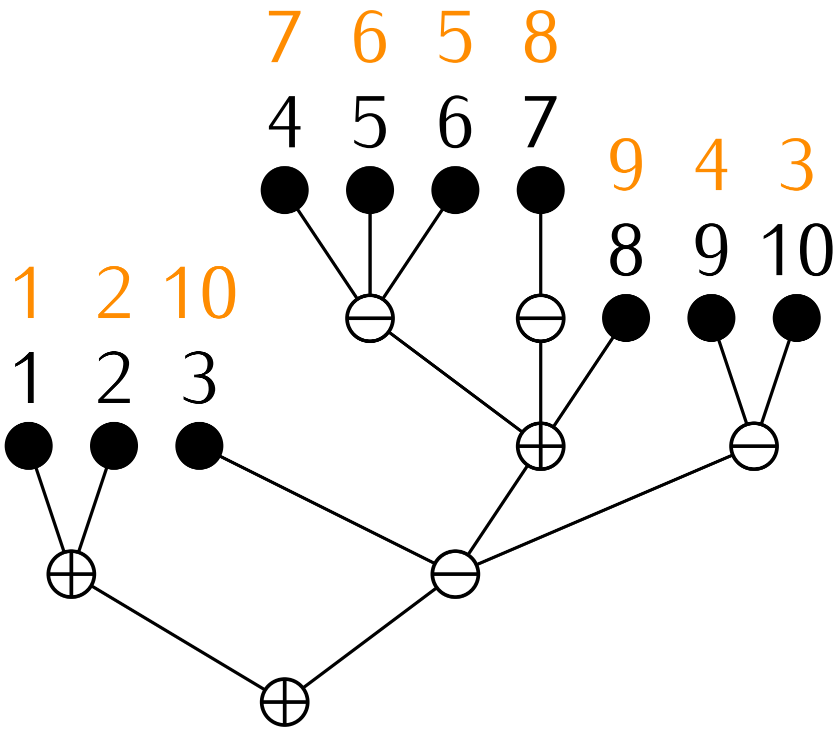

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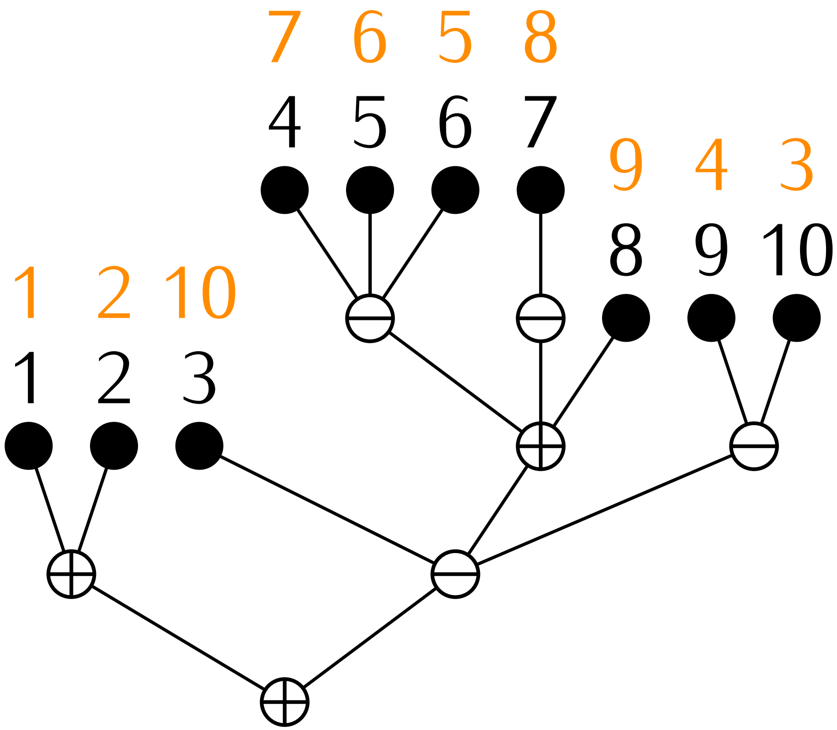

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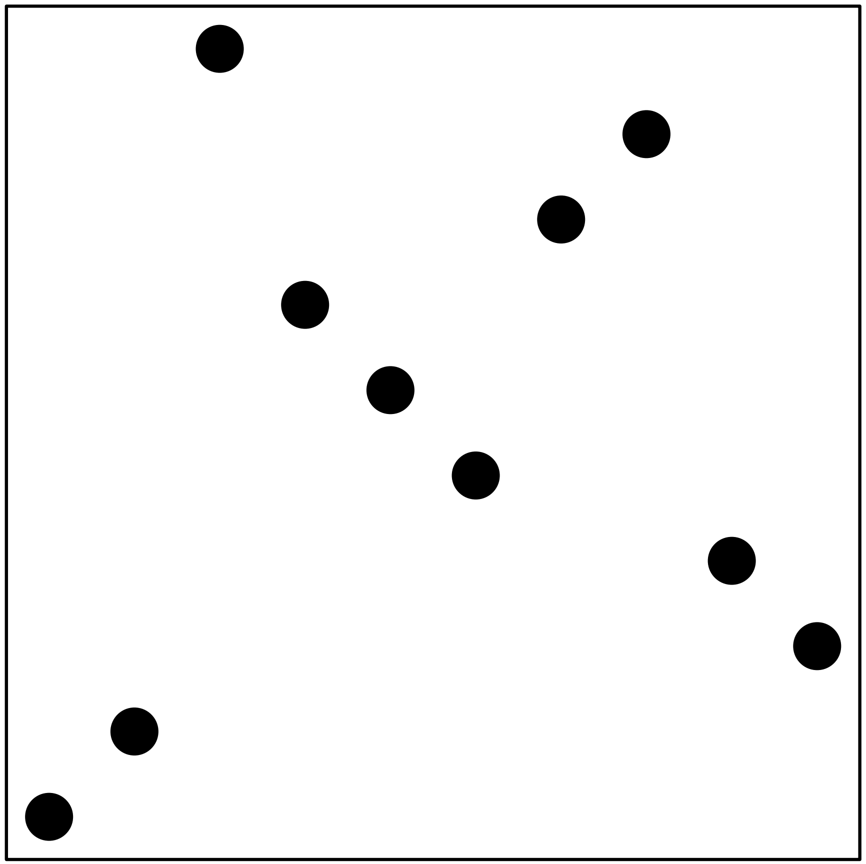

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$\mathrm{perm}(\tau) = (1\;2\;10\;7\;6\;5\;8\;9\;4\;3)$

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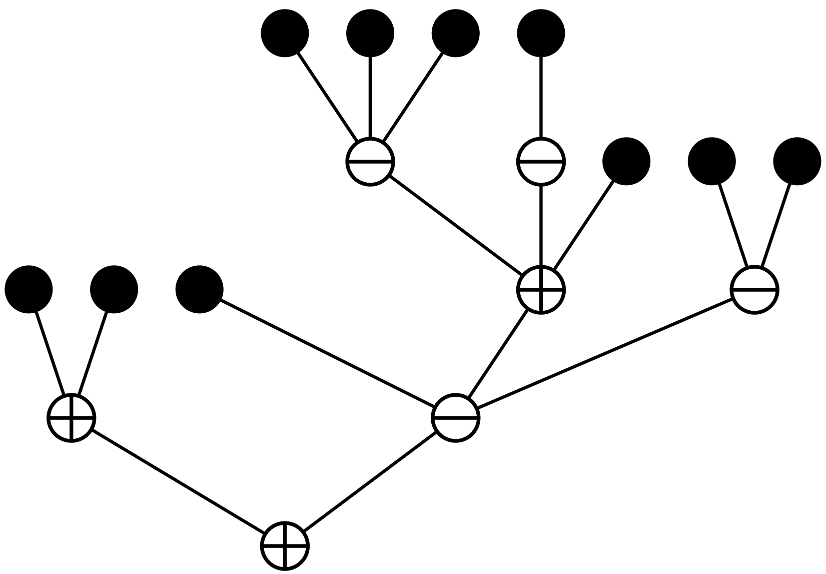

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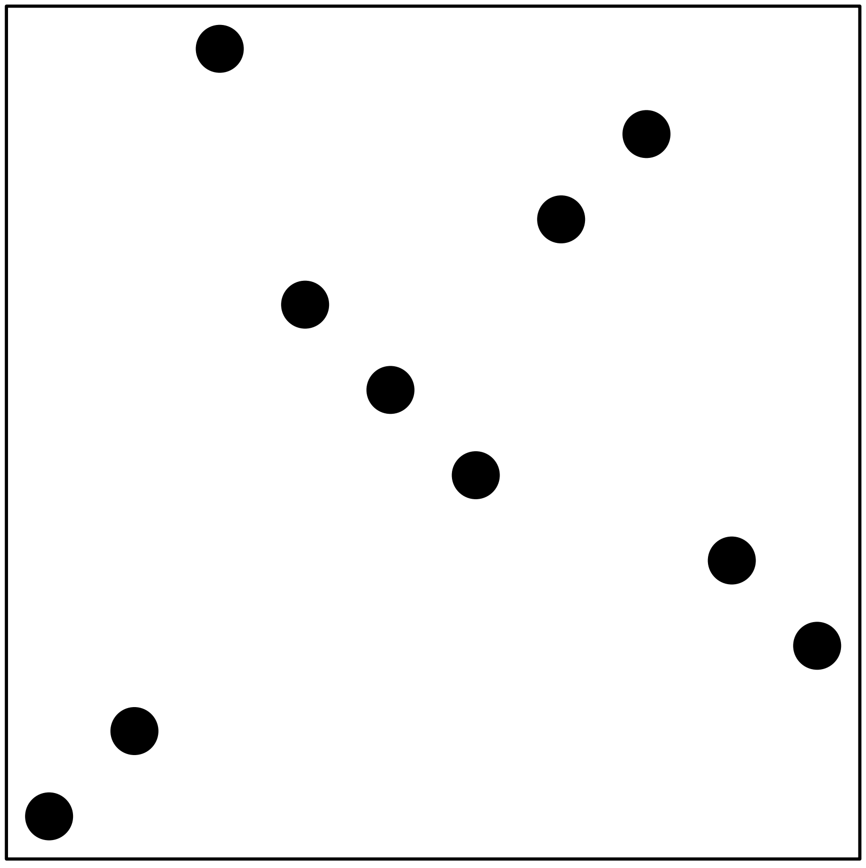

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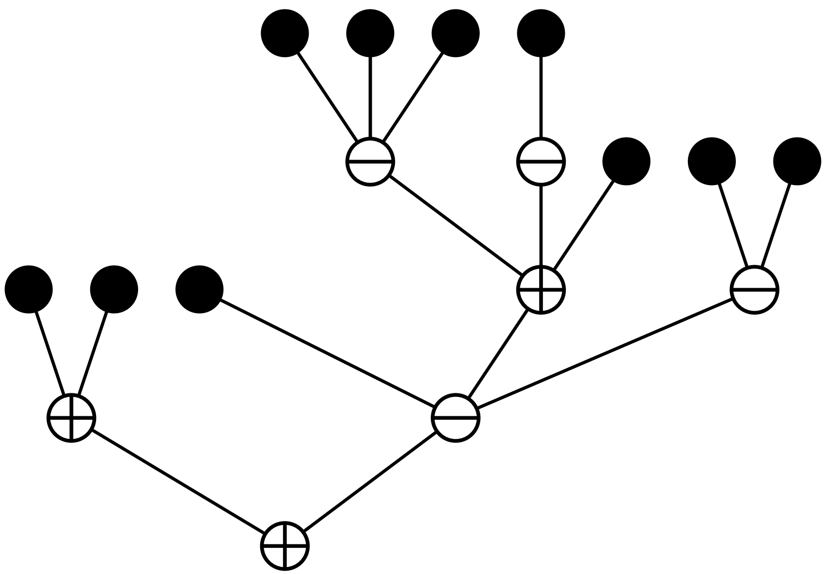

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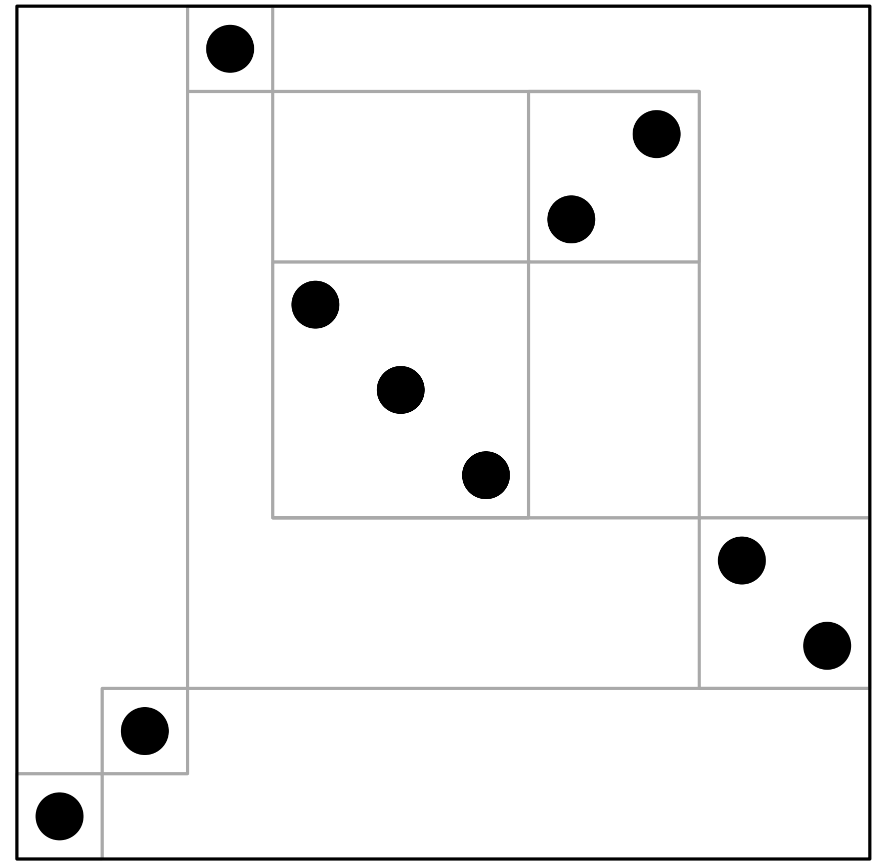

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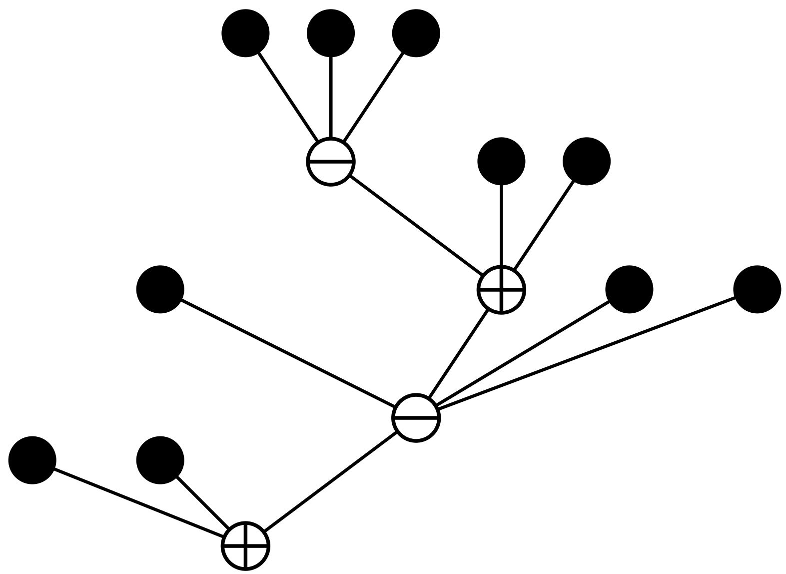
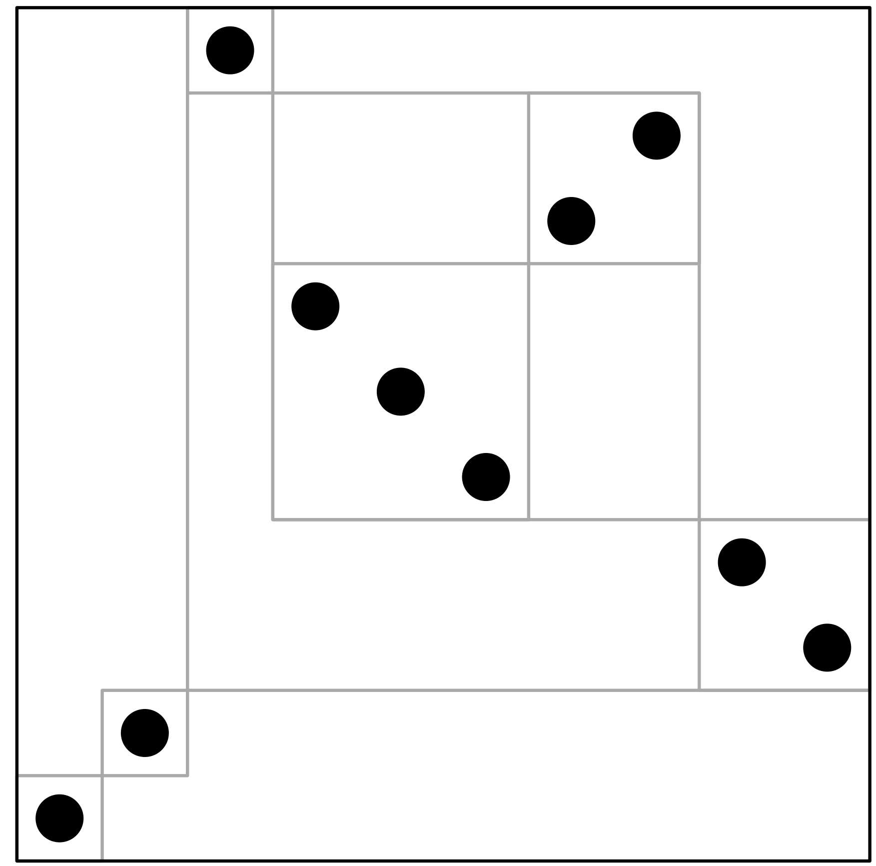


Alternating-signs Schröder tree



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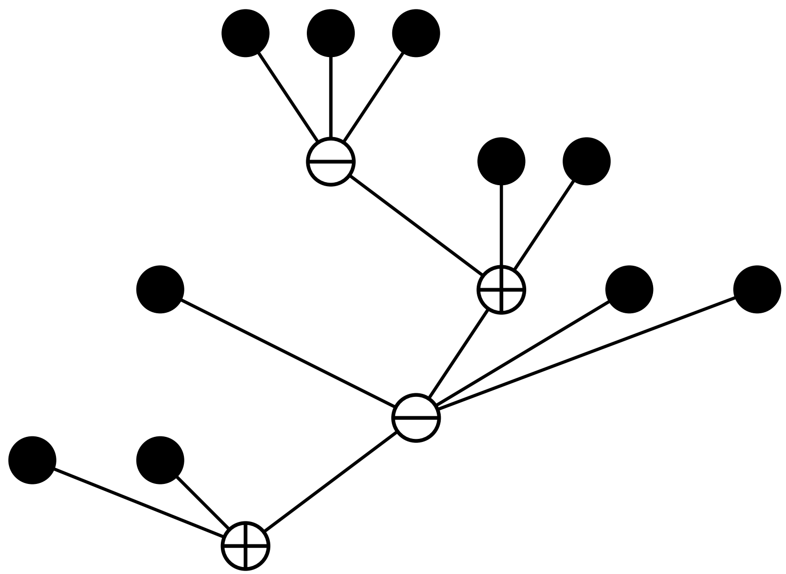

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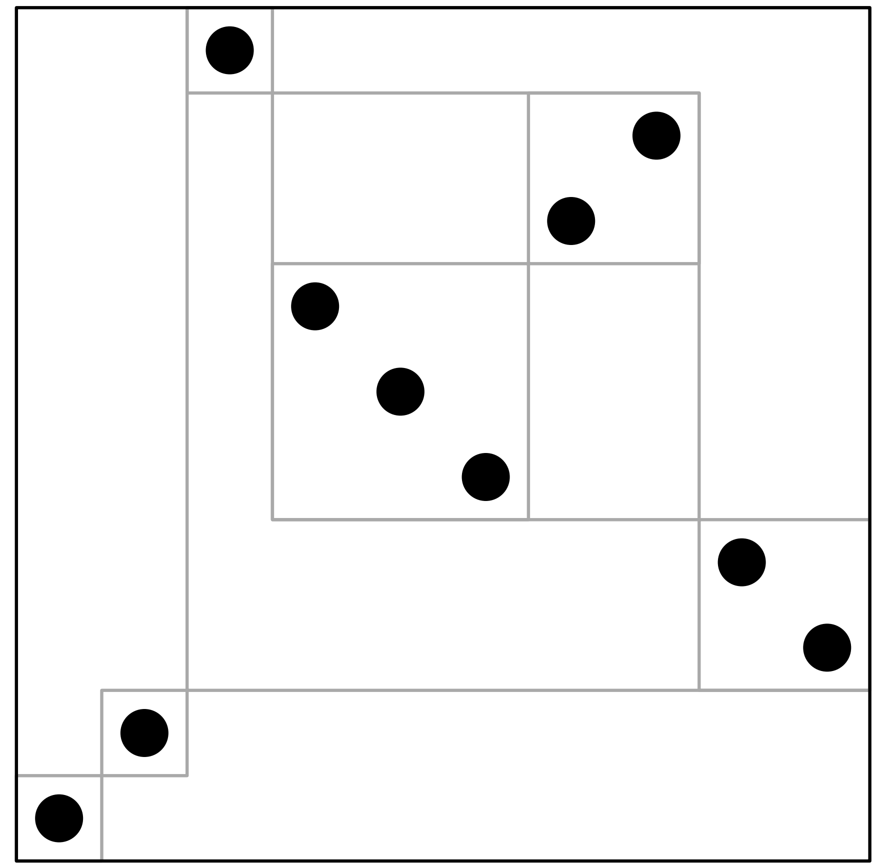

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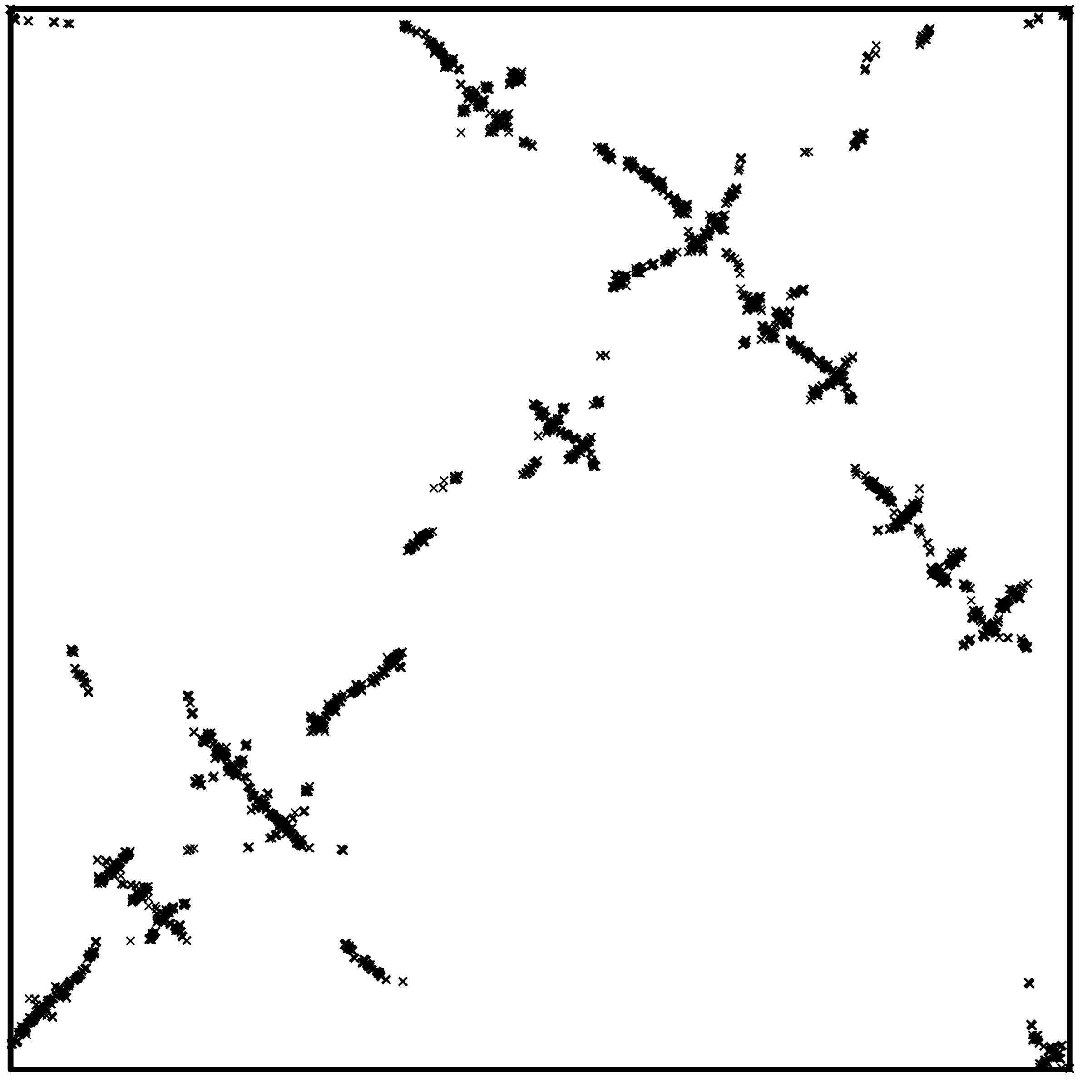
Counted by large Schröder numbers
$1, 2, 6, 22, 90, 394, 1806, 8558, \ldots \asymp (3+\sqrt{8})^n n^{-3/2}$

