A new family of bijections for planar maps

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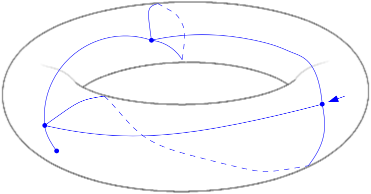
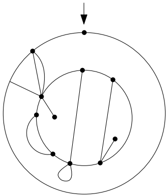
Introduction : maps

Map = connected multigraph G embedded in a compact oriented surface S up to homeomorphism
Face = connex components of $S \setminus G$, homeomorphic to disks
Genus $g = \#$handles



Rooted maps : distinguish a corner (kill symmetries)

Planar maps ($g = 0$) = drawn in the plane w/ root in the outer face



Context : KP hierarchy and existing formulas

KP hierarchy = infinite set of PDE's coming from mathematical physics
F = GF of maps, i-th variable counts vertices of degree i

$$F_{3,1} = F_{2,2} + \frac{1}{2}F_{1,1}^2 + \frac{1}{12}F_{1^4}$$
$$F_{4,1} = F_{3,2} + F_{1,1}F_{2,1} + \frac{1}{6}F_{1^3,2}$$
$$\dots$$

Context : KP hierarchy and existing formulas

Two (very combinatorial) quadratic recurrences come from KP :

Goulden-Jackson '08 (for triangulations)

$$(n+1)T(\underbrace{n}_{\frac{\#edges}{3}},\underbrace{g}_{genus}) = 4n(3n-2)(3n-4)T(n-2,g-1)+4(3n-1)T(n-1,g) + 4\sum_{\substack{i+j=n-2\\ i,j\geq 0}}\sum_{\substack{g_1+g_2=g\\ g_1,g_2\geq 0}}(3i+2)(3j+2)T(i,g_1)T(j,g_2)+2\mathbb{1}_{n=g=1},$$

Carell-Chapuy '14 (for general maps)

$$(n+1)Q_{\underbrace{g}_{genus}}(\underbrace{n}_{\#edges},\underbrace{f}_{\#faces}) = 2(2n-1)Q_g(n-1,f)+2(2n-1)Q_g(n-1,f-1) + (2n-3)(n-1)(2n-1)Q_{g-1}(n-2,f) + 3\sum_{\substack{i+j=n-2\\ i,j\geq 0}}\sum_{\substack{f_1+f_2=f\\ f_1,f_2\geq 1}}\sum_{\substack{g_1+g_2=g\\ g_1,g_2\geq 0}}(2i+1)(2j+1)Q_{g_1}(i,f_1)Q_{g_2}(j,f_2)$$

Bijections

Bijective explanation :
Only case known before : $f = 1$ [Chapuy-Féray-Fusy '12] :

Harer-Zagier recurrence formula '86

$$(n+1)\epsilon_g(n) = 2(2n-1)\epsilon_g(n-1) + (n-1)(2n-1)(2n-3)\epsilon_{g-1}(n-2)$$

For $g = 0$ (my contribution) :

Carell-Chapuy planar case

$$\begin{aligned}(n+1)Q(n,f) =& 2(2n-1)Q(n-1,f) + 2(2n-1)Q(n-1,f-1) \\ &+ 3 \sum_{\substack{i+j=n-2 \\ i,j\geq 0}} \sum_{\substack{f_1+f_2=f \\ f_1,f_2\geq 1}} (2i+1)(2j+1)Q(i,f_1)Q(j,f_2)\end{aligned}$$

Two bijections that recombine :
Cut-slide bijection :

$$\underbrace{(f-1)}_{??}Q(n,f) = \sum_{\substack{i+j=n-1\\ i,j\geq 0}} \sum_{\substack{f_1+f_2=f\\ f_1,f_2\geq 1}} \underbrace{v_1}_{\substack{vertex\\ (v_1=2+i-f_1)}} Q(i,f_1)\underbrace{(2j+1)}_{corner}Q(j,f_2)$$

Generalized Rémy bijection :

$$vQ(n,f) = 2(2n-1)Q(n-1,f) + \sum_{\substack{i+j=n-1\\ i,j\geq 0}} \sum_{\substack{f_1+f_2=f\\ f_1,f_2\geq 1}} v_1 Q(i,f_1) v_2 Q(j,f_2)$$

(Rémy bijection = growing trees recursively)