A large uniform separable permutation



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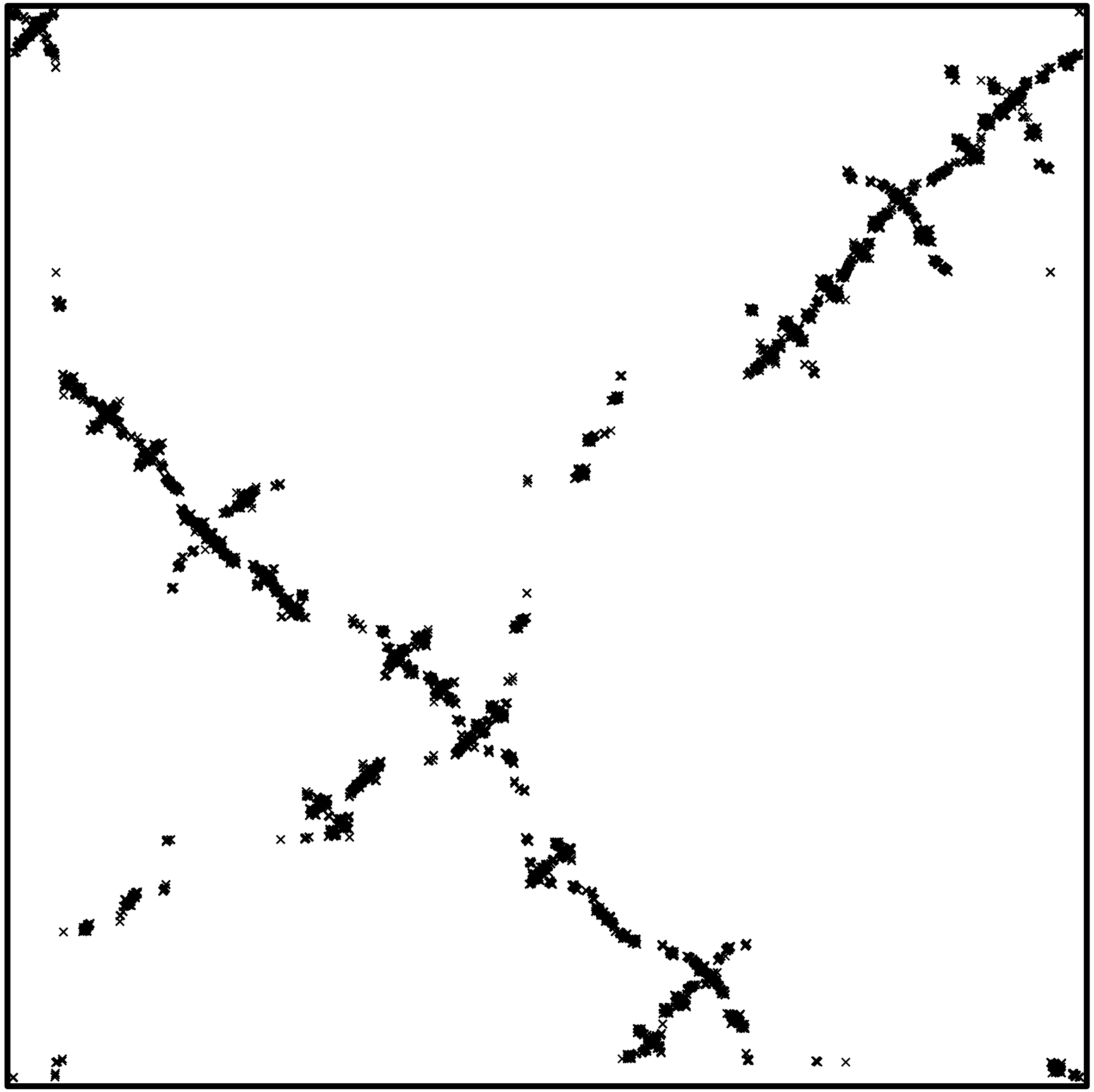
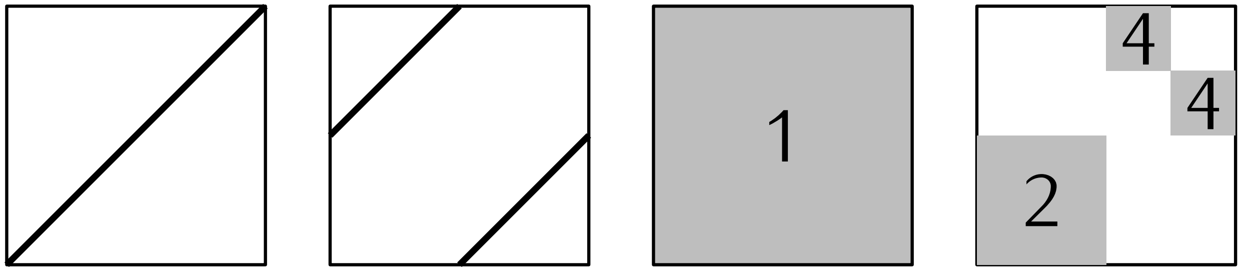


Permutons

A permuton is a probability measure on $[0, 1]^2$ with both marginals uniform.



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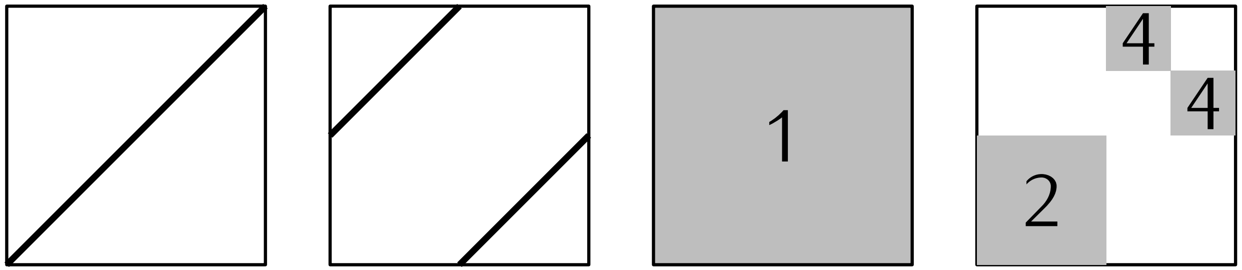

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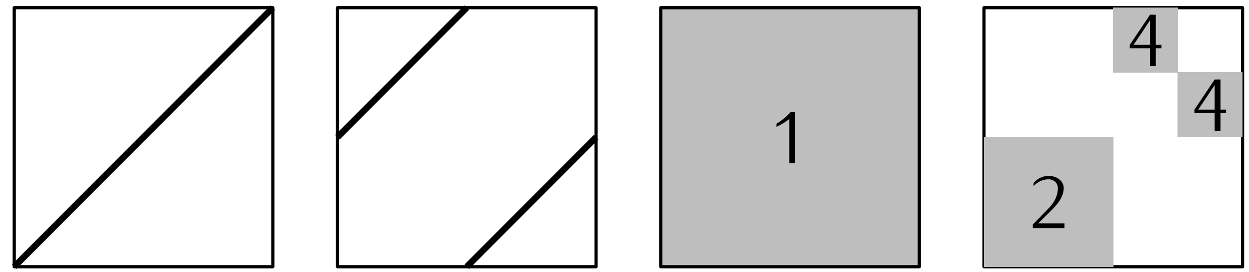

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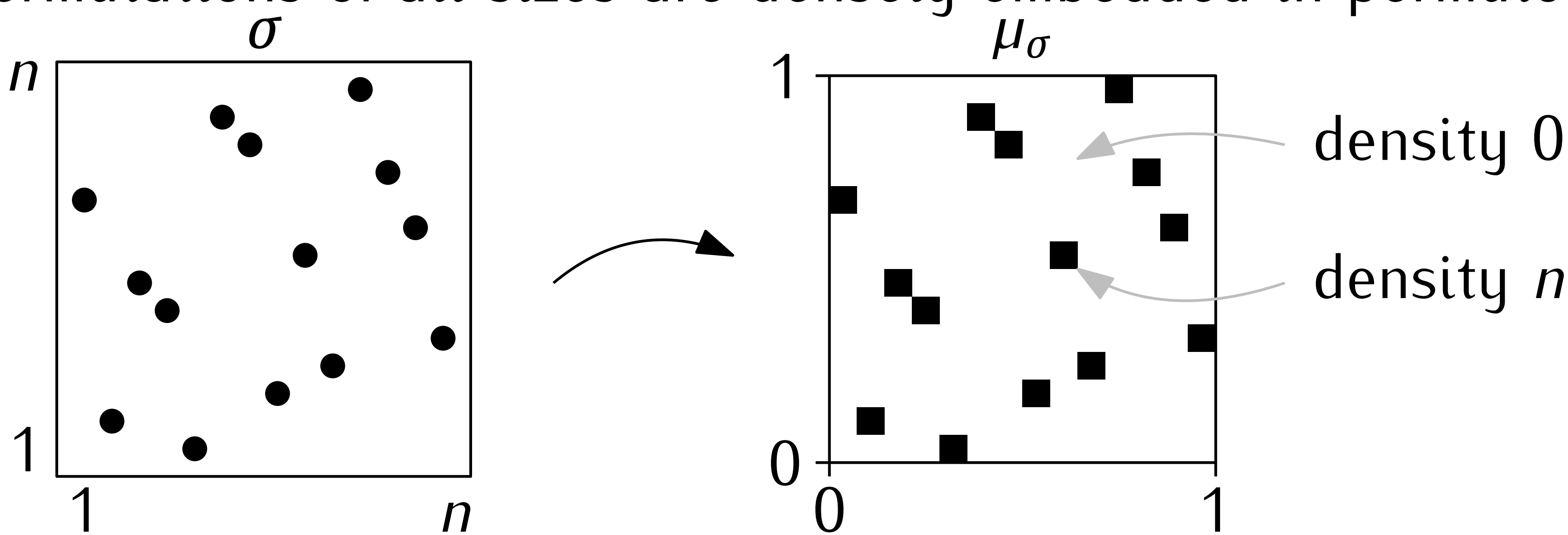
Permutations of all sizes are densely embedded in permutons.



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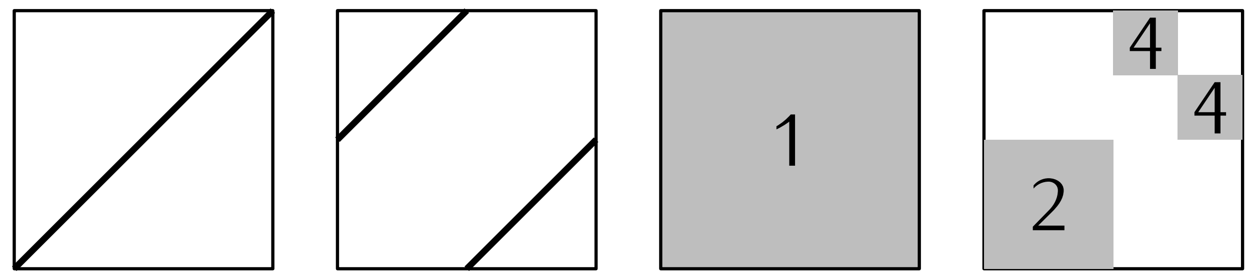

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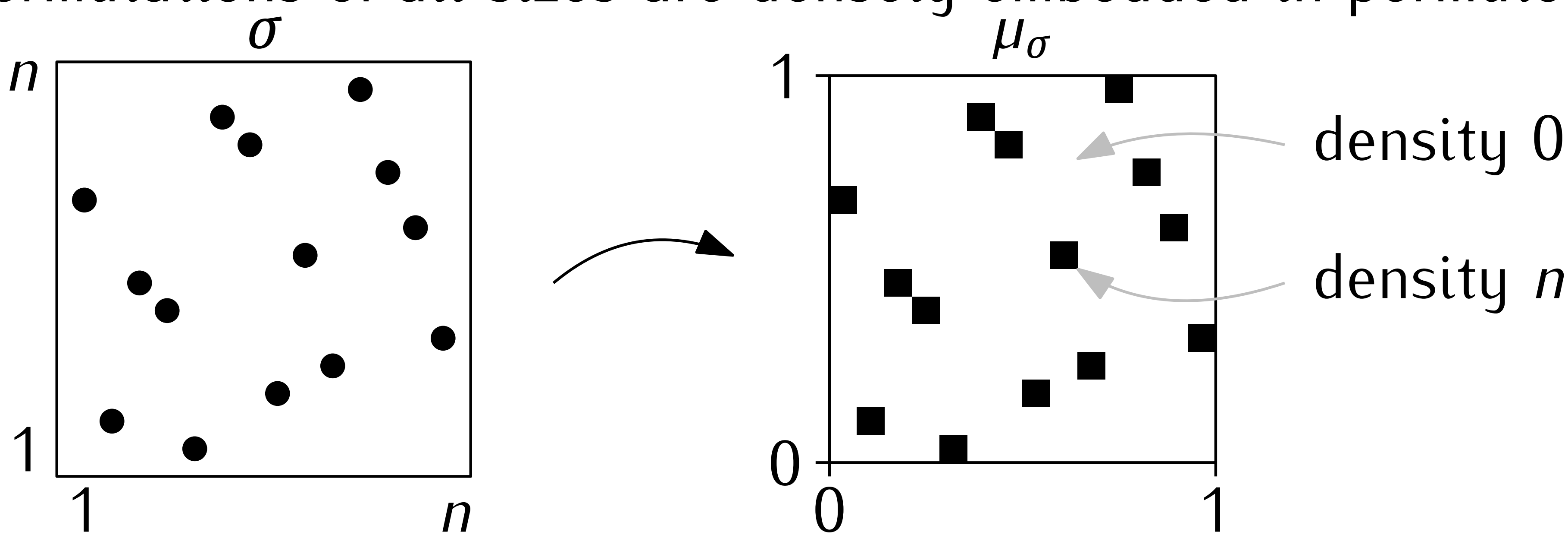

We say that a sequence (σ_n) converges to μ when $\mu_{\sigma_n} \xrightarrow[n\to\infty]{w} \mu$.

Density of patterns

For $\sigma \in \mathfrak{S}_n, \pi \in \mathfrak{S}_k, \widetilde{\mathrm{occ}}(\pi, \sigma)$ is the proportion of π among the $\binom{n}{k}$ subpermutations of σ of size k.

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Similarly, if μ is a permuton, the density of π in μ is the probability that k independent points drawn from μ are ordered like π.

$$\widetilde{\mathrm{occ}}(\pi, \mu) = \int_{[0,1]^{2k}} \mu(dx_1 dy_1) \cdots \mu(dx_k dy_k) \mathbf{1}[(\vec{x}, \vec{y}) \sim \pi].$$

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Theorem (Hoppen *et. al.*, 2013)
The sequence (σ_n) converges to μ iff for every $\pi \in \mathfrak{S}$, $\widetilde{\mathrm{occ}}(\pi, \sigma_n) \to \widetilde{\mathrm{occ}}(\pi, \mu)$.

Sequences of random permutations

If σ_n is a sequence of random permutations, we can consider the convergence in distribution of the random permutons μ_{σ_n}.
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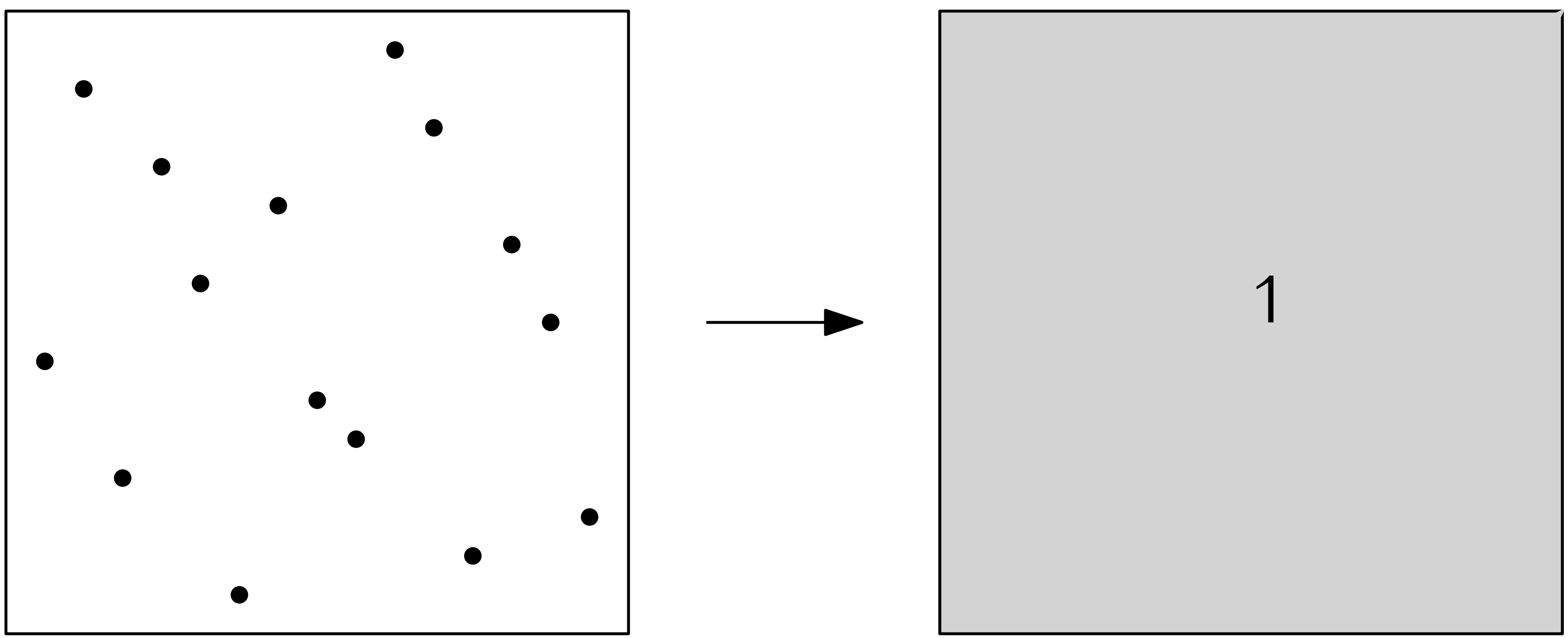
$\mathcal{C} = \mathfrak{S}$: $\sigma_n \xrightarrow{\mathbb{P}} \mathrm{Leb}_{[0,1]^2}$.



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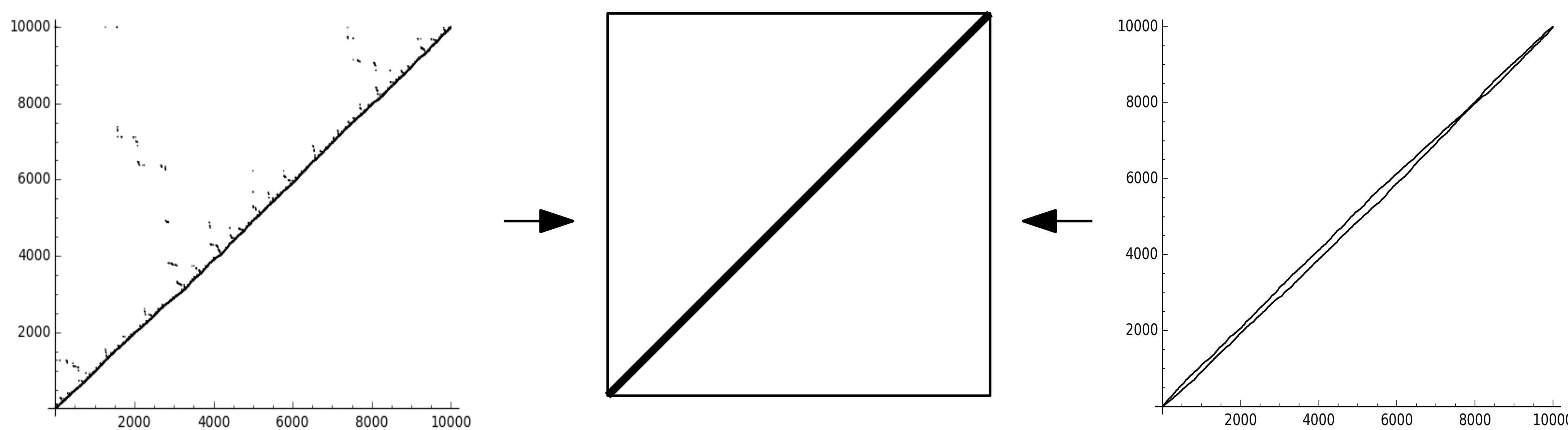
$$\mathcal{C} = \mathrm{Av}(231) \text{ or } \mathrm{Av}(321) : \ \sigma_n \xrightarrow{\mathbb{P}} (\mathrm{id}, \mathrm{id})_* \mathrm{Leb}_{[0,1]}.$$



Pictures by C. Hoffman, D. Rizzolo, E. Slivken

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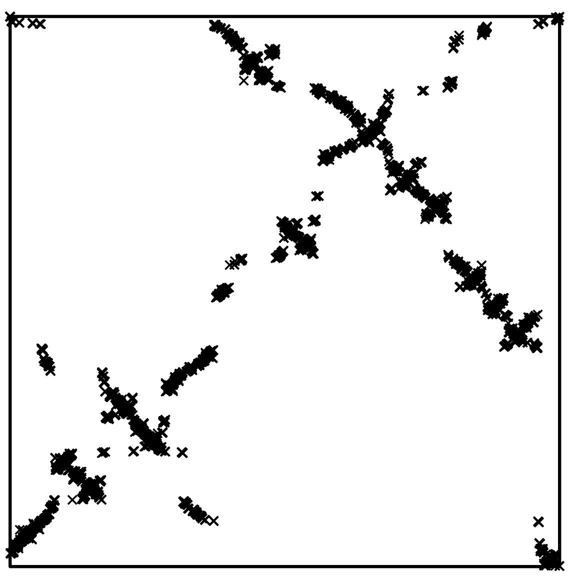
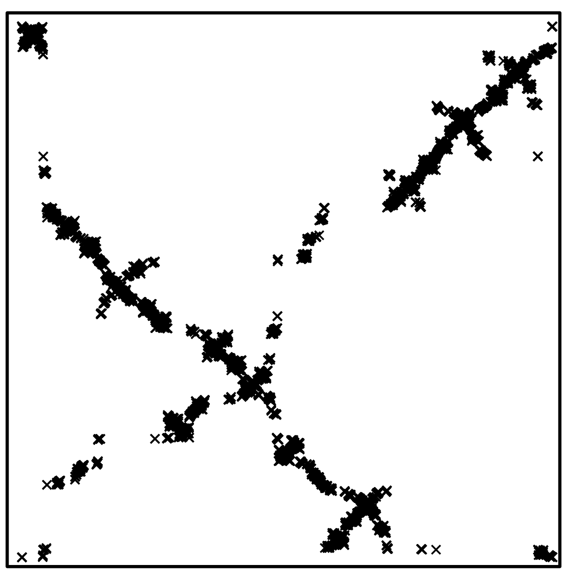
$\mathcal{C} = \mathrm{Av}(2413, 3142) = \{\text{separables}\}$:

Theorem (Bassino, Bouvel, Féray, Gerin, Pierrot 2016)

σ_n converges in distribution to some random permuton μ, called the Brownian separable permuton.





A portmanteau theorem for random permutons

Theorem (Bassino, Bouvel, Feray, Gerin, M., Pierrot. 2017)
The following are equivalent:

1. The random measure μ_{σ_n} converges in distribution to some random permuton μ.
2. The random variables $\widetilde{\mathrm{occ}}(\pi, \sigma_n)$ converge in distribution, jointly in $\pi \in \mathfrak{S}$.
3. $\mathbb{E}[\widetilde{\mathrm{occ}}(\pi, \sigma_n)]$ converges to some Δ_π for every $\pi \in \mathfrak{S}$.

Moreover, the law of μ is characterized by
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$\mathbb{E}[\widetilde{\mathrm{occ}}(\pi, \mu)] = \Delta_\pi, \pi \in \mathfrak{S}$

Remark: $\mathbb{E}[\widetilde{\mathrm{occ}}(\pi, \sigma_n)] = \mathbb{P}[\mathrm{pat}_{I_n^k}(\sigma_n) = \pi]$, where I_n^k is a uniform k-subset of $[n]$, independent of σ_n.

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When σ_n uniform in $\mathcal{C}_n$, this is only a matter of enumeration: how many of the $\binom{n}{k}|C_n|$ pairs $(I, \sigma) \in [n] \times \mathcal{C}_n$ are such that $\mathrm{pat}_I(\sigma) = \pi$?

Idea of proof

Use the bijection with signed Schröder trees: $\sigma_n = \mathrm{perm}(t_n)$, where t_n is a uniform signed Schröder tree with n leaves.

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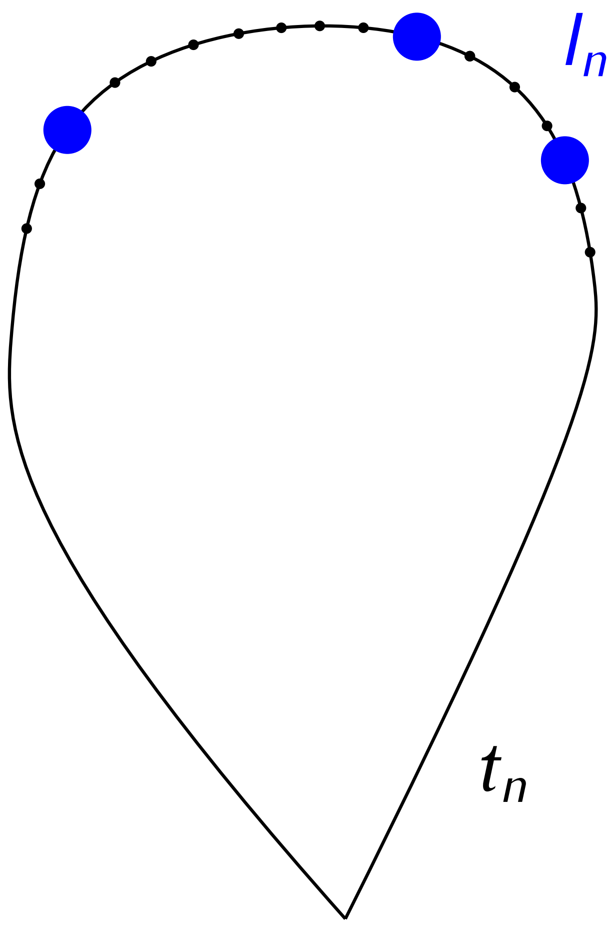

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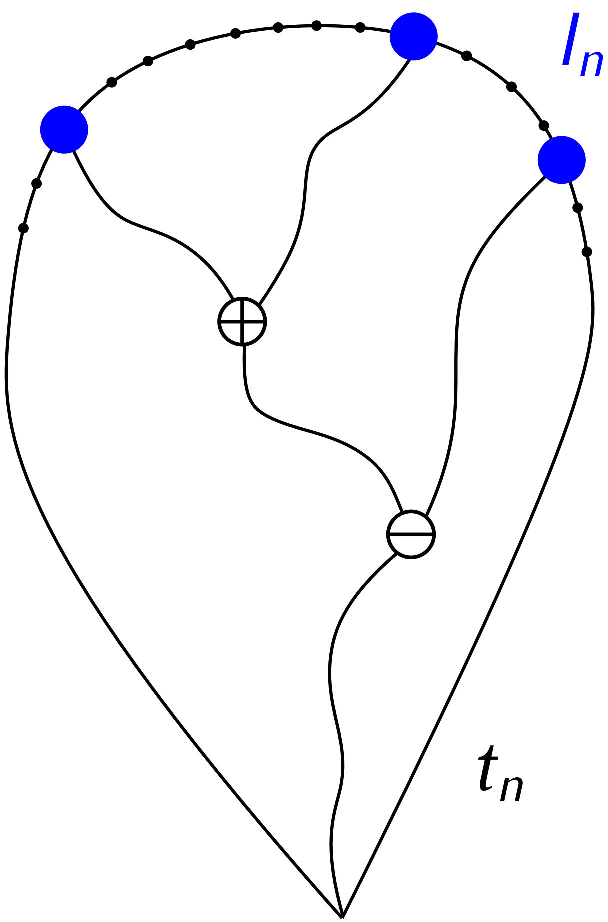

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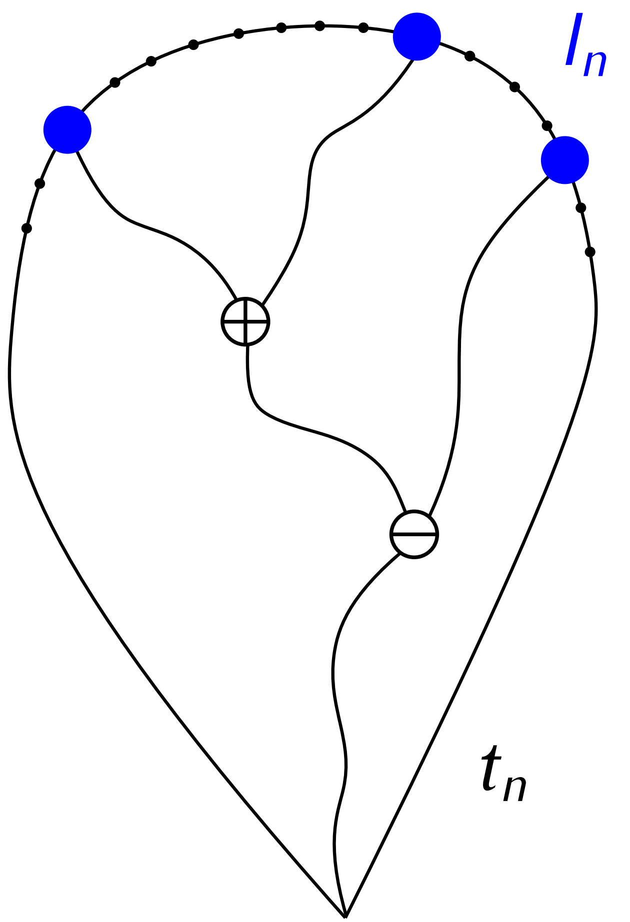

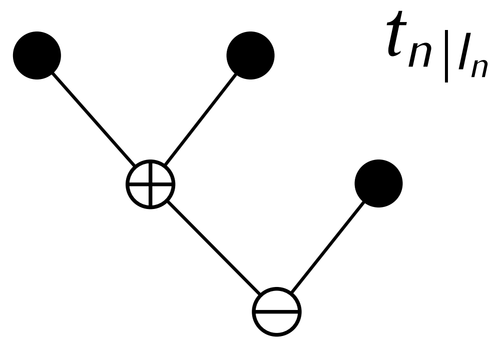

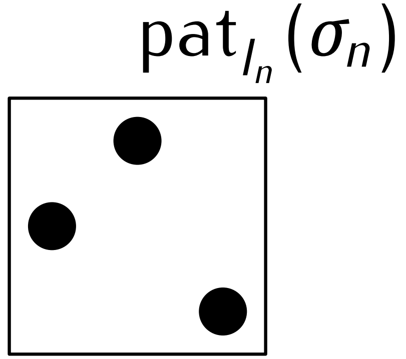

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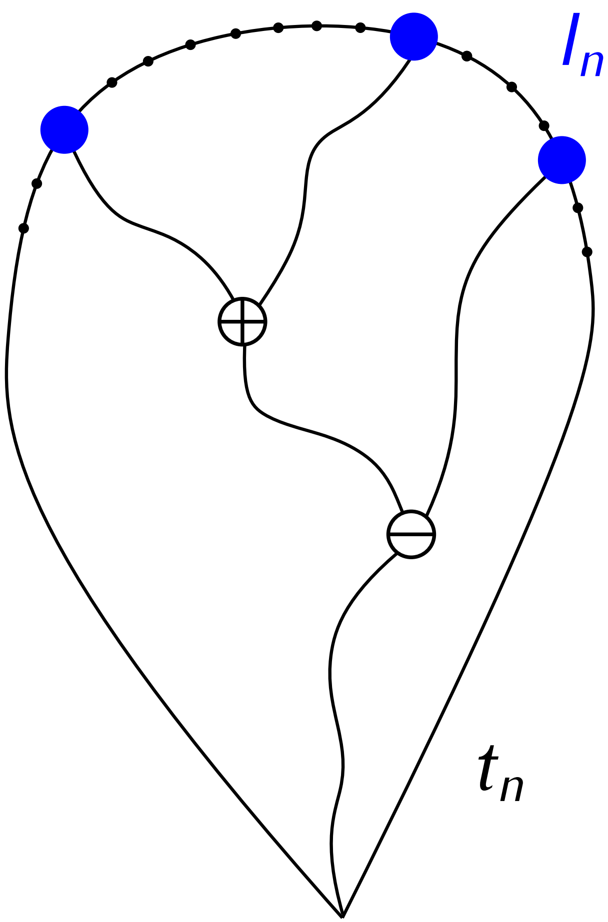

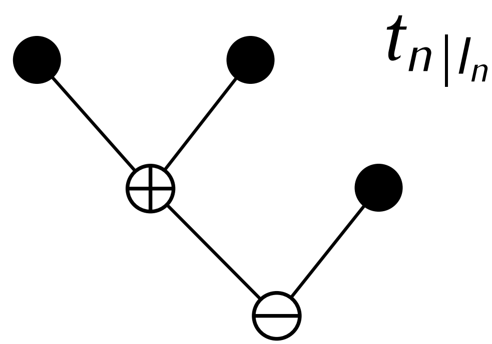

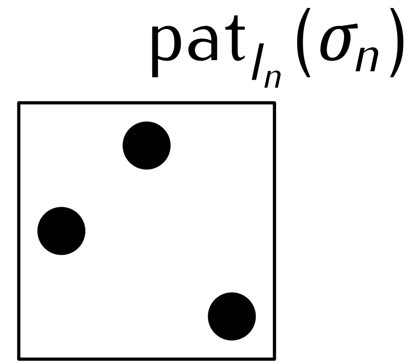

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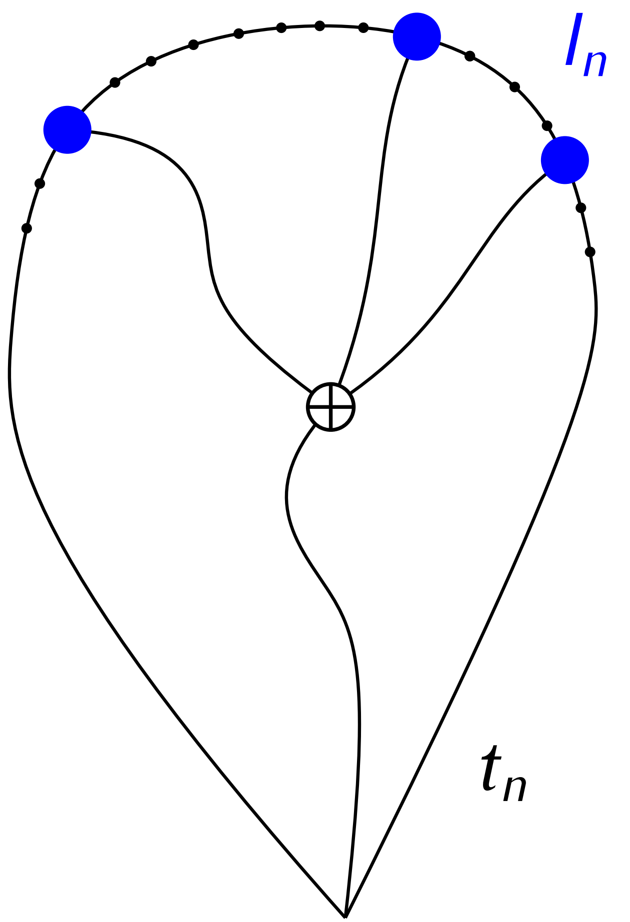

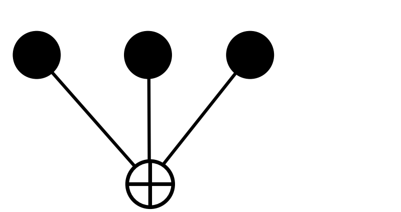

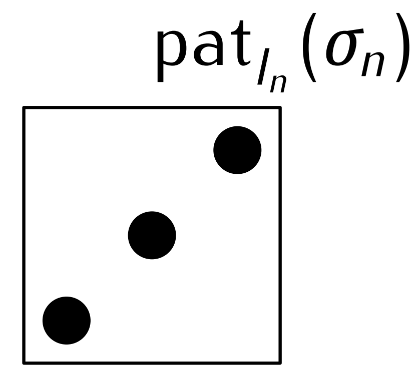

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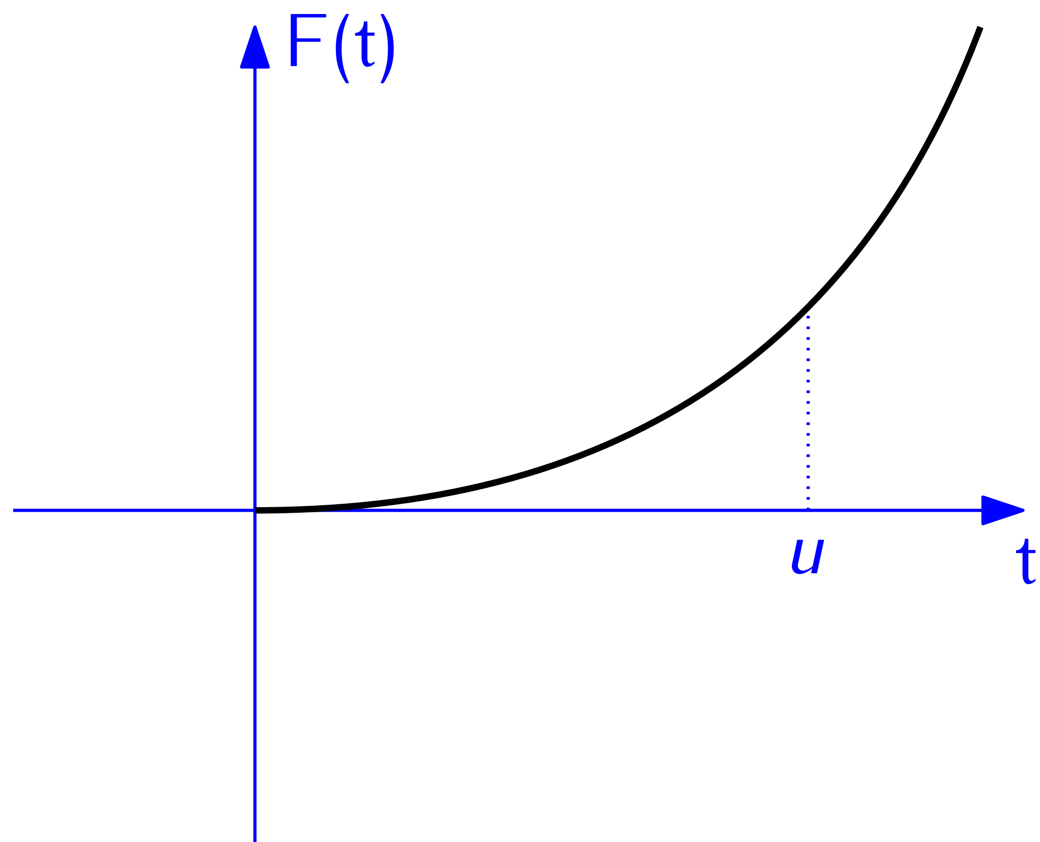

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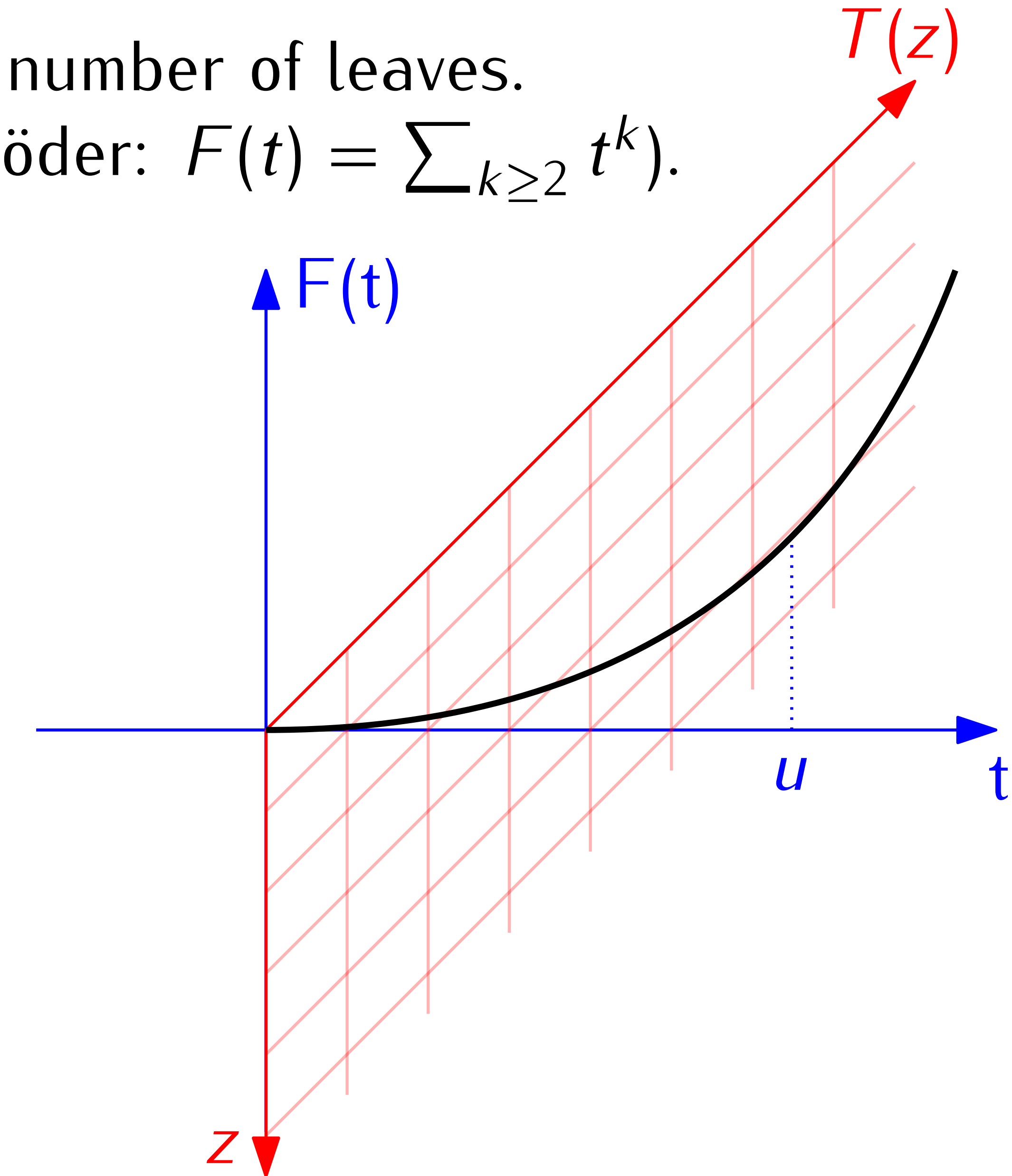

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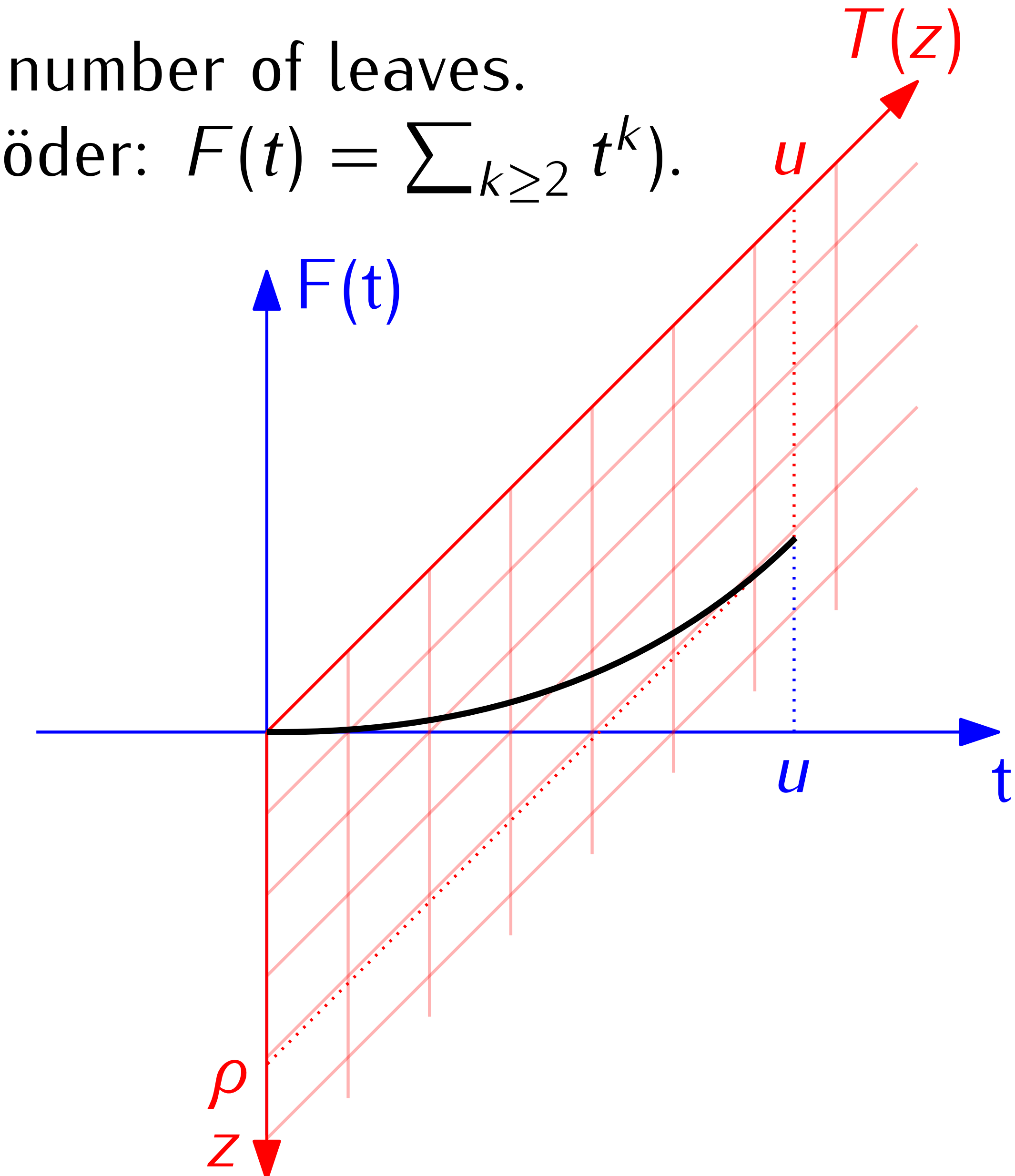

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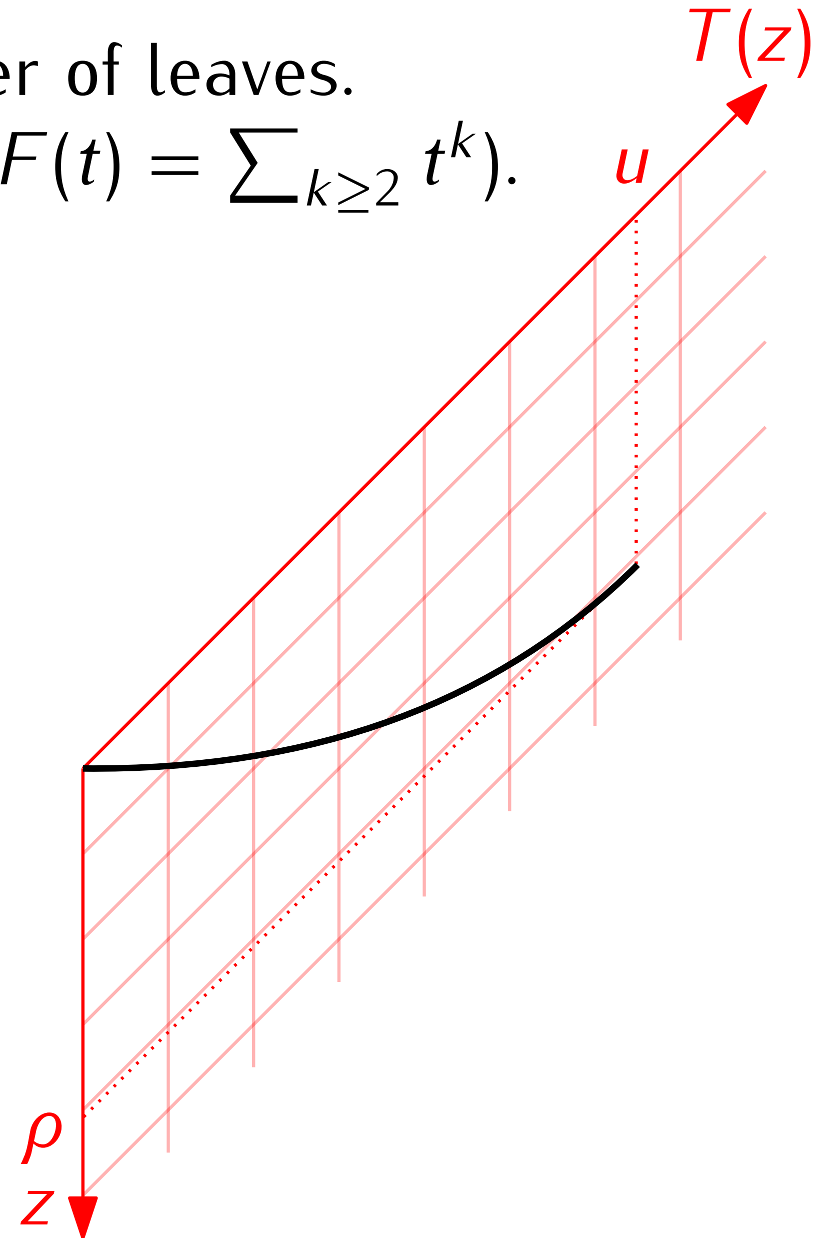

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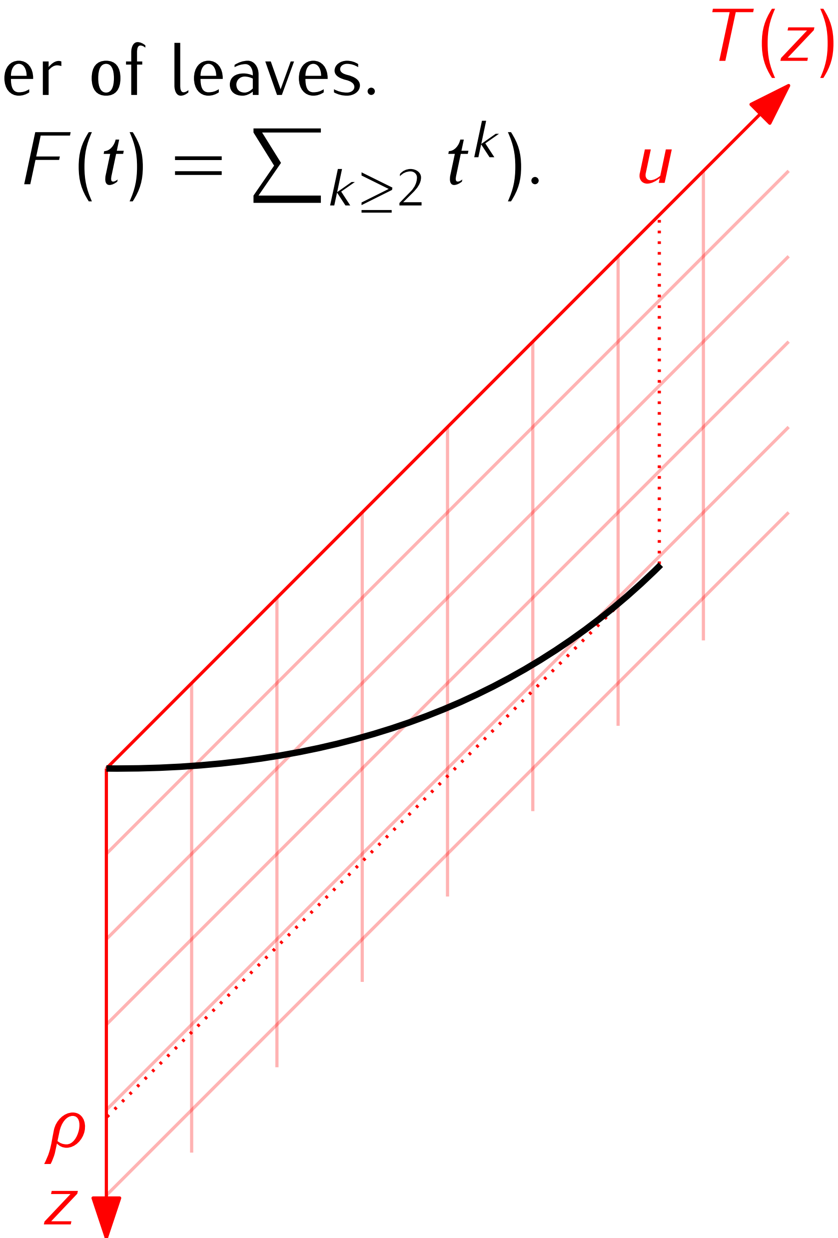
This is the case for Schröder (F rational)