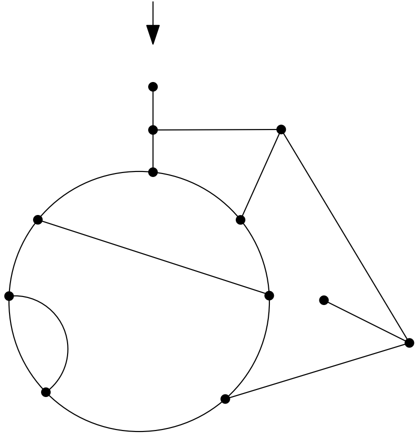
Bijections

Two steps :

- exploration of the map
- cut and slide operation

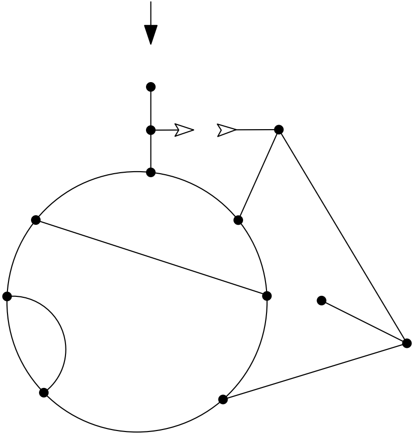
Exploration

Start from the root and go along the edges . . .



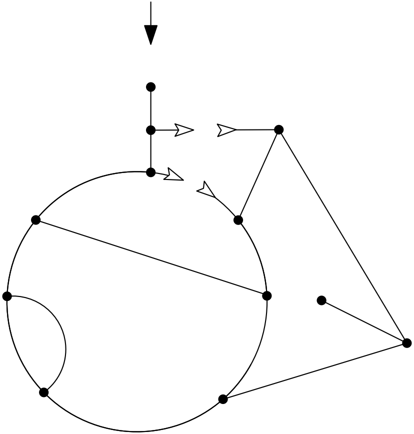
Exploration

. . . open an edge whenever another face is found . . .



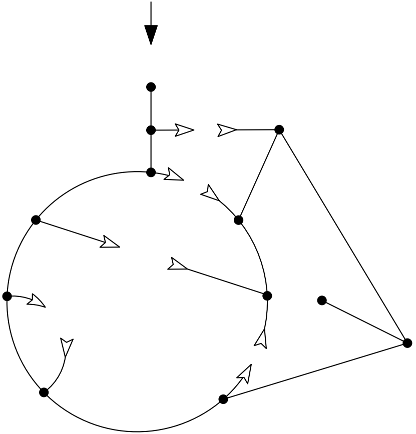
Exploration

... go on ...



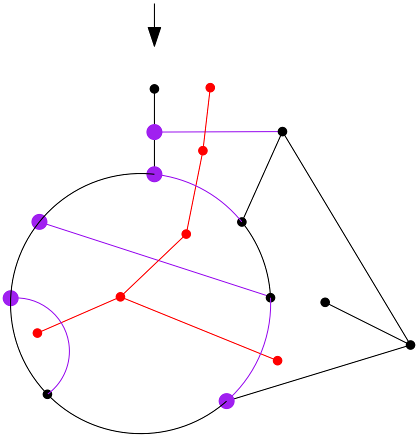
Exploration

... until you're back at the root ...



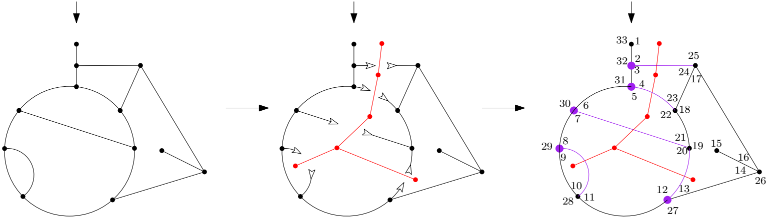
Exploration

. . . and close the map back.