Uniform k-subtree in large unsigned trees

T has square-root singularity at ρ and F analytic at $T(\rho)$. Then, the $g.f$ of trees with k marked leaves that induce the k-tree τ is

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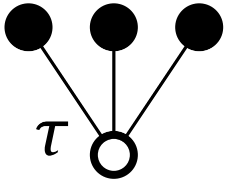

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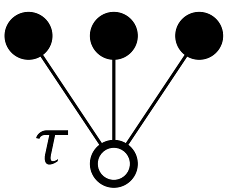

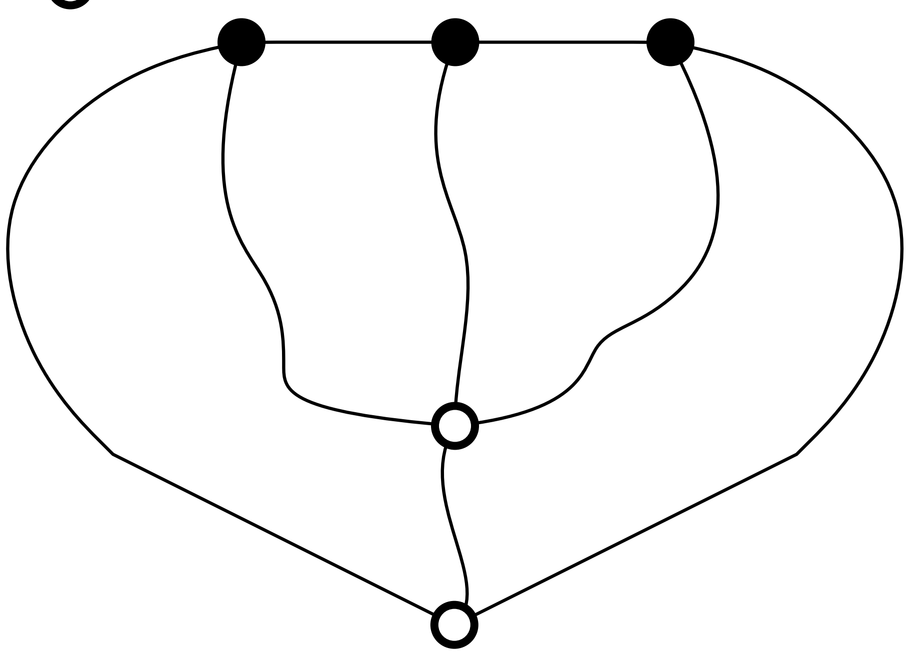

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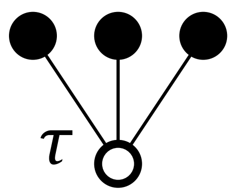

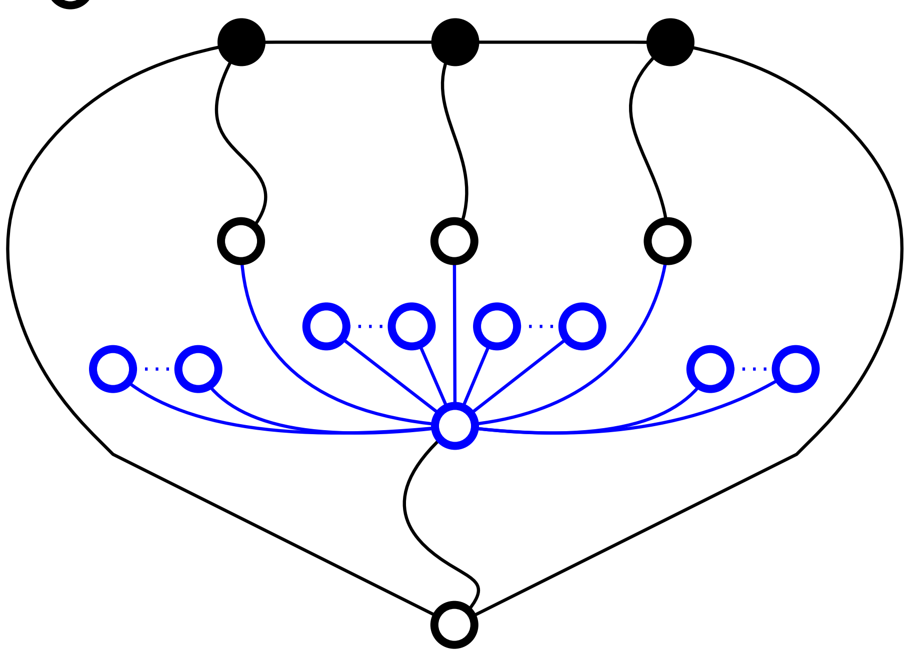

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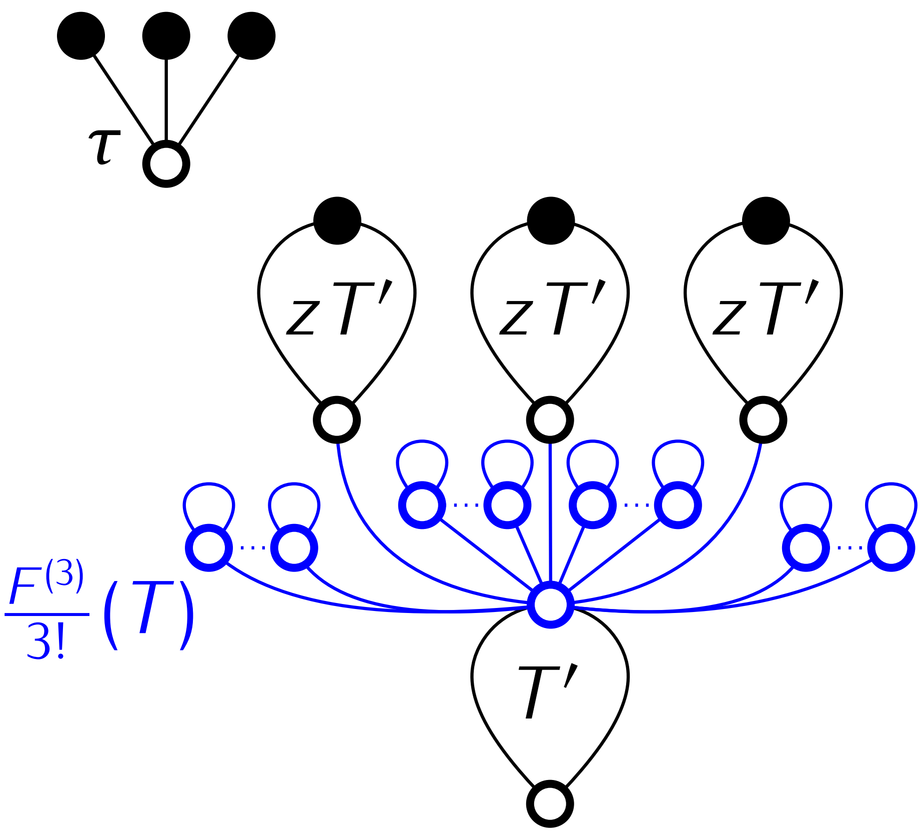

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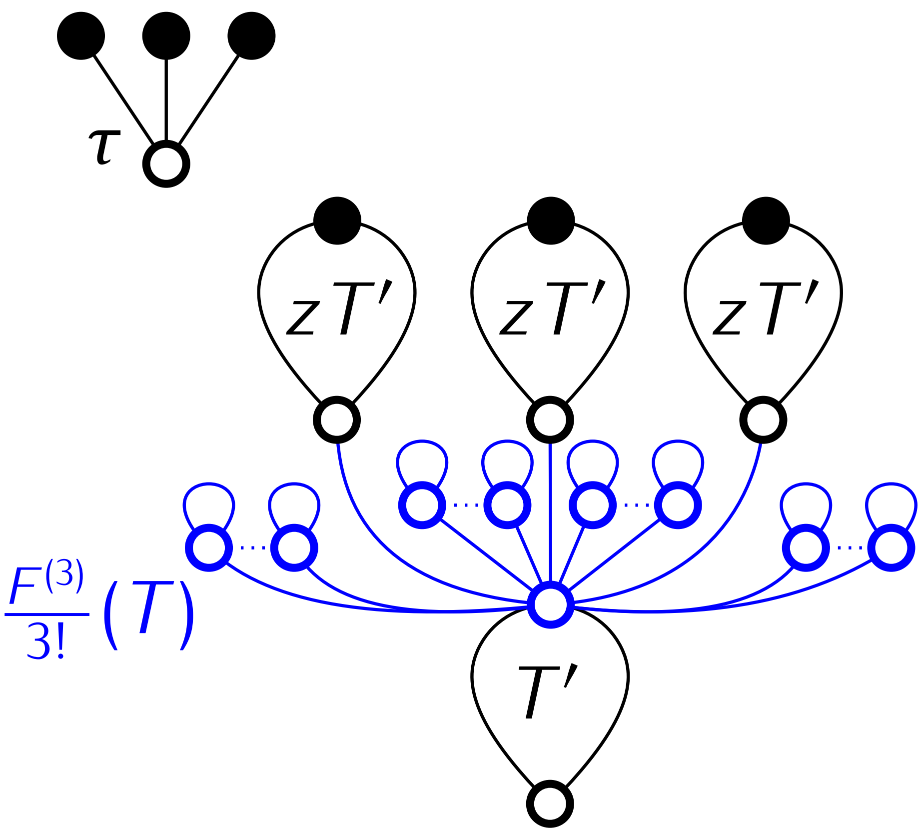

$\sim_\rho C_\tau (1 - \frac{z}{\rho})^{-\#\{\text{nodes in } \tau\}/2}$.
Dominates when τ binary.
(Then C_τ doesn't depend on τ).
Transfer: $t_{n|I_n^k}$ converges in distribution to a uniform binary tree.

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Hence all signed binary trees have the same asymptotic probability. Hence convergence in distribution to a permuton.

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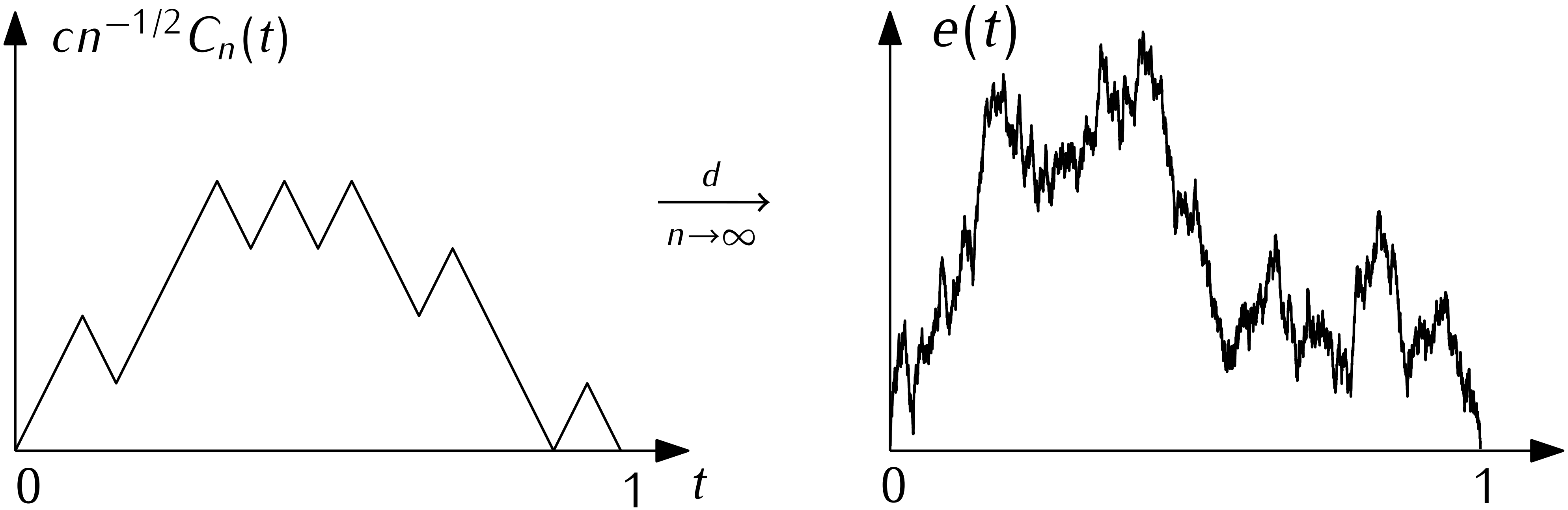

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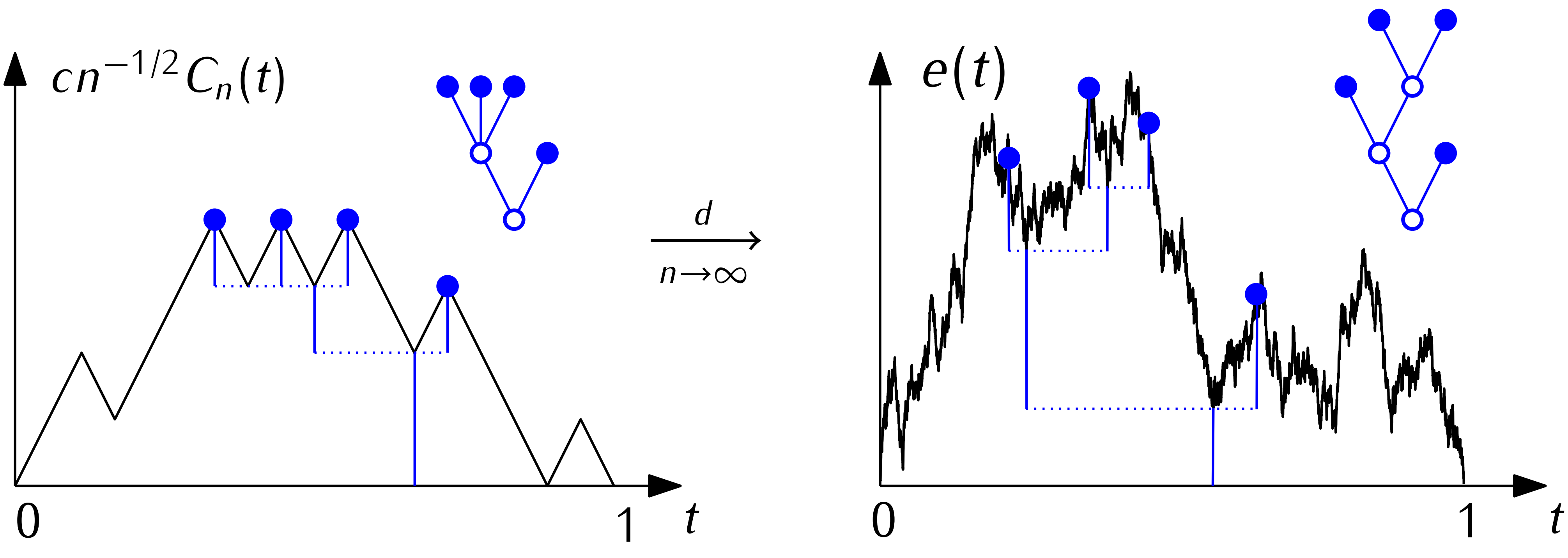
So uniform extracted subtrees from C_n converge to uniform extracted subtrees from the Brownian excursion, which are uniform binary trees (Aldous '93, Le Gall '93)



2 – Universality of permuton limits in substitution-closed classes.

Joint work with F. Bassino, M. Bouvel, V. Féray, L. Gerin and A. Pierrot [arXiv:1706.08333]

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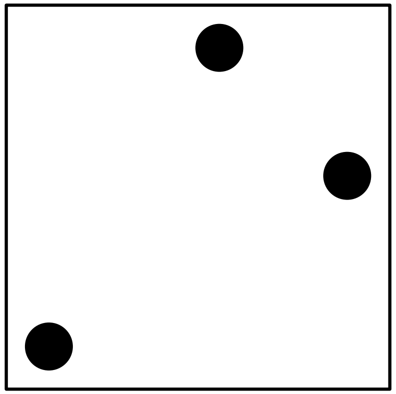

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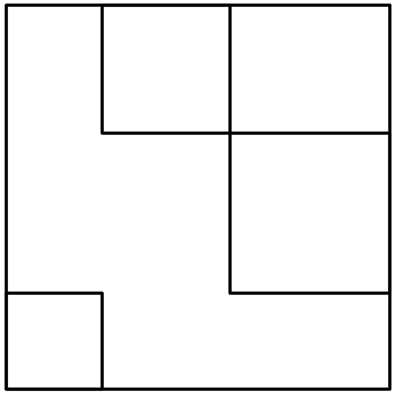

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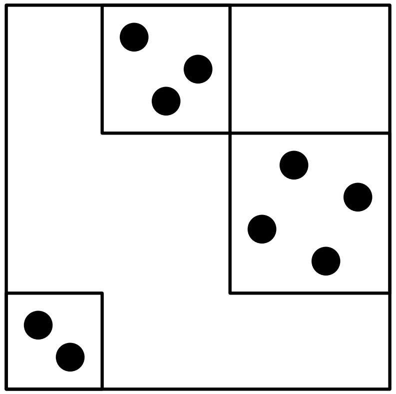

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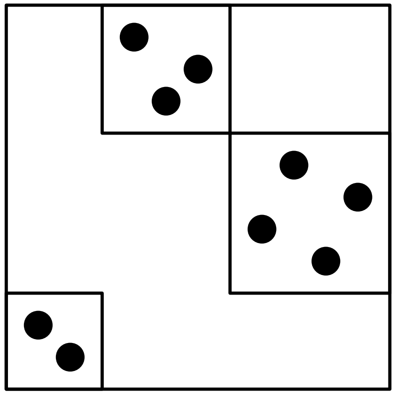

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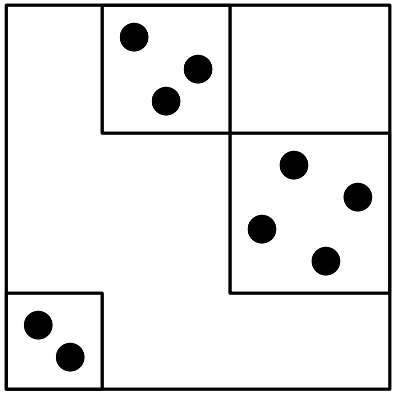

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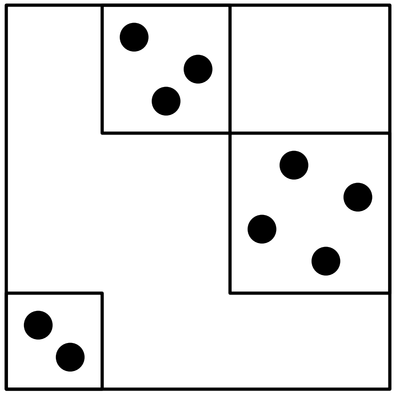

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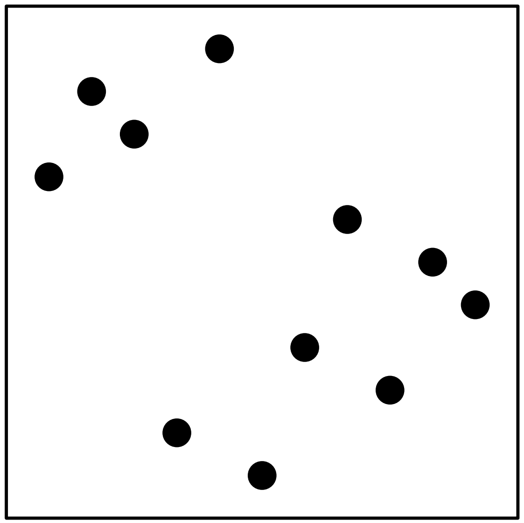
- Or σ can't be decomposed by a nontrivial substitution : σ is a **simple permutation**. Ex : $1, 12, 21, 2413, 3142, 31524, \ldots \sim \frac{n!}{e^2}$.

Substitution decomposition



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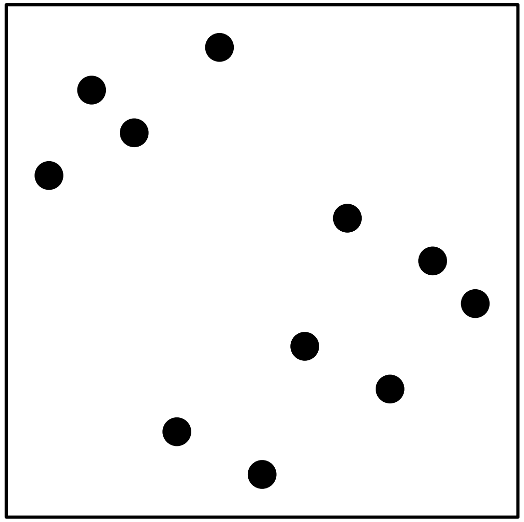

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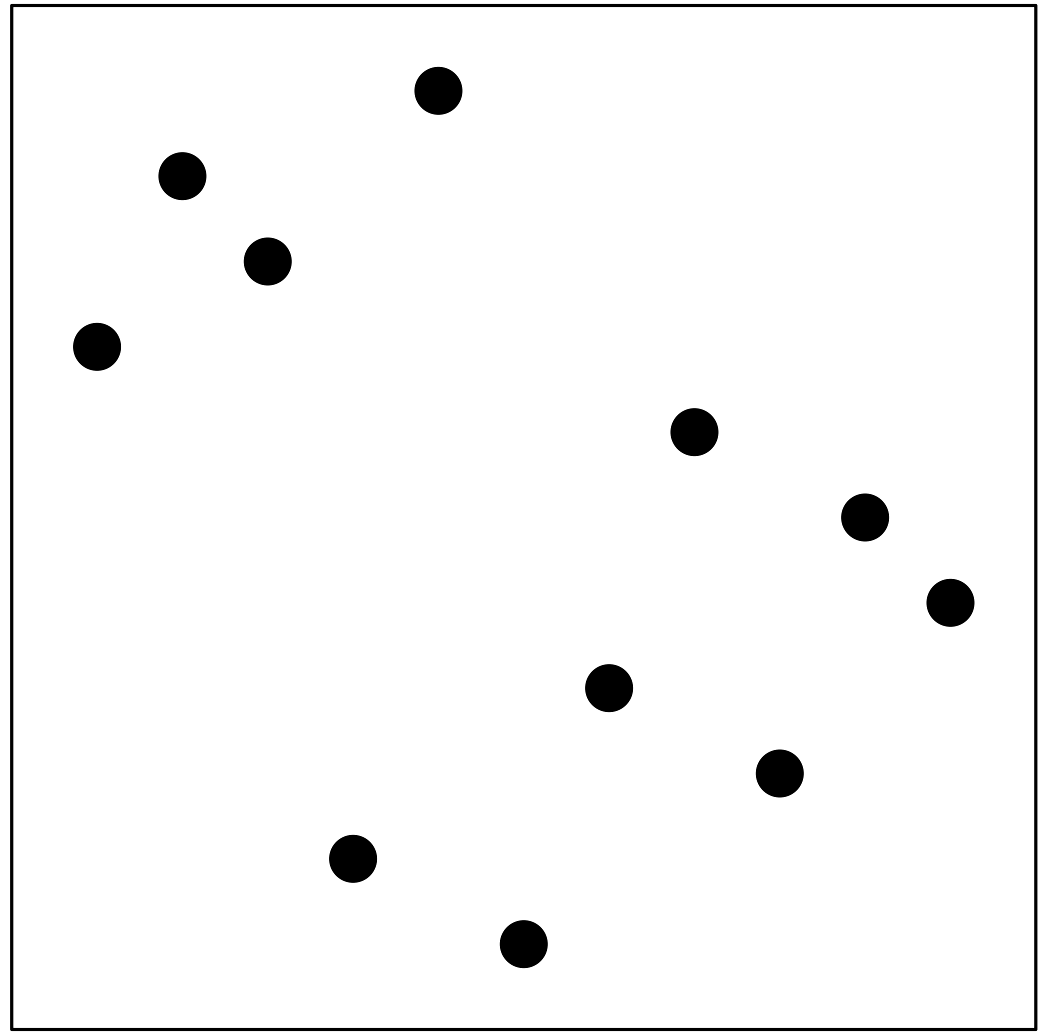

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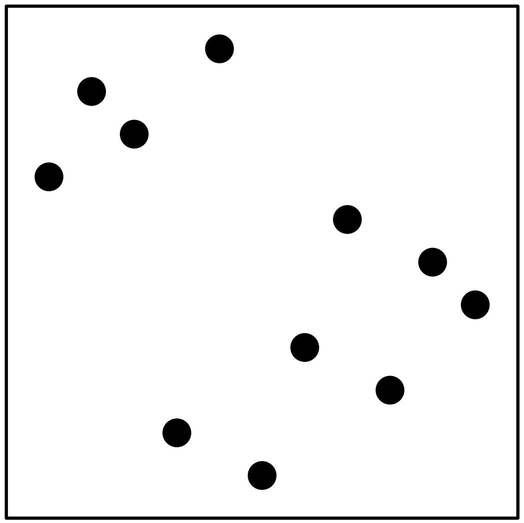

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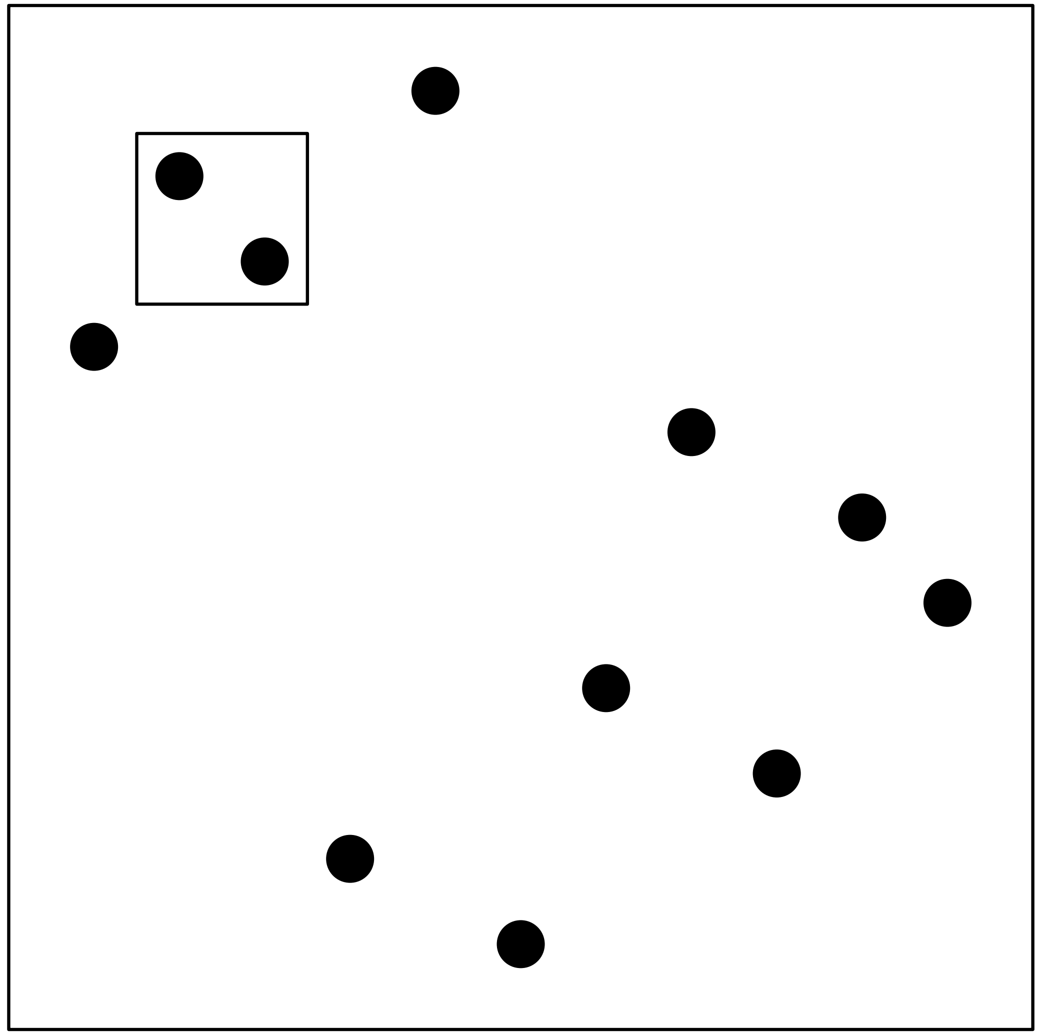

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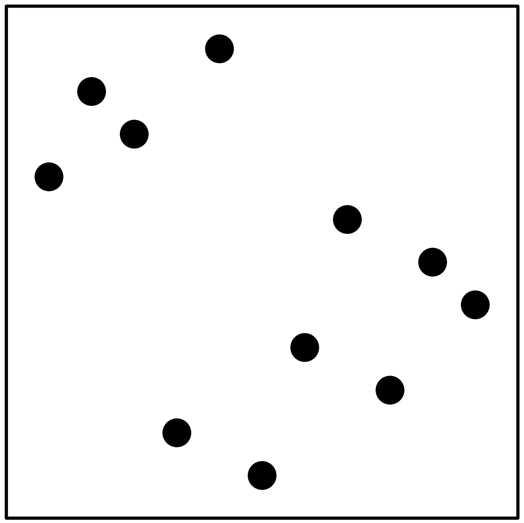

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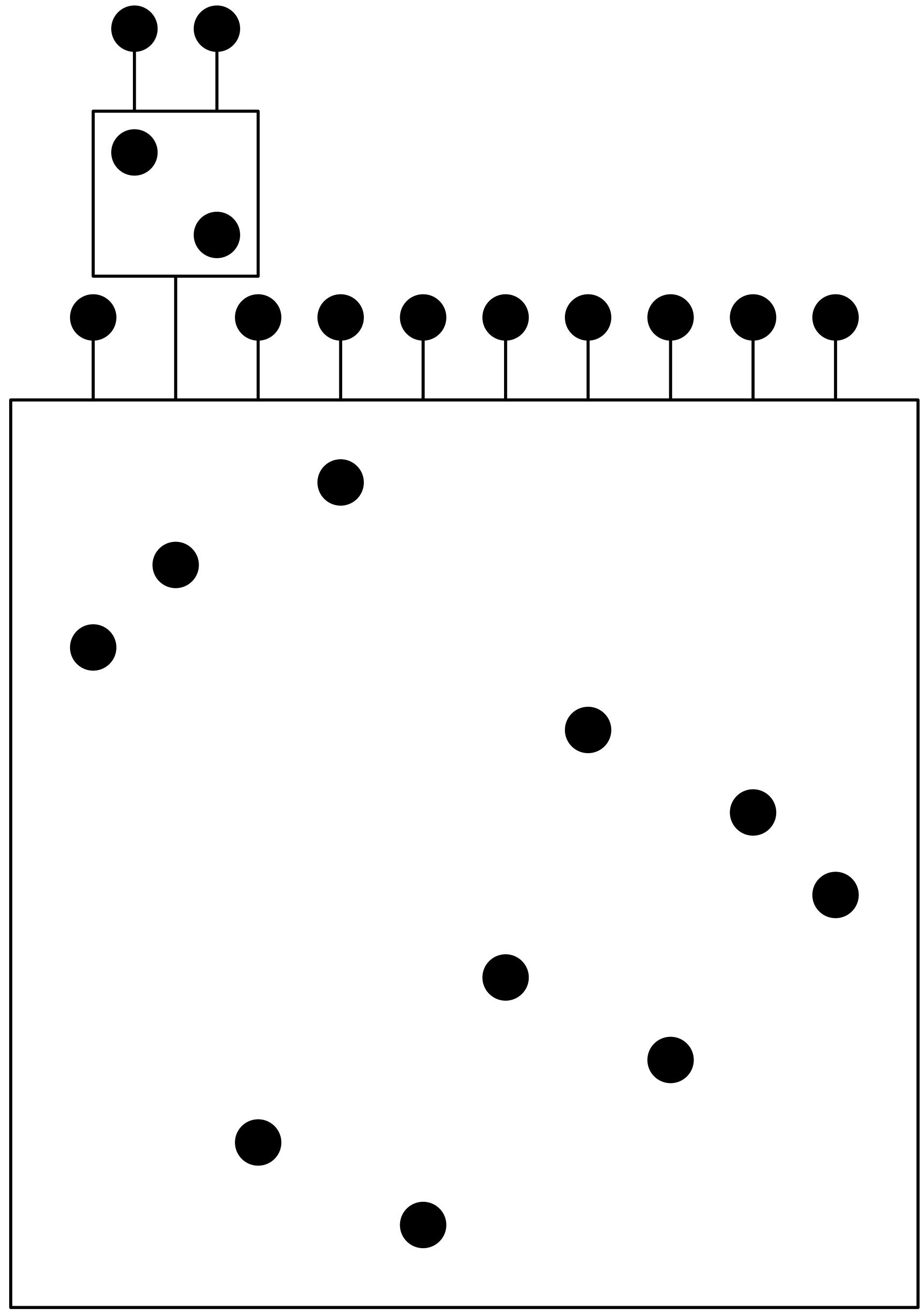

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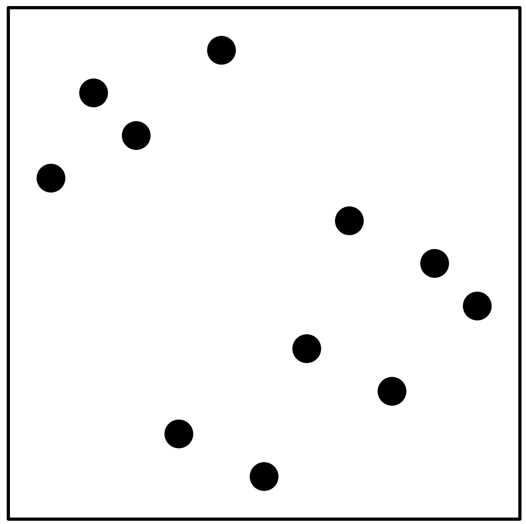
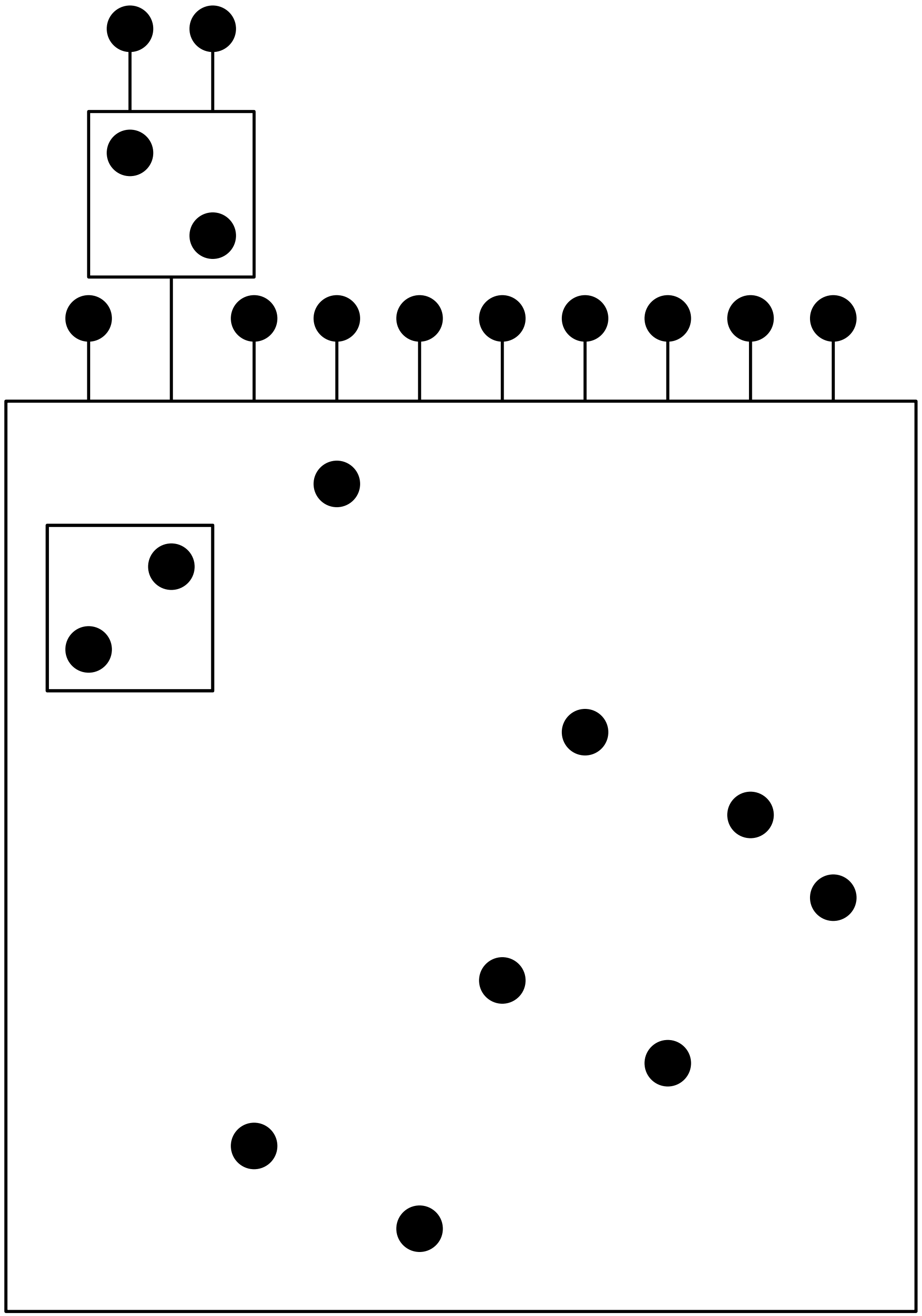


(8, 10, 9, 2, 11, 1, 4, 7, 3, 6, 5)



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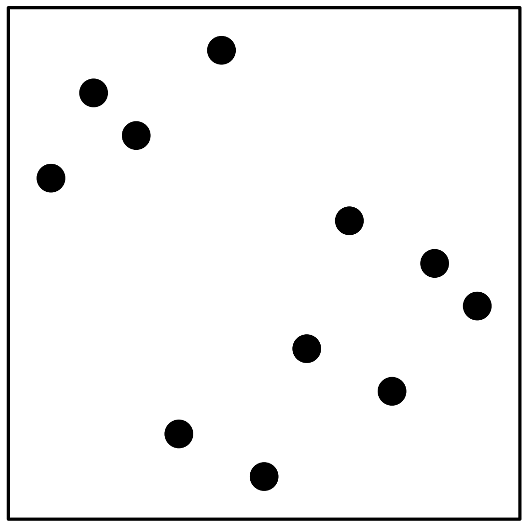

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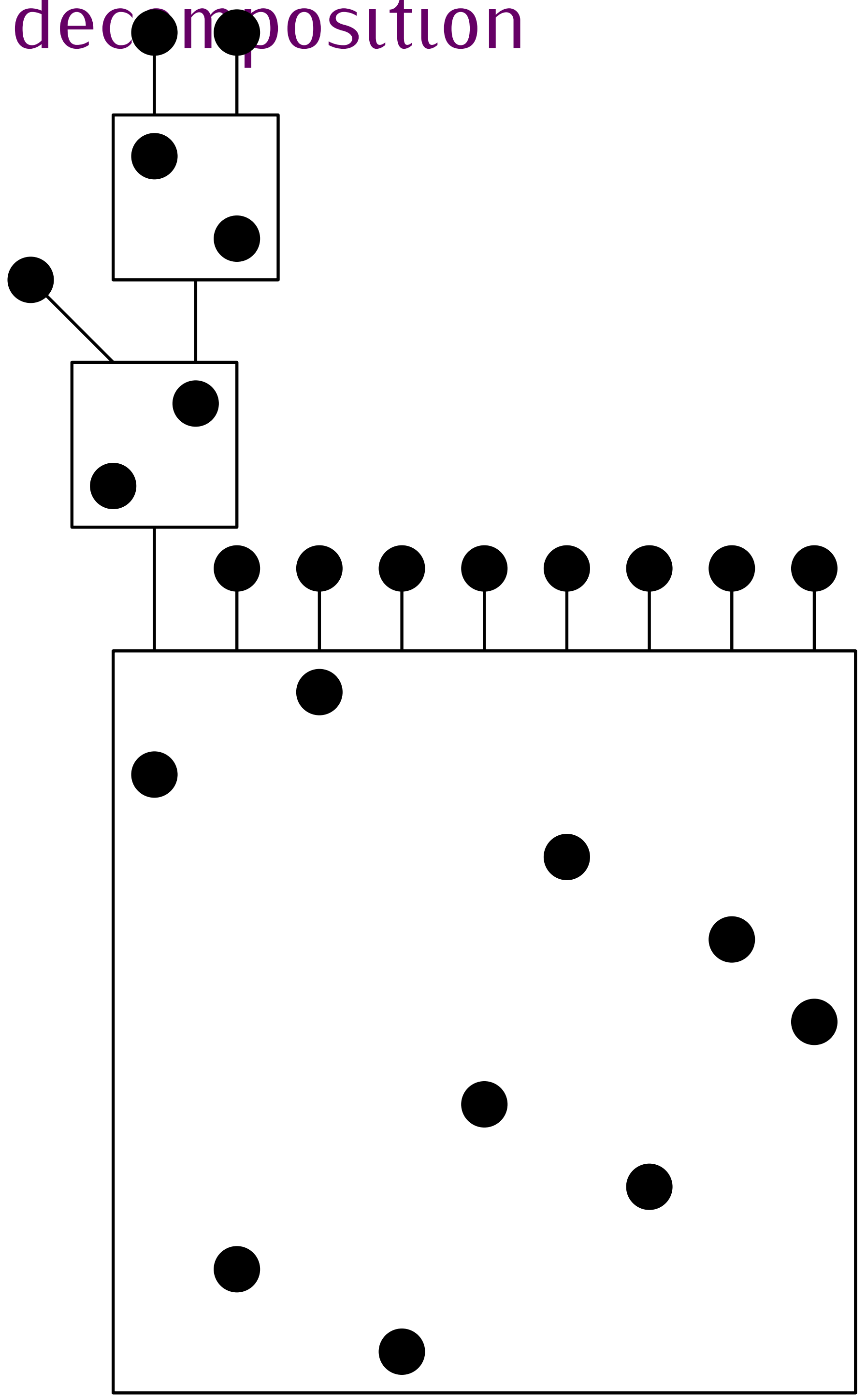

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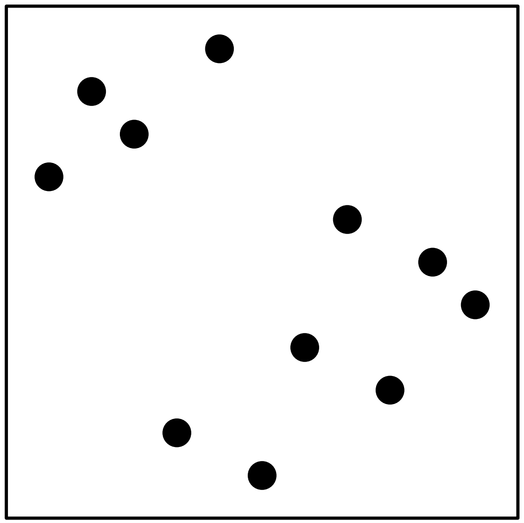

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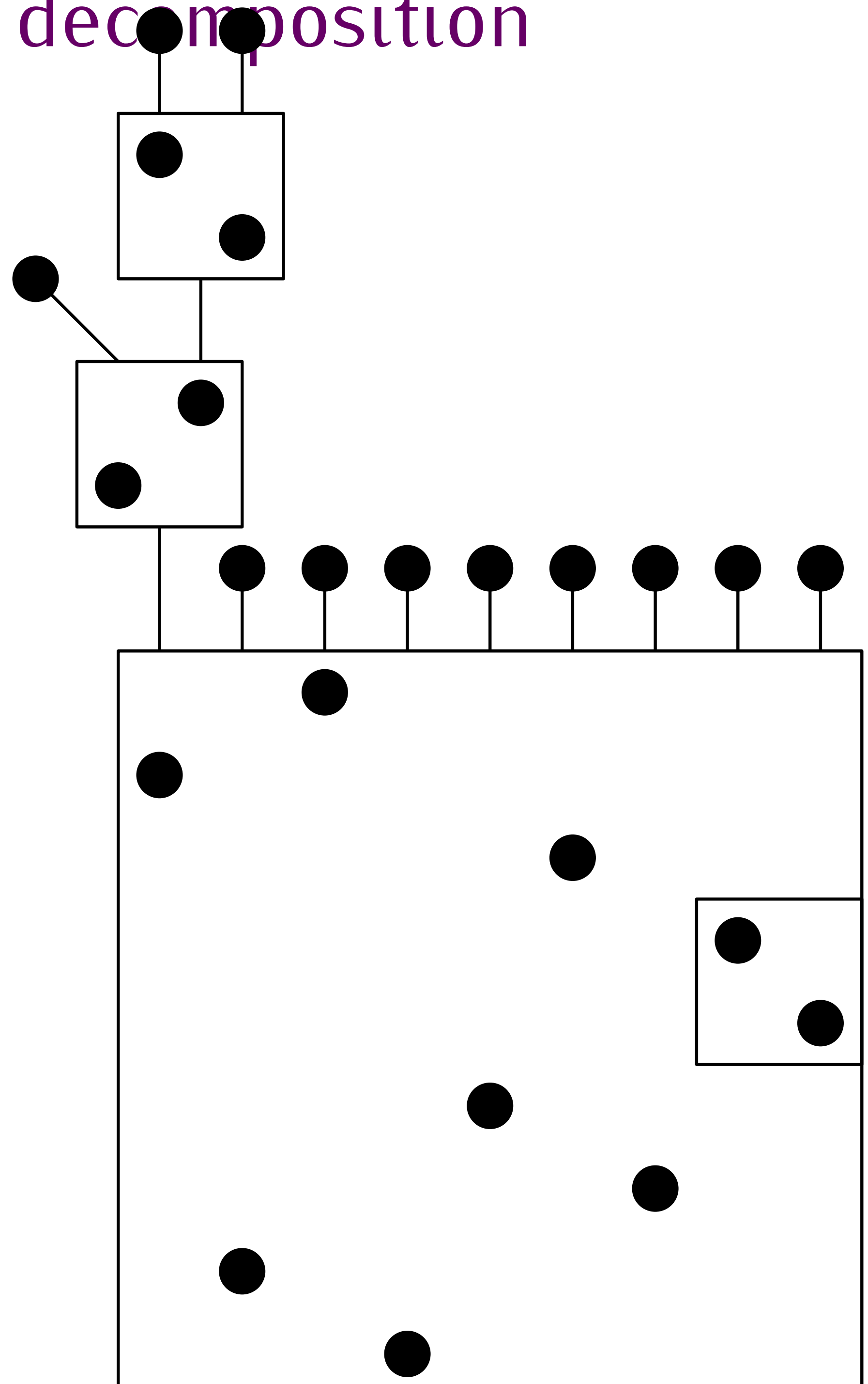

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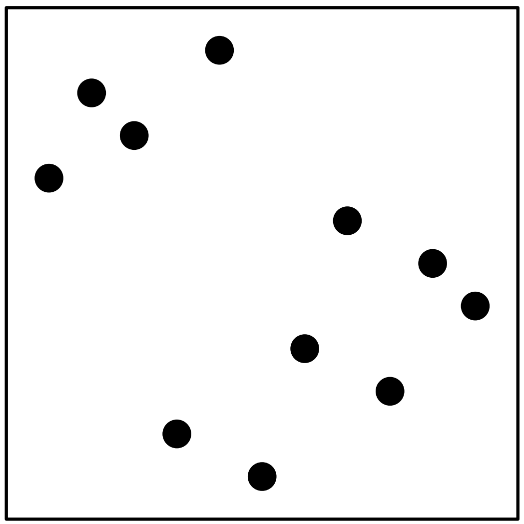

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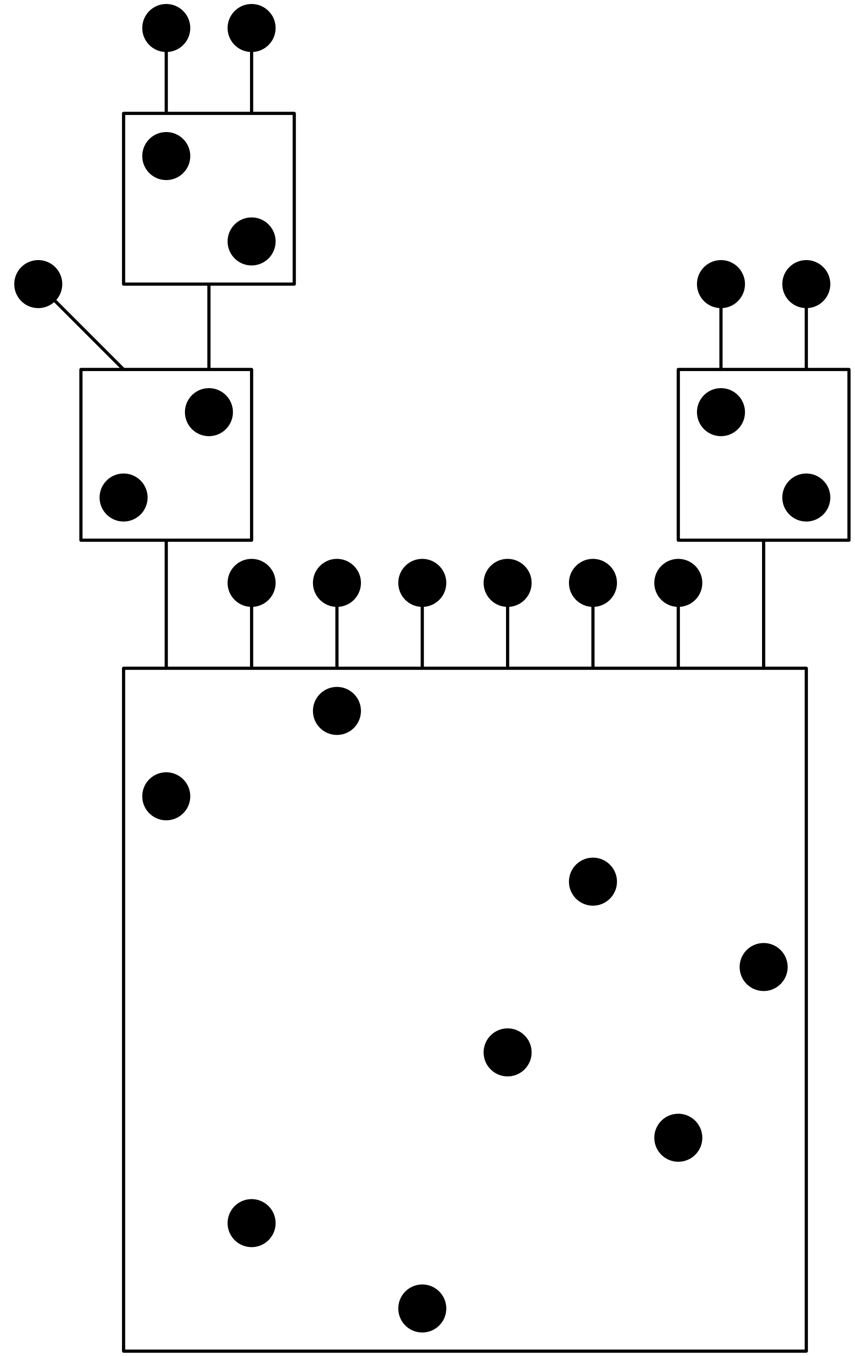

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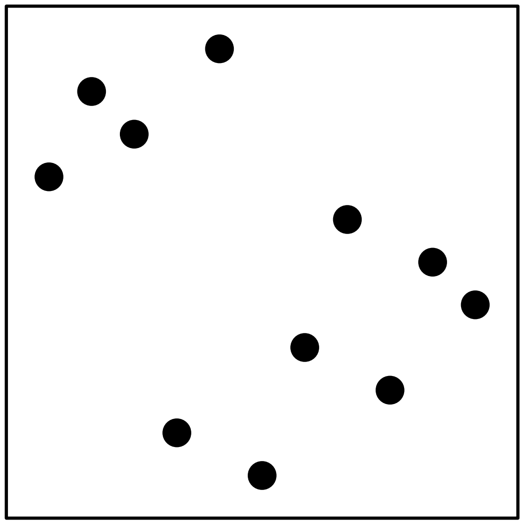