Exploration

- Equivalent to a rightmost DFS on the dual
- Edges opened during the exploration = discoveries ($f - 1$ in total)
- Order on the corners, preorder on the faces (and discoveries)



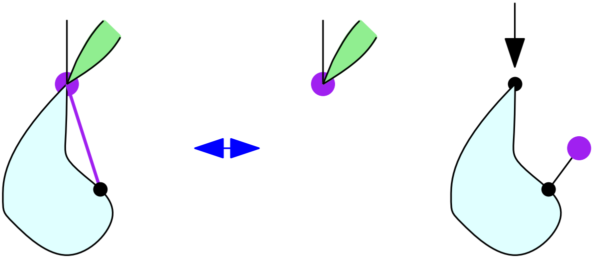
$$\underbrace{(f-1)Q(n,f)}=\underbrace{\sum_{\substack{i+j=n-1\\ i,j\geq 0}}\ \sum_{\substack{f_1+f_2=f\\ f_1,f_2\geq 1}} v_1 Q(i,f_1)(2j+1)Q(j,f_2)}$$

a map with a marked discovery

a pair of maps, one with a marked vertex, the other with a marked corner

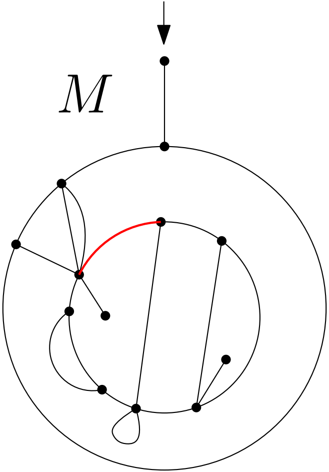
Cut and slide operation

Disconnecting discovery e : the corner before e and the last corner of the discovery vertex v lie in the same face
Can split the map in two at v



Cut and slide operation

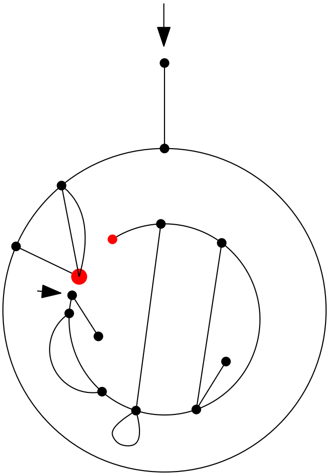
Start with M and a marked disconnecting discovery . . .



Case 1 : disconnecting discovery

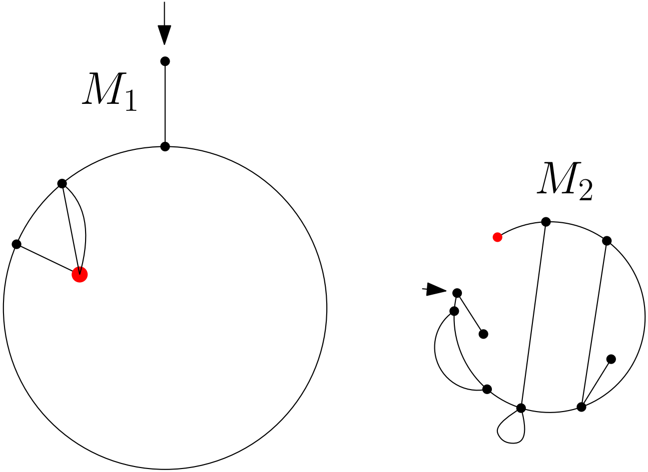
Cut and slide operation

... split the discovery ...



Case 1 : disconnecting discovery

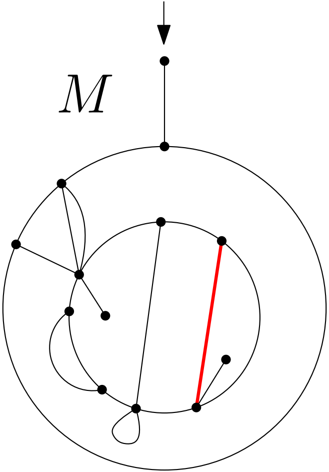
... and obtain M_1 (marked vertex) and M_2 (marked leaf)



Case 1 : disconnecting discovery

Cut and slide operation

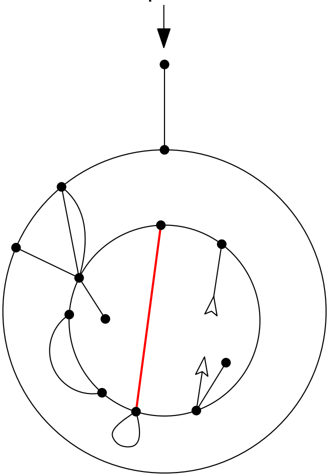
Start with M and a marked non-disconnecting discovery . . .