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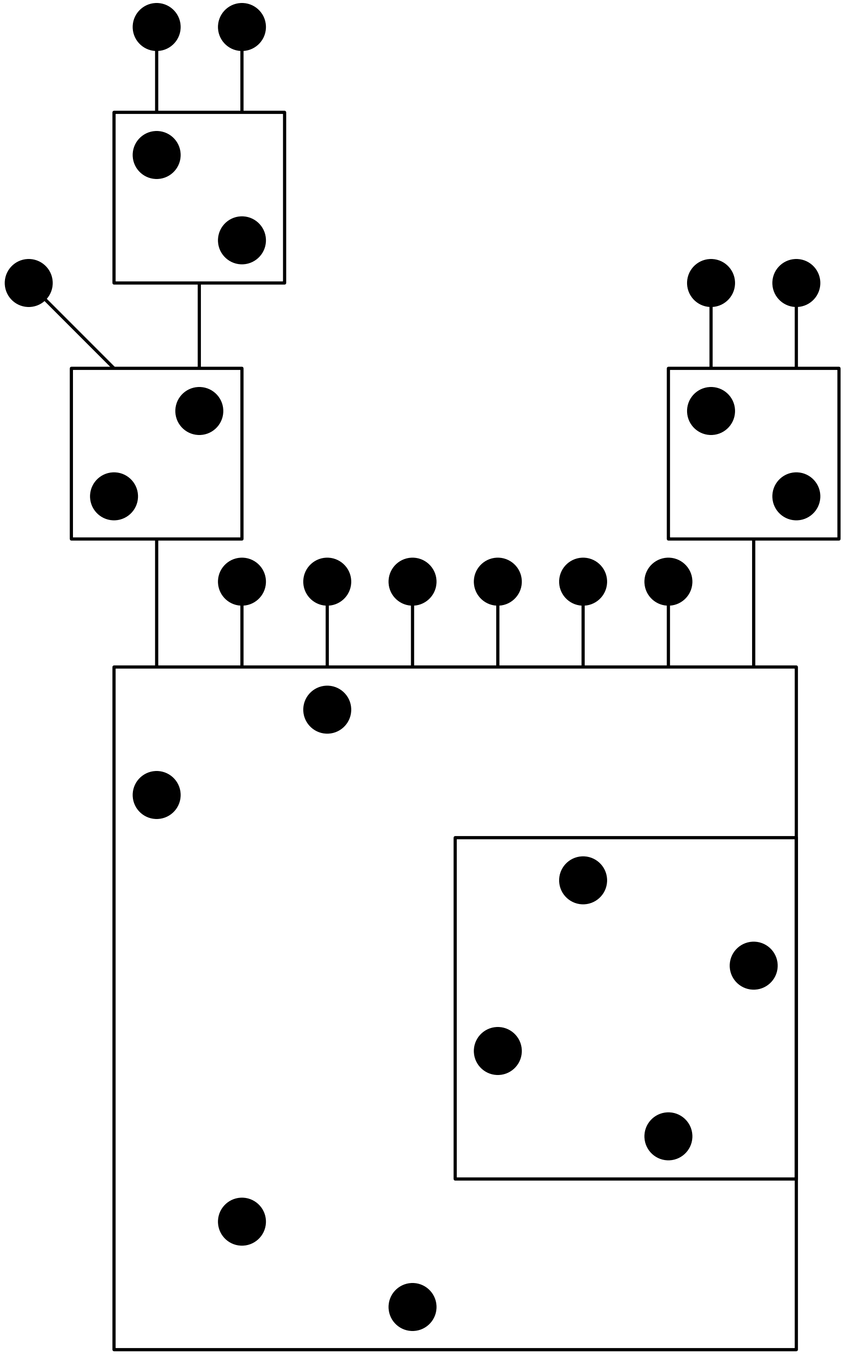

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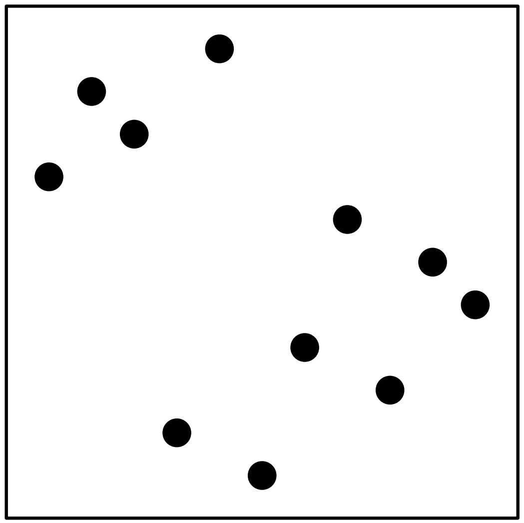

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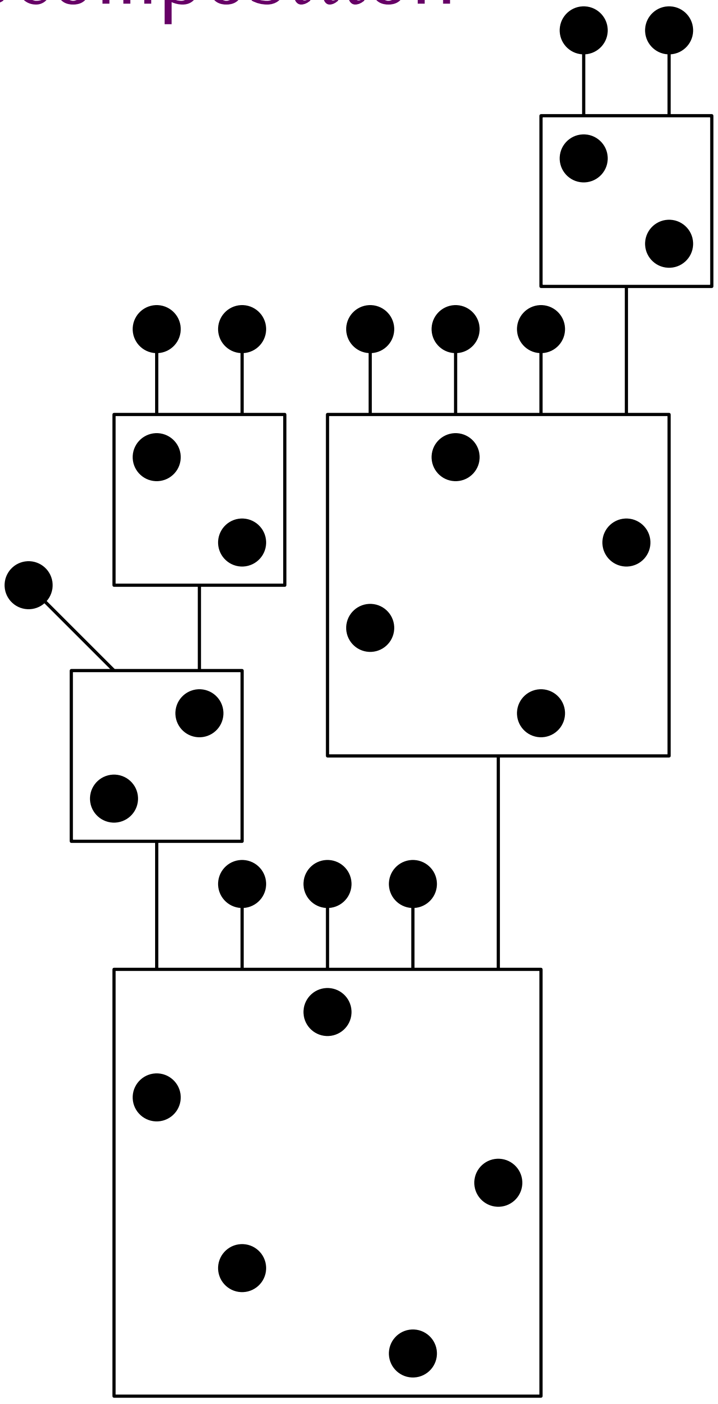

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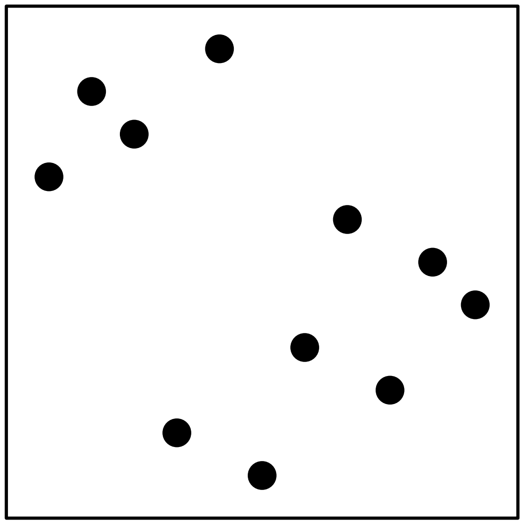

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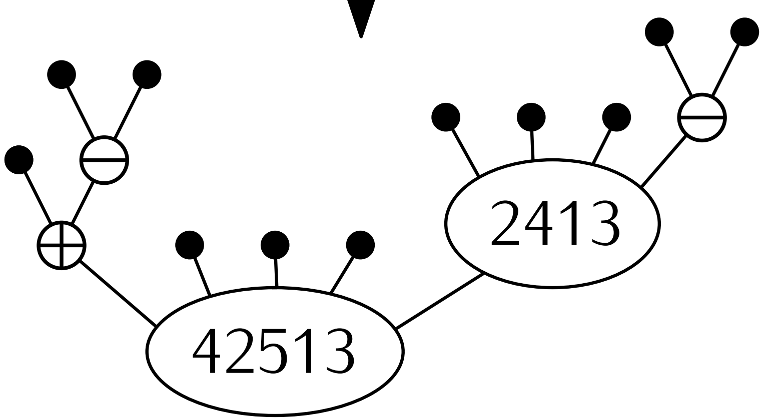

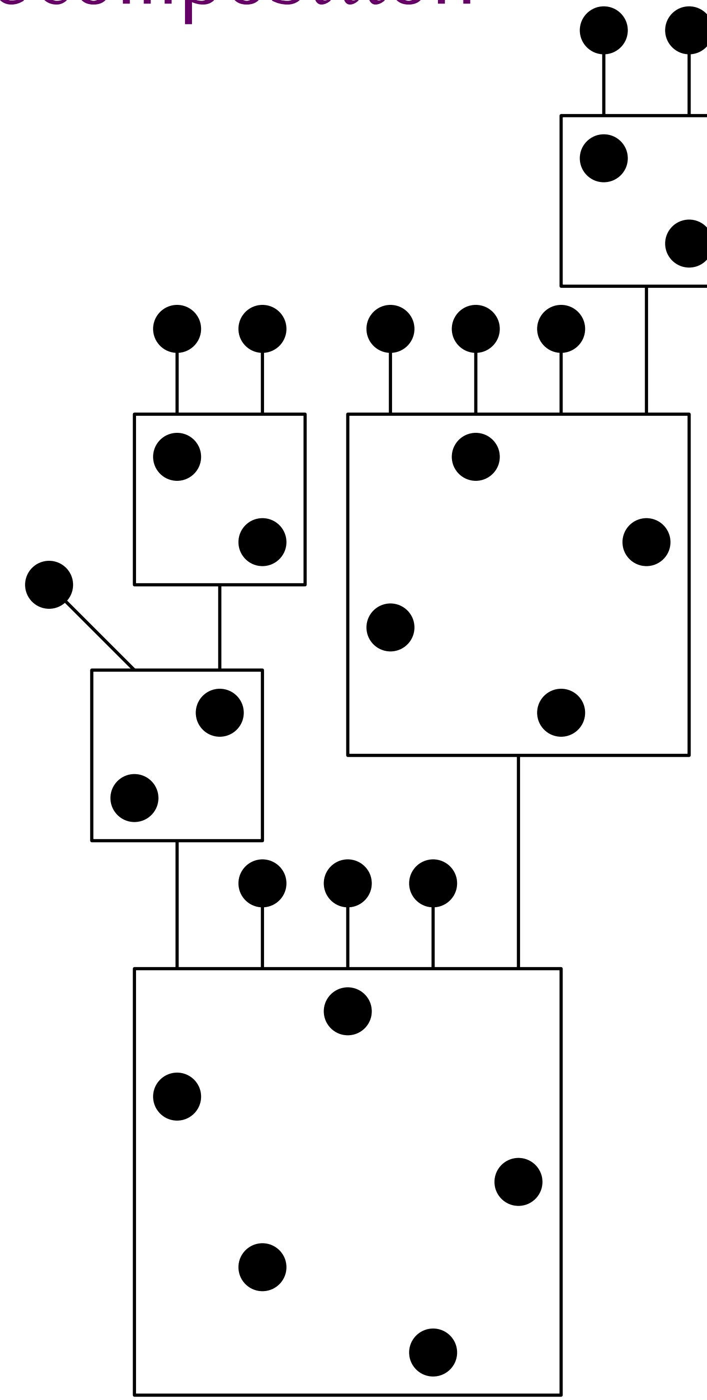

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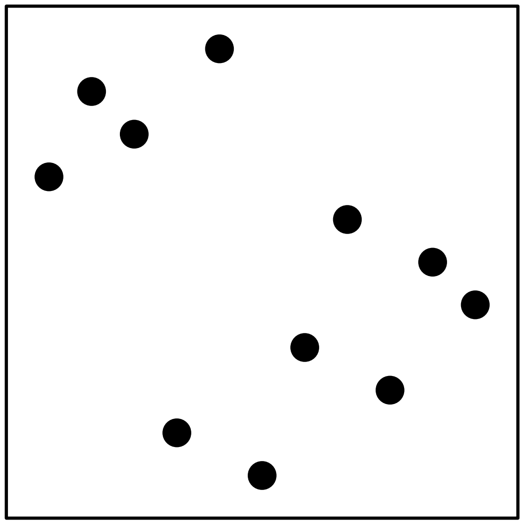

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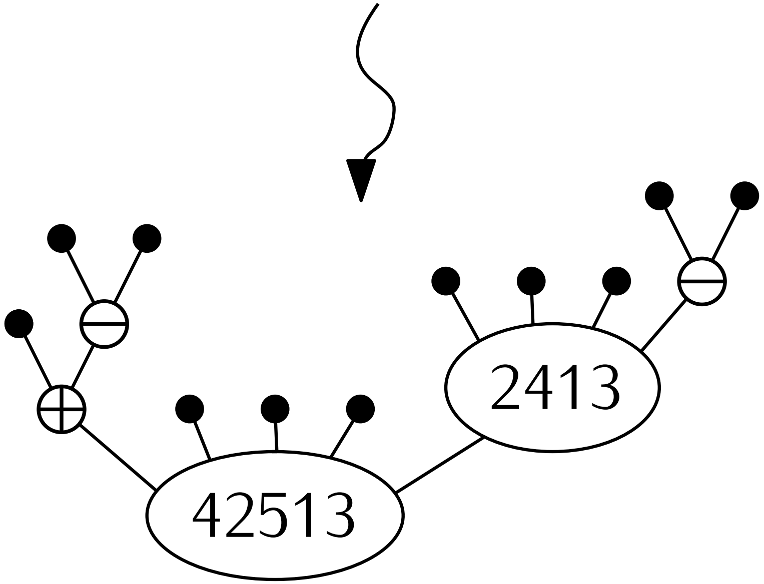

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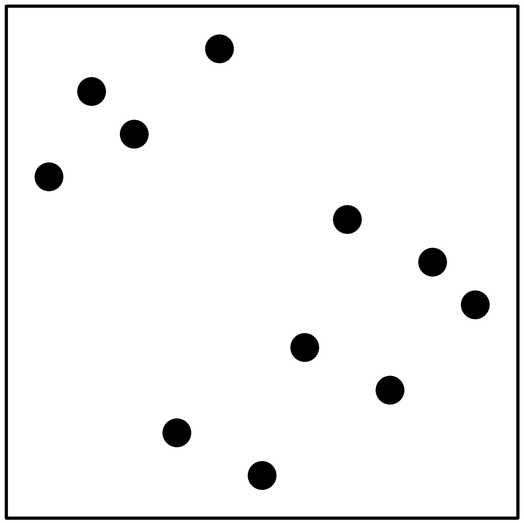

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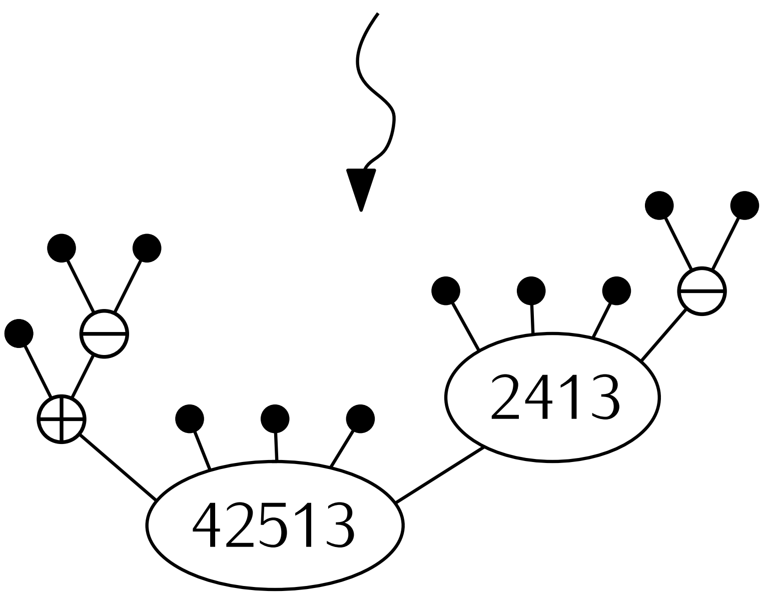

Theorem (Albert, Atkinson 2005): Any permutation can be decomposed into a substitution tree with $\oplus$, $\ominus$ nodes, and simple nodes of length ≥ 4, unique as long as adjacent $\oplus$ and $\ominus$ are merged.

Substitution-closed families

$\mathcal{S} \subset \{$simple permutations of length $\geq 4\}$.

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Let σ_n be a uniform permutation of size n in $\langle \mathcal{S} \rangle$.

$S(z) = \sum_{\alpha \in \mathcal{S}} z^{|\alpha|}$ generating function of the simples, radius R. Set $a = S'(R) - 2/(1+R)^2 + 1$ and $b = S''(R)$

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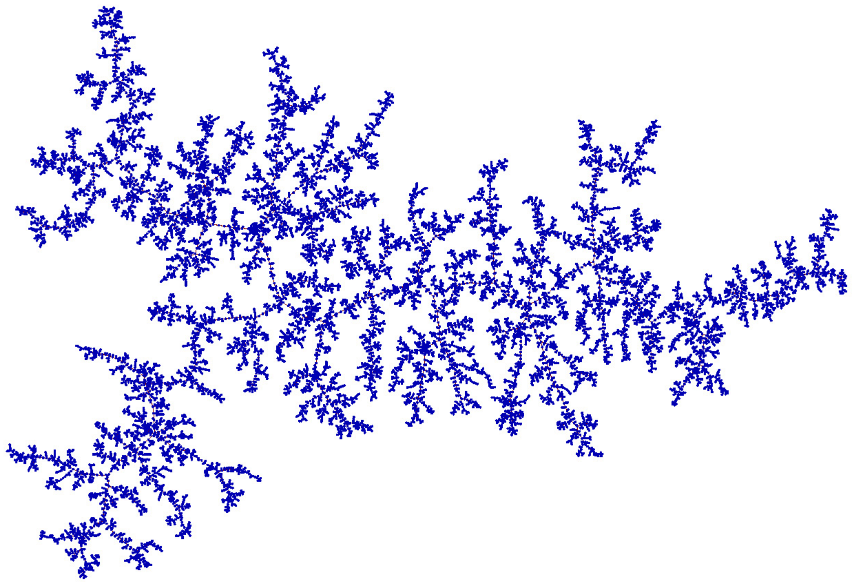
Theorem (Bassino, Bouvel, Féray, Gerin, M., Pierrot 2017)
The limit in distribution of σ_n is

- a **biased** Brownian separable permuton if $a > 0$ or $a = 0, b < \infty$,
- the same limit ν as an uniform simple permutation in $\mathcal{S}$ if $a < 0$,
- a **stable permuton** if $a = 0$, $b = \infty$.

When $a \leq 0$ additional hypotheses are needed.

Biased Brownian separable permuton

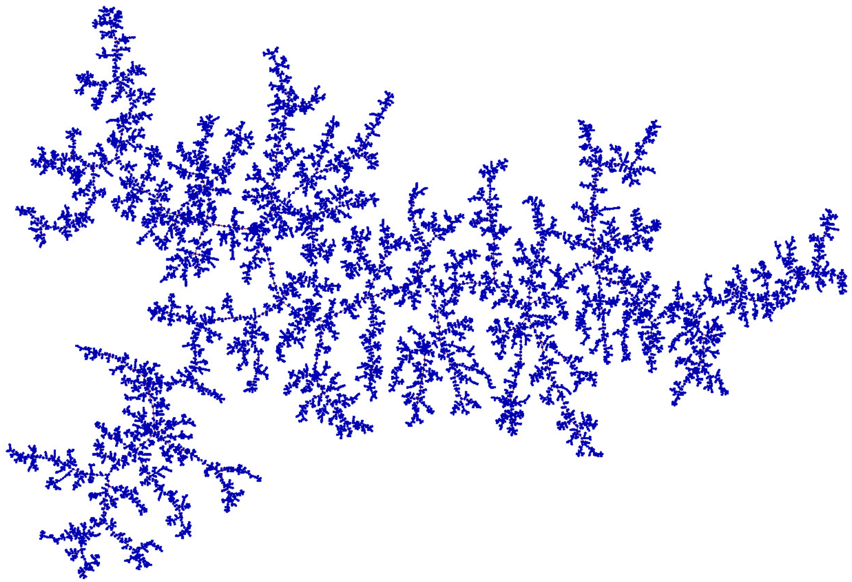
Regime where the decomposition tree converges to a Brownian CRT.



Picture by I. Kortchemski

Biased Brownian separable permuton

Regime where the decomposition tree converges to a Brownian CRT.



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The signs in a uniform subtree are biased: $\mathbb{P}(\oplus) = p$, and p depends explicitely on $\mathcal{S}$.
Here $p = 0.2$.

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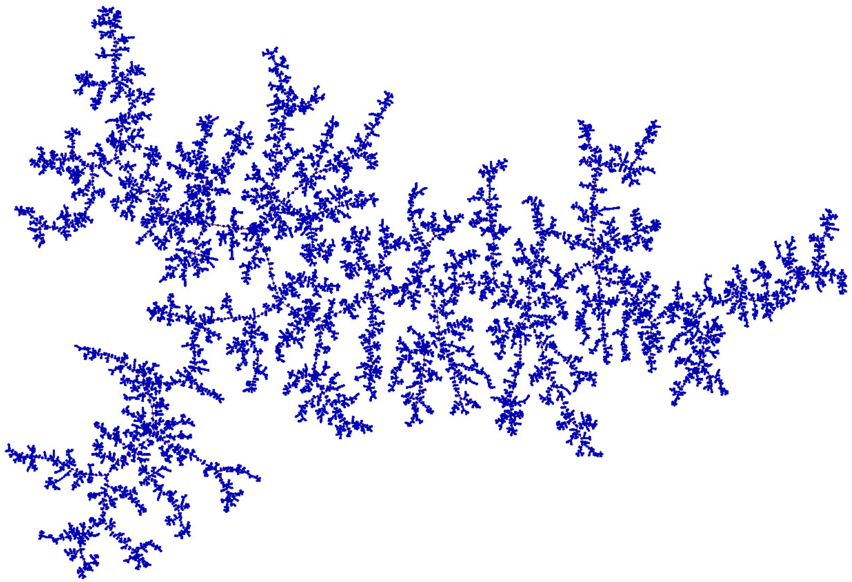

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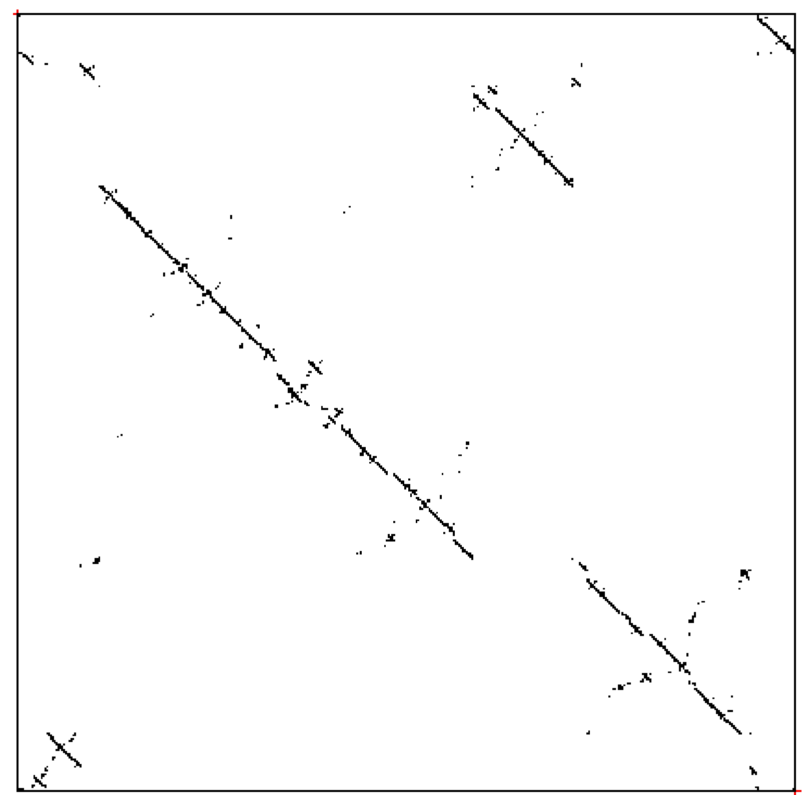

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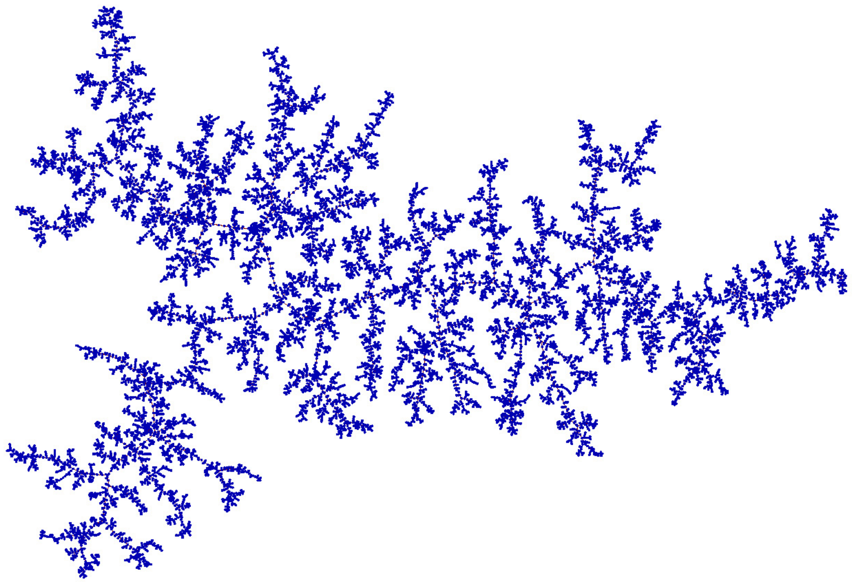

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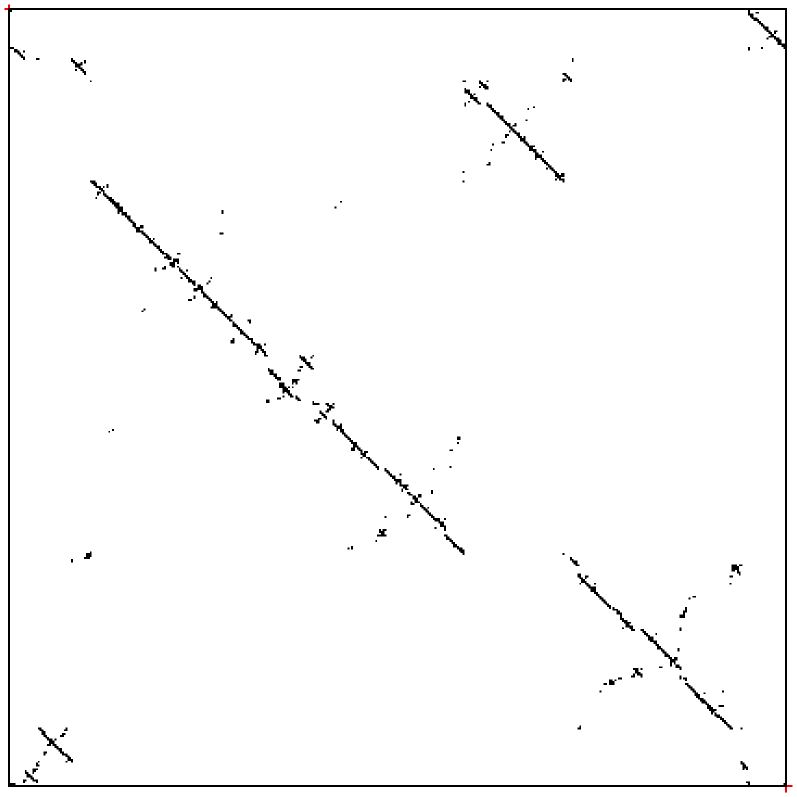
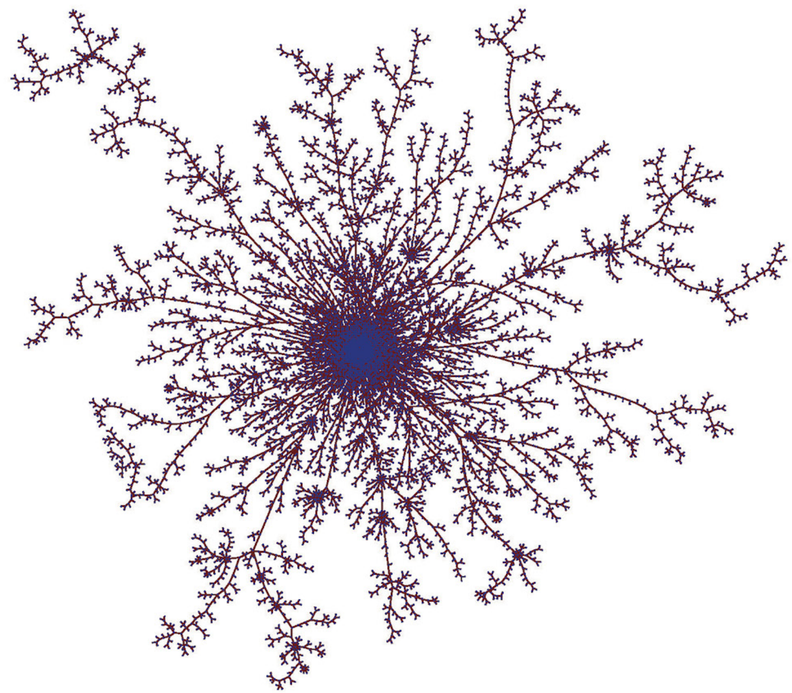


The regime $a > 0$ covers most known substitution-closed classes: $\mathcal{S}$ finite or subexponential, S rational,...

Degenerate case a<0

Regime where the decomposition tree exhibits a condensation phenomenon. Roughly, σ_n looks like a large uniform simple permutation in $\mathcal{S}$ and converges to the same limit ν.



Picture by I. Kortchemski

Example: Av(2413). We still need to understand the permuton limit of large simples in this class (+ technical hypotheses) to apply our theorem.

Stable permutons

Regime where the decomposition tree converges to a α-stable tree, α explicit.

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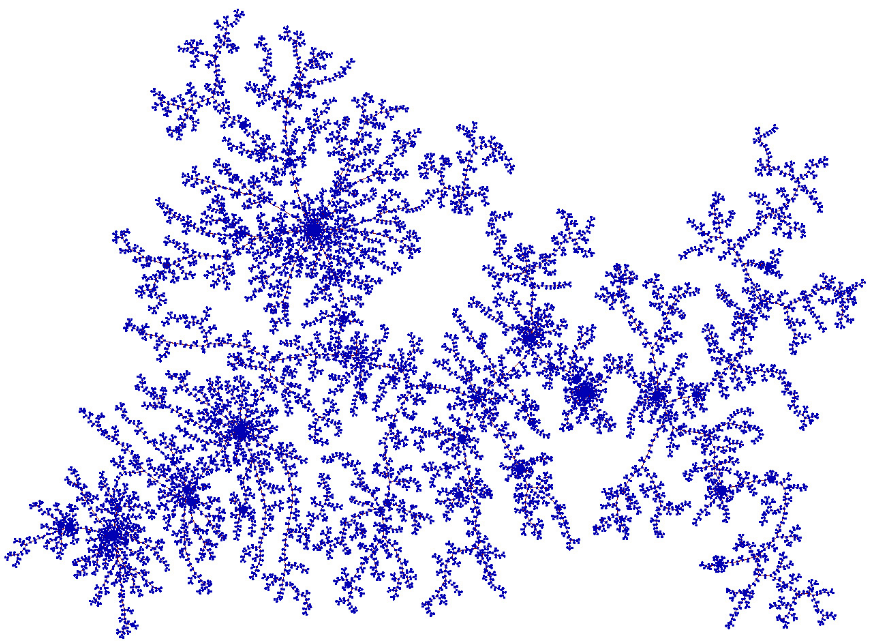
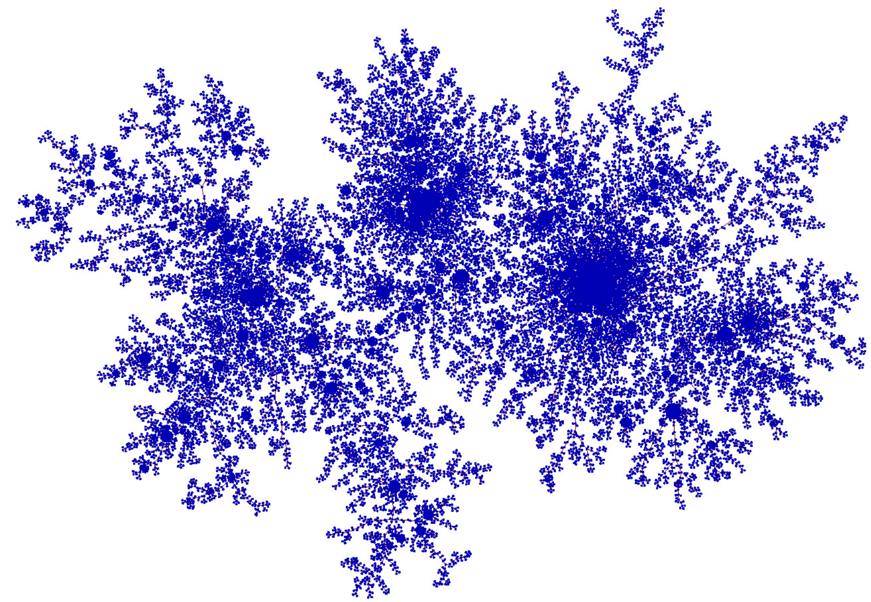
$\alpha = 1.5$



$\alpha = 1.1$



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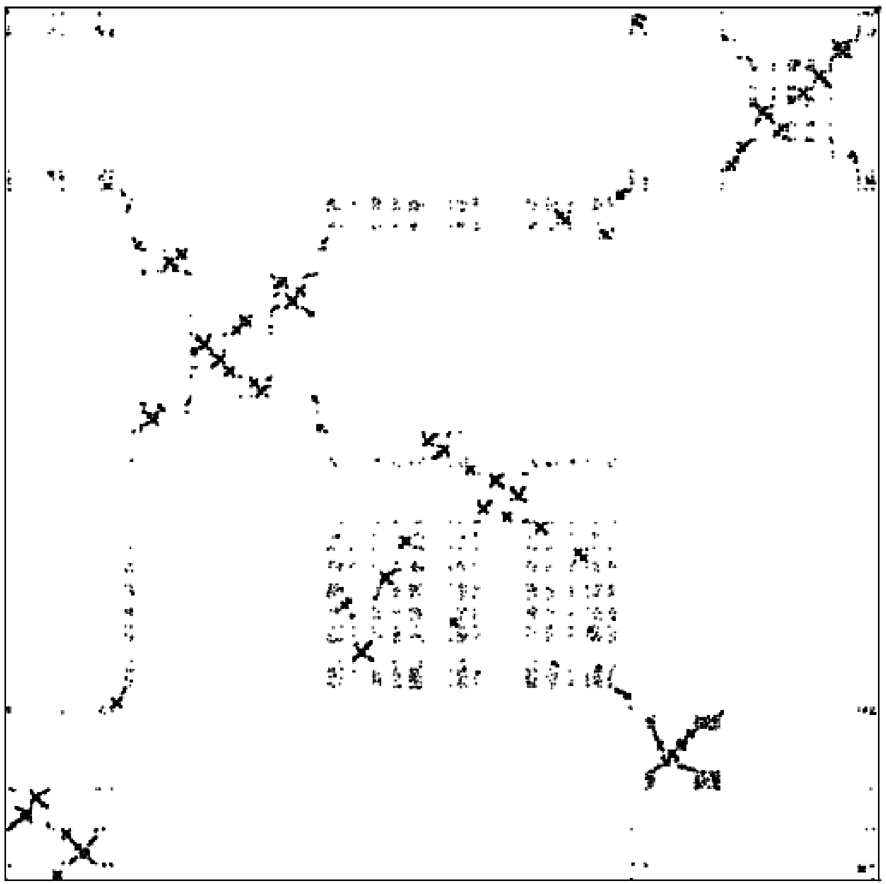

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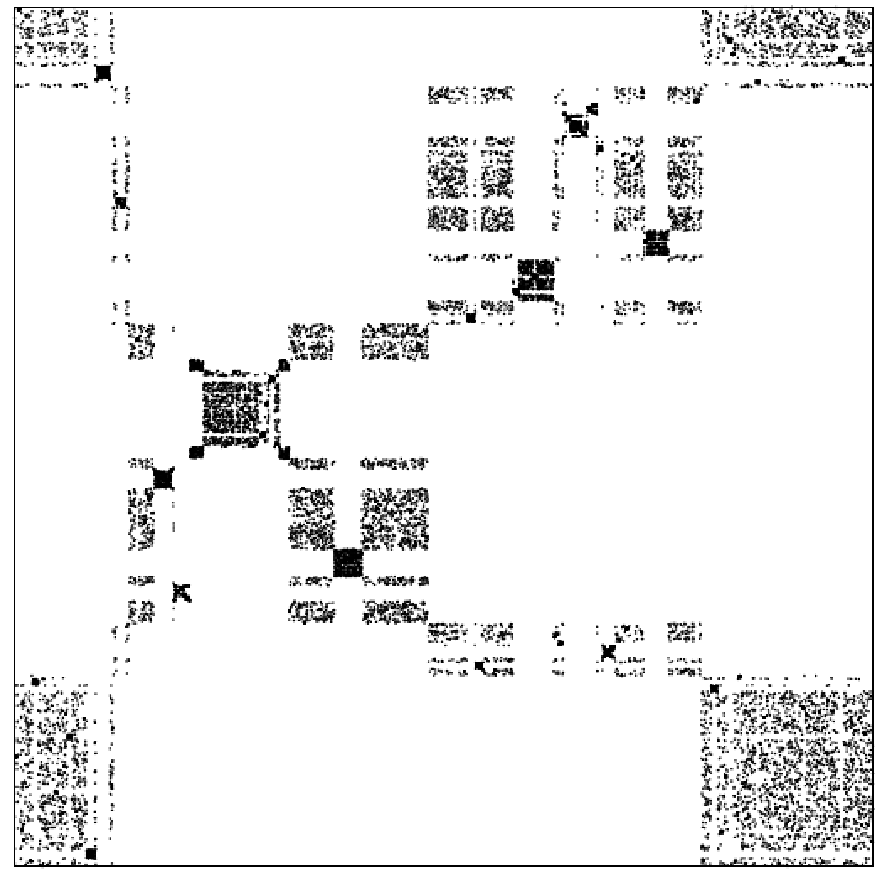
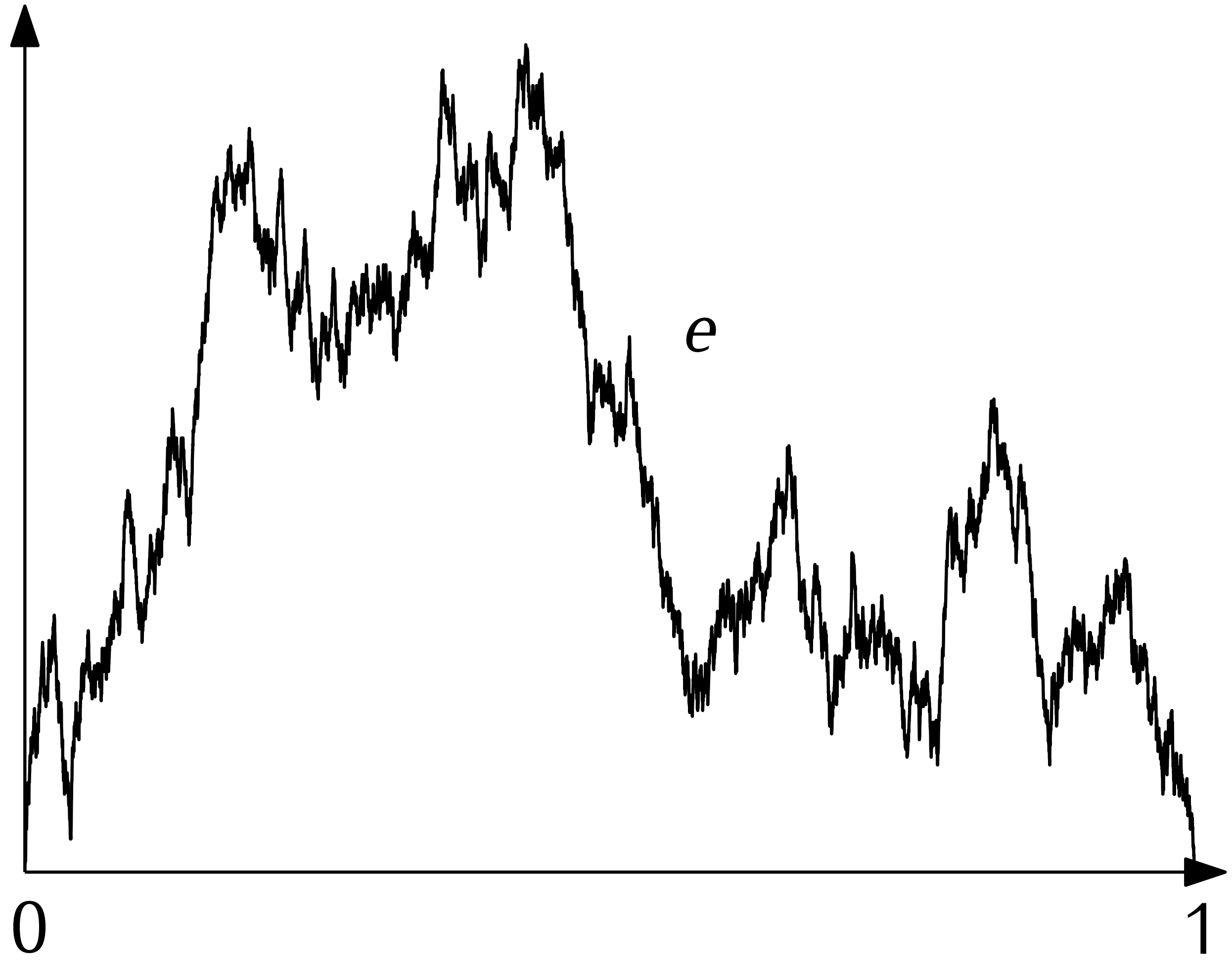


Branches from each infinite-degree point are reordered according to an independent copy of ν (the limit of large simples in the class)

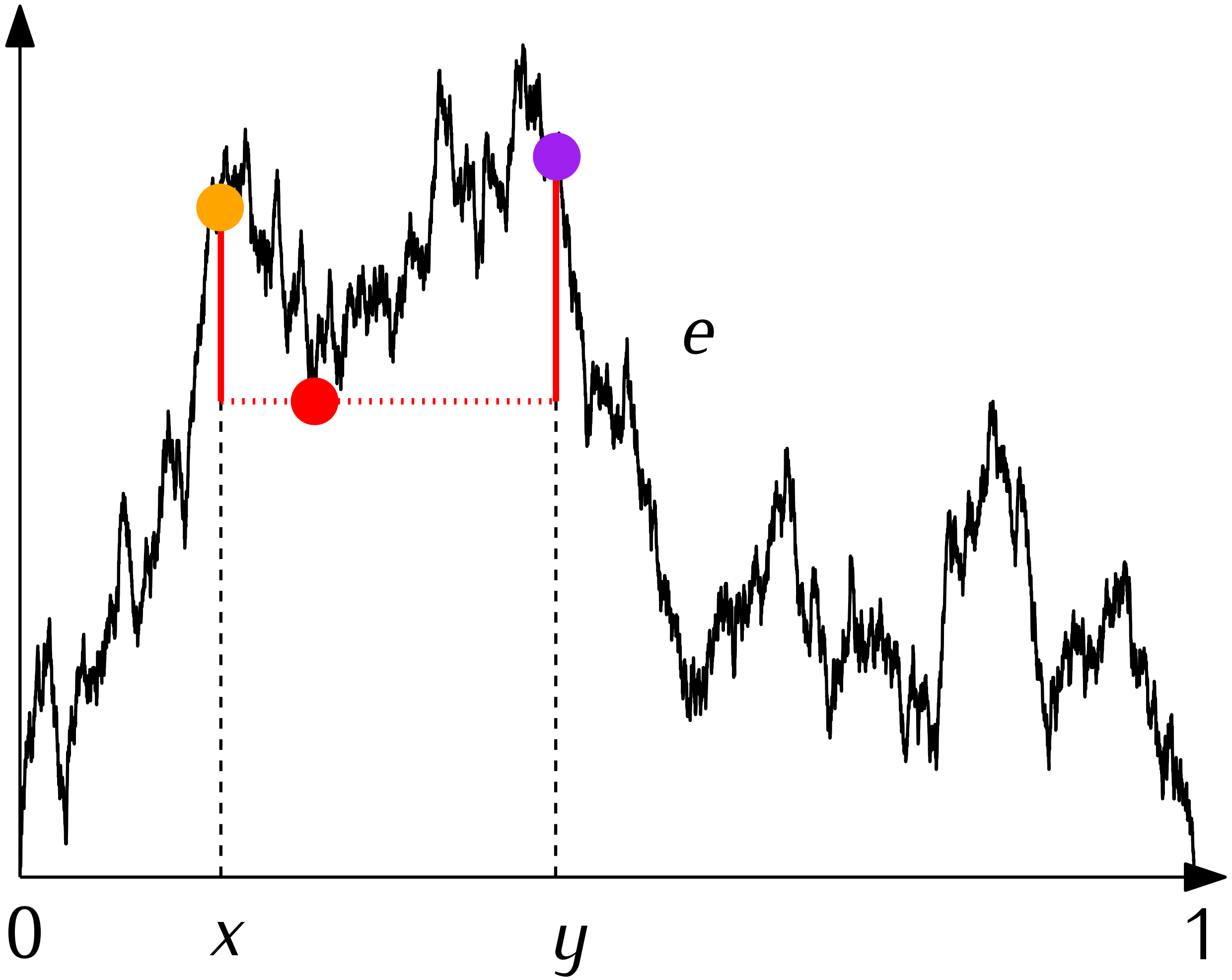
3 – Construction of the Brownian Permuton

[arXiv:1711.08986]

The (signed) Brownian excursion and CRT



The (signed) Brownian excursion and CRT



$$d_e(x, y) = e(x) + e(y) - 2\min_{[x,y]} e$$

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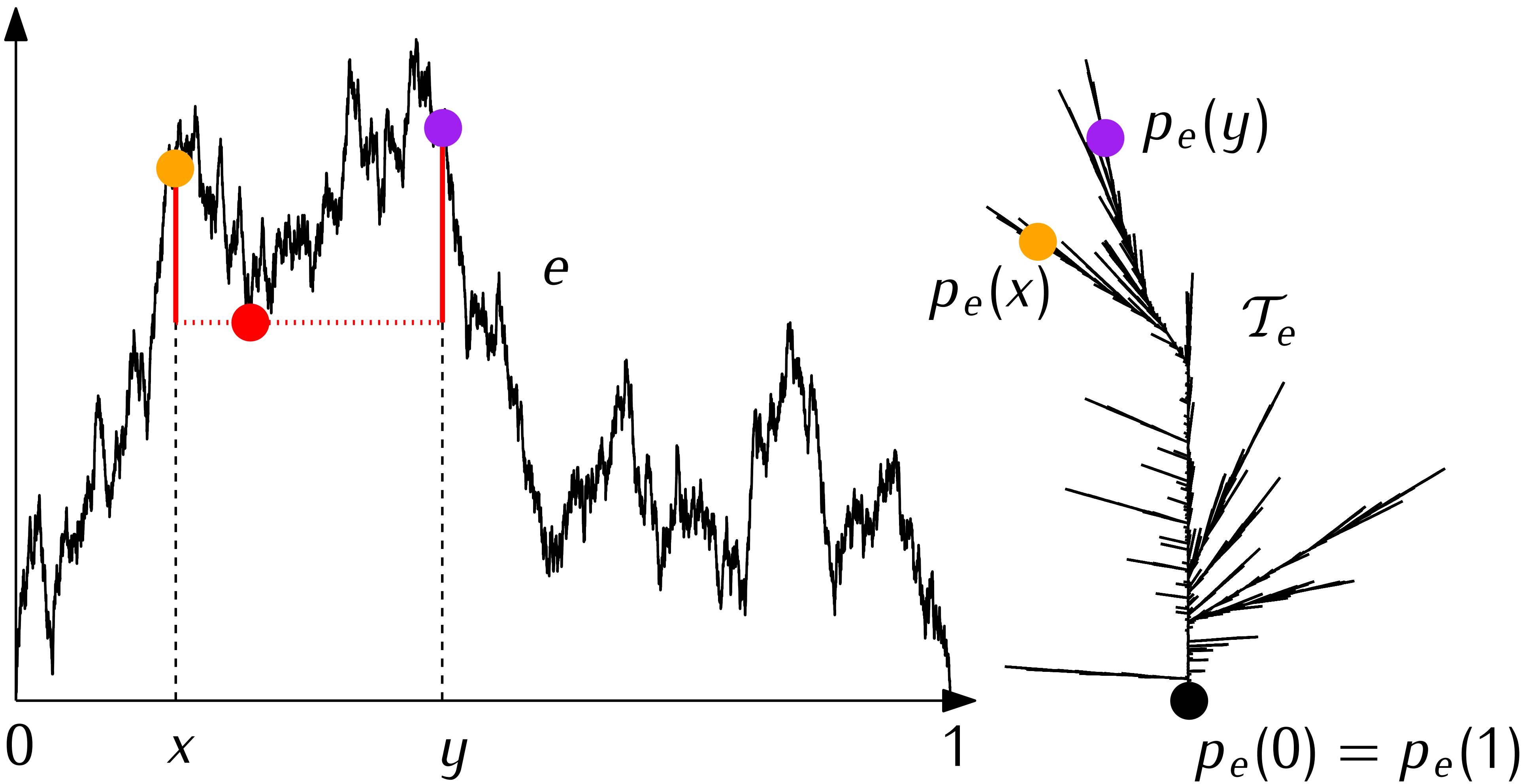

$d_e(x, y) = e(x) + e(y) - 2 \min_{[x,y]} e$ $\quad$ $\mathcal{T}_e = ([0,1]/\{d_e = 0\}, d_e)$

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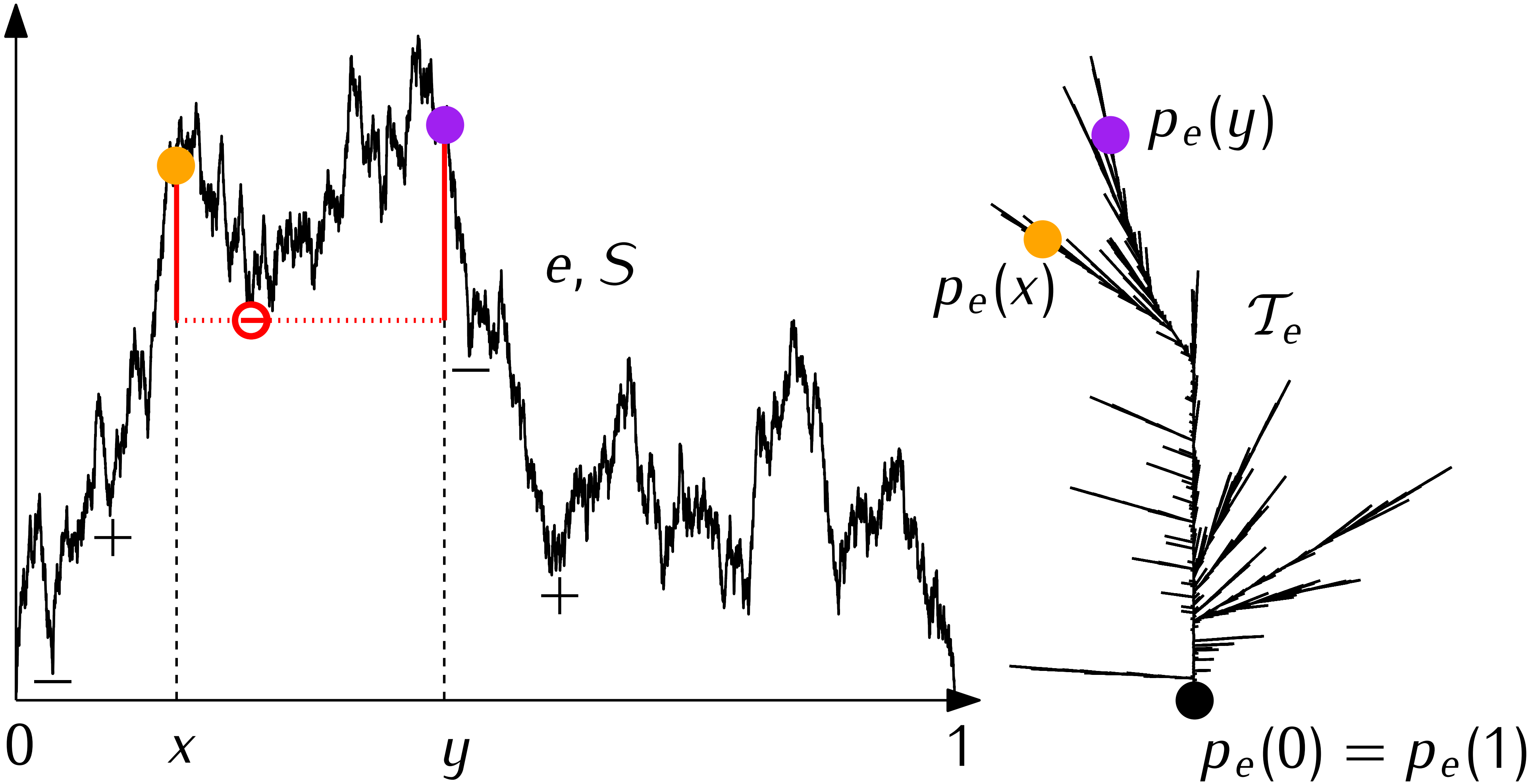
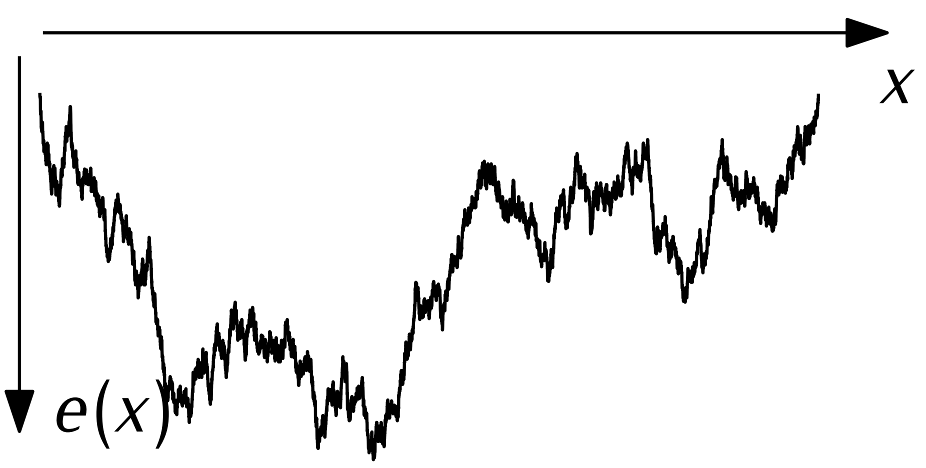


$d_e(x,y) = e(x) + e(y) - 2\min_{[x,y]} e \qquad \mathcal{T}_e = ([0,1]/\{d_e = 0\}, d_e)$

Constructing the Brownian permuton



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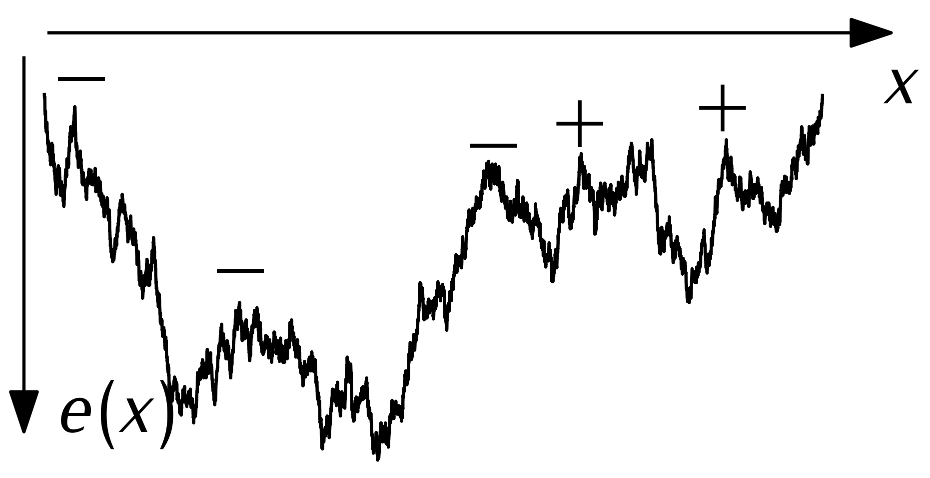
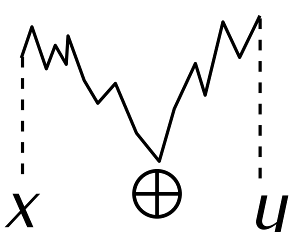


Constructing the Brownian permuton

e Brownian excursion, S i.i.d. signs indexed by the local minima of e.

Define a shuffled pseudo-order on $[0,1]$: $x \lhd_e^S y$ if and only if



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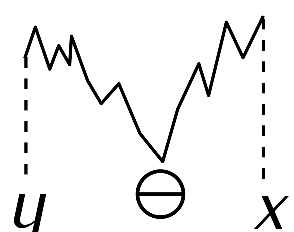

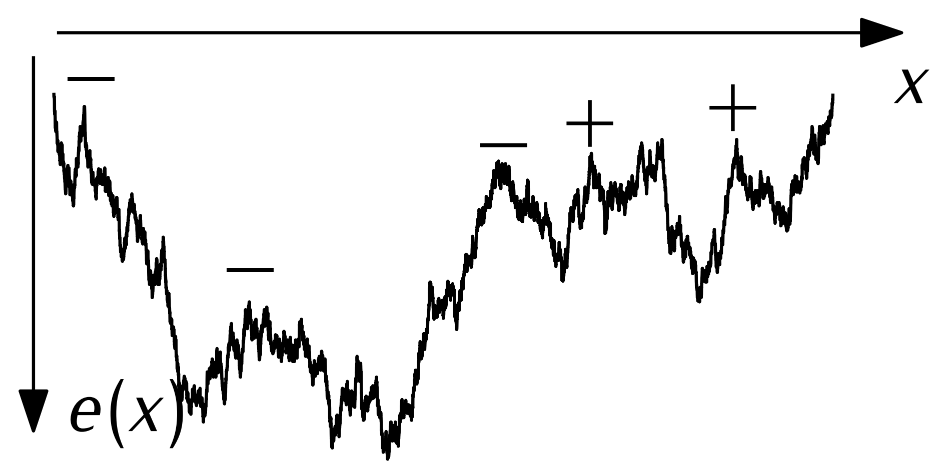

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Theorem (M. 2017)
Amost surely there exists a Lebesgue-preserving, $\lhd_e^S$-increasing function (unique up to a.e. equality). The random measure $(\mathrm{id}, \varphi)_\star\mathrm{Leb}$ has the law of the Brownian separable permuton.

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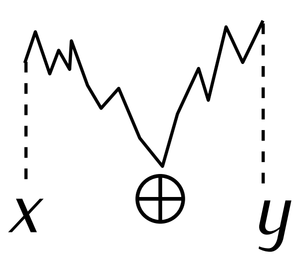

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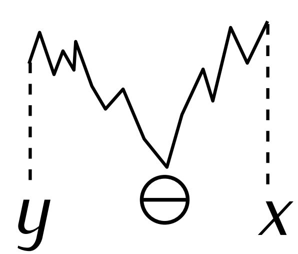

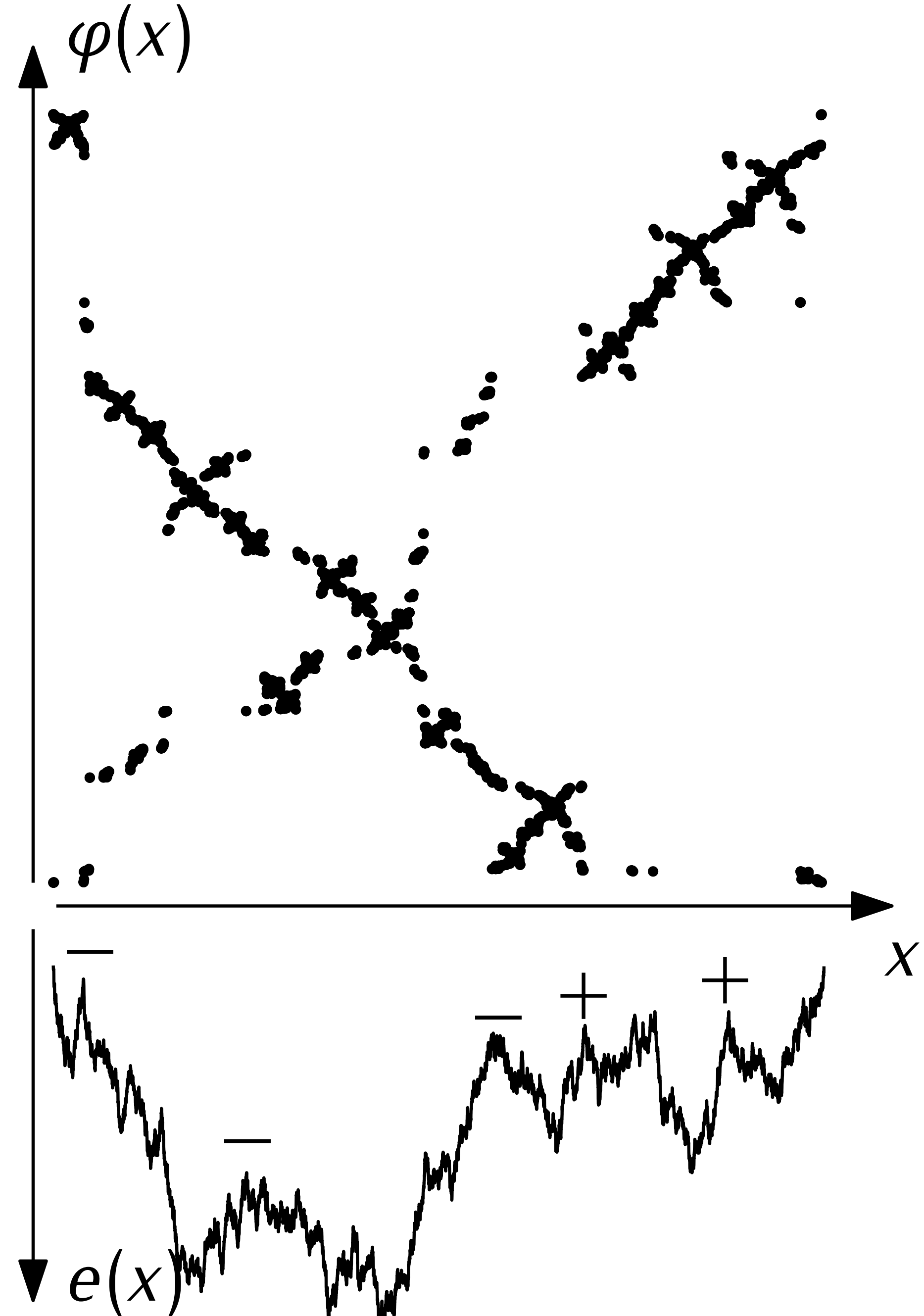

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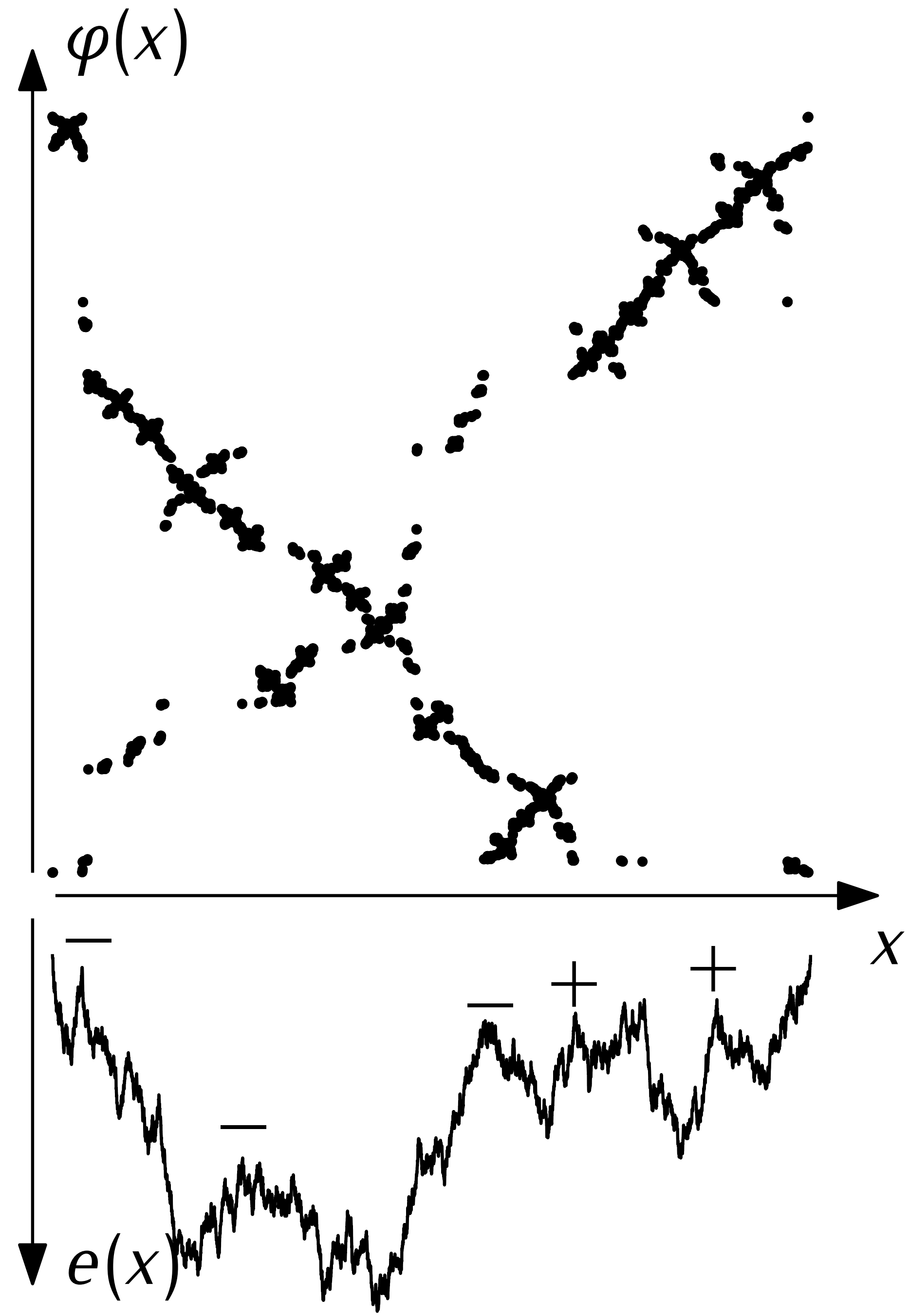
φ is continuous at every leaf (point which is not a one-sided local minimum) of e (full Lebesgue measure).

$\rightsquigarrow$ The support of μ is of Hausdorff dimension 1



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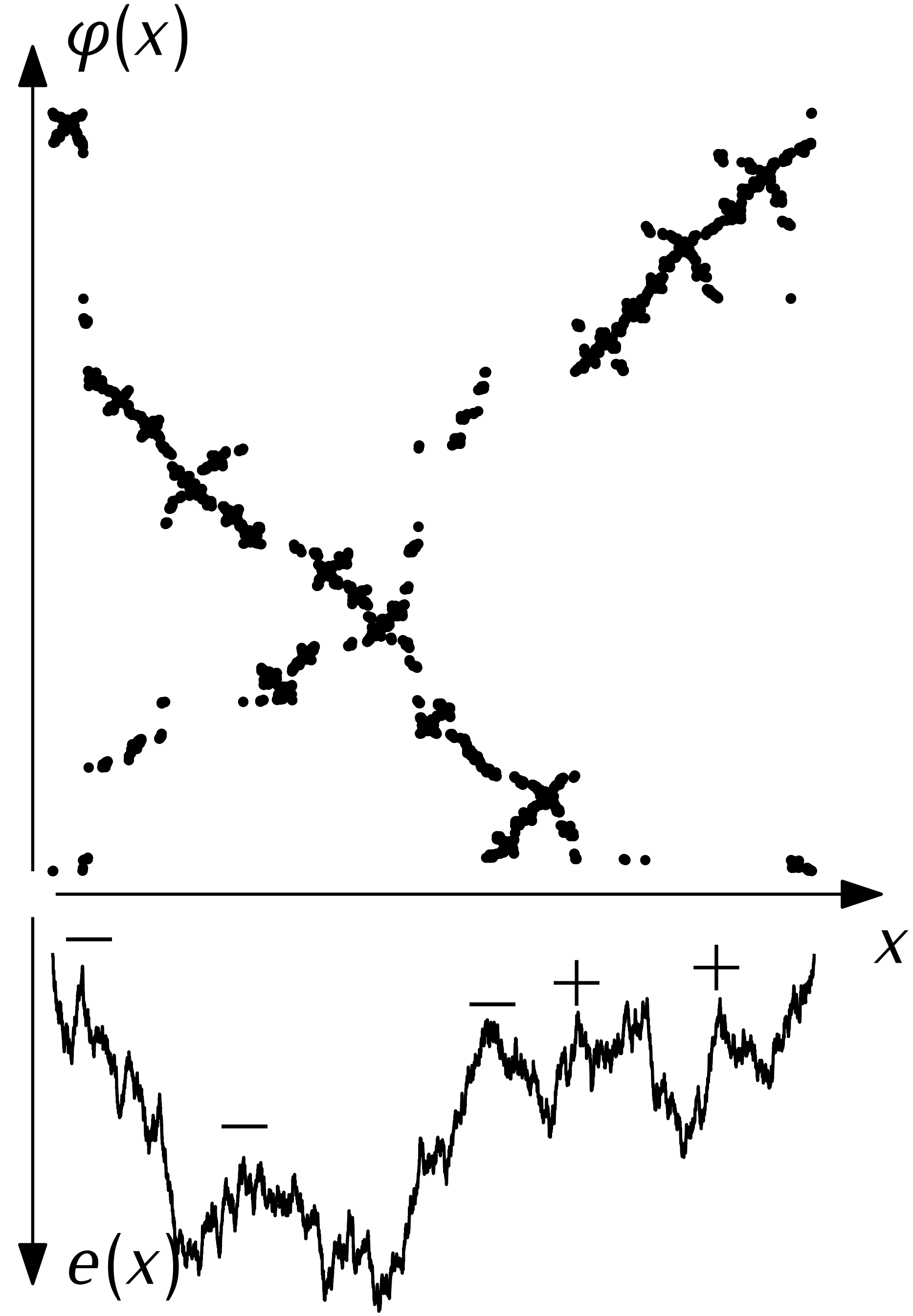
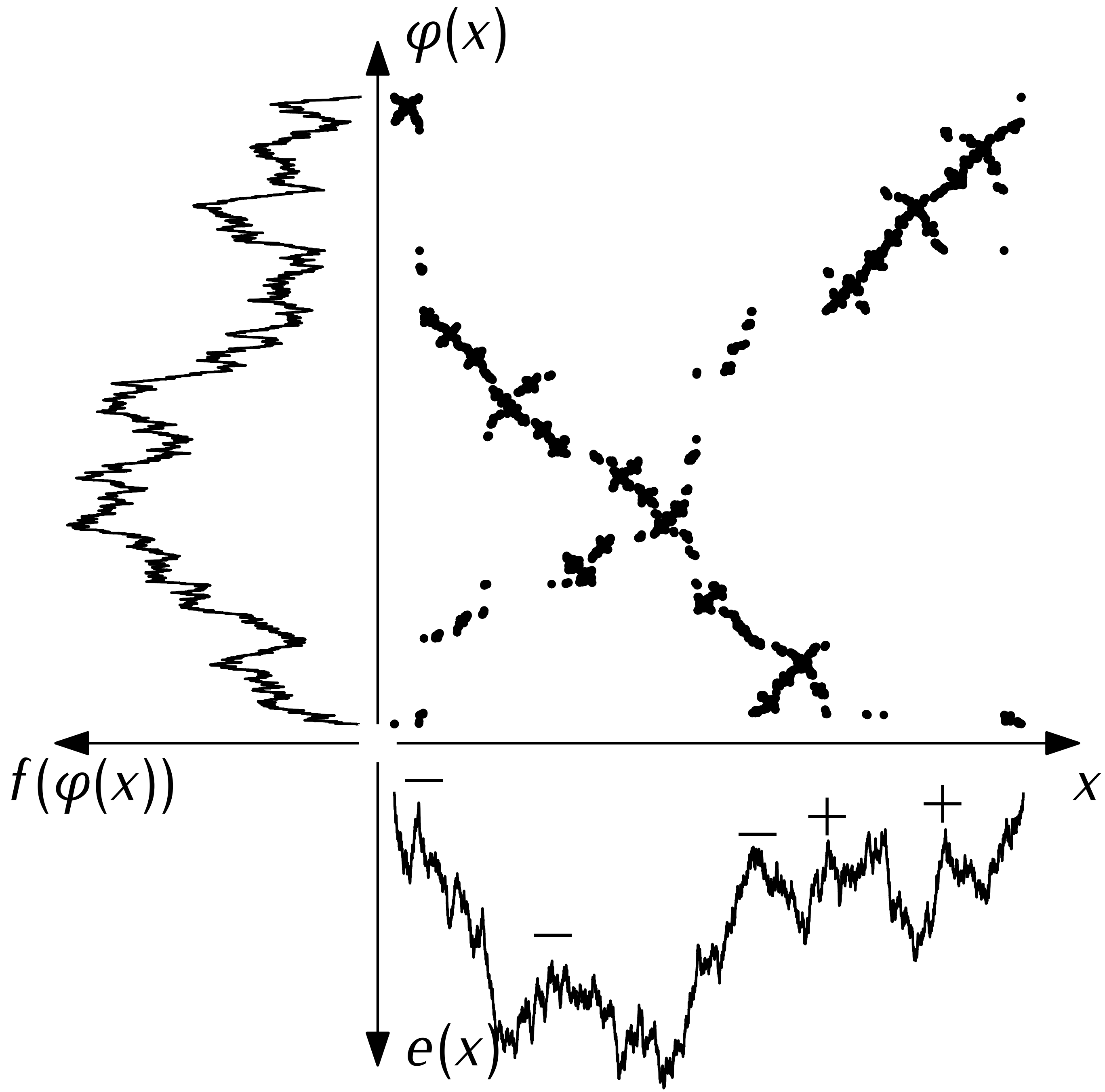
Discontinuities at every strict local minima of e (dense)
$\rightsquigarrow$ The support of μ is totally disconnected.



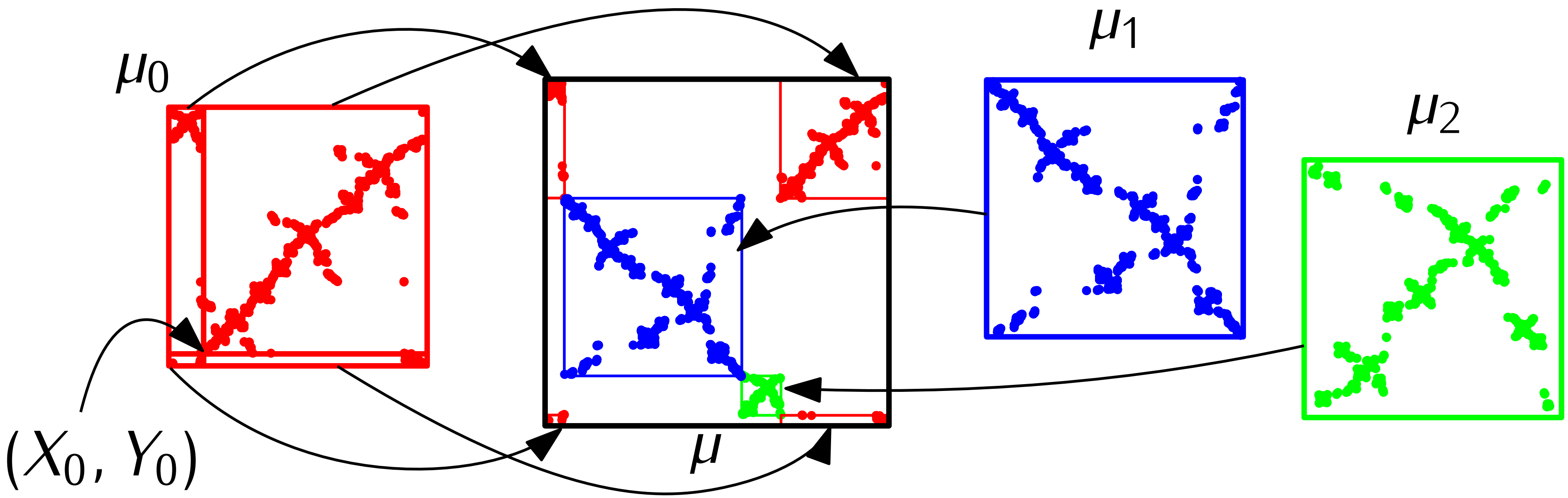
Constructing the Brownian permuton

There exists a Brownian excursion f defined on the same probability space such that $f \circ \varphi = e$.
a.s., $\mathcal{T}_f$ is isometric to $\mathcal{T}_e$.



Self-similarity

The Brownian permuton can be obtained by cut-and-pasting three independent copies in distribution of itself. The first copy μ_0 is cut according to a sample $(X_0, Y_0) \sim \mu_0$.
The scaling is an independent Dirichlet$(1/2, 1/2, 1/2)$ vector.
The relative position of μ_1 and μ_2 is chosen independently and uniformly between $\oplus$ and $\ominus$.



Thank you!