Case 2 : non-disconnecting discovery

Cut and slide operation

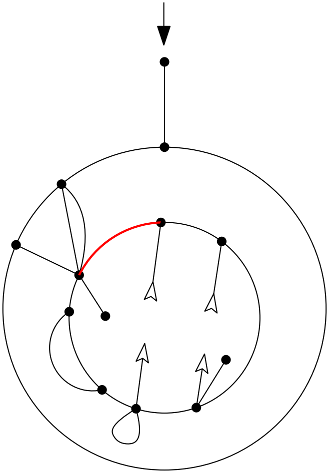
... open the discovery and look at the previous discovery ...



Case 2 : non-disconnecting discovery

Cut and slide operation

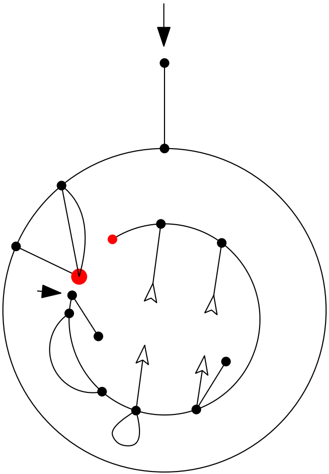
... iterate until a disconnecting discovery is found ...



Case 2 : non-disconnecting discovery

Cut and slide operation

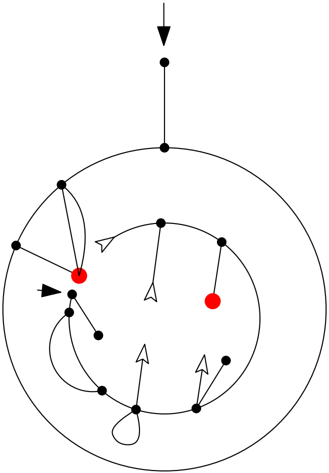
. . . then split it . . .



Case 2 : non-disconnecting discovery

Cut and slide operation

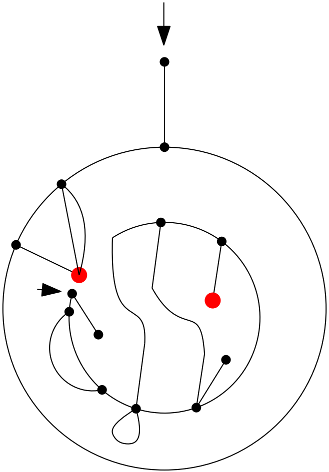
. . . "slide" . . .



Case 2 : non-disconnecting discovery

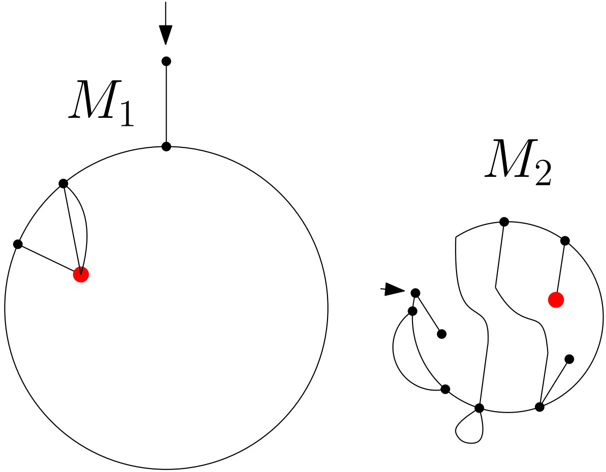
Cut and slide operation

... and glue back the edges ...



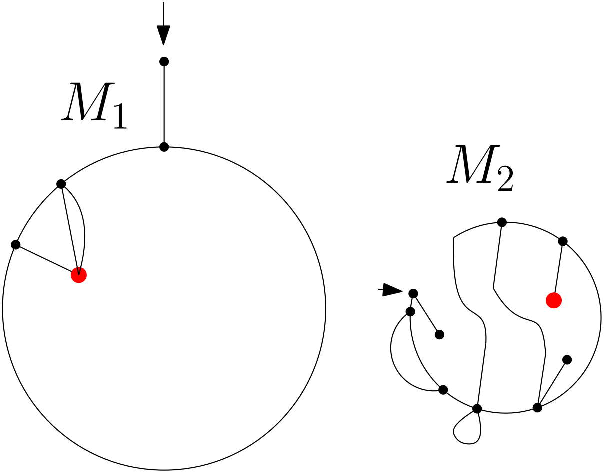
Case 2 : non-disconnecting discovery

. . . and obtain M_1 (marked vertex) and M_2 (marked leaf)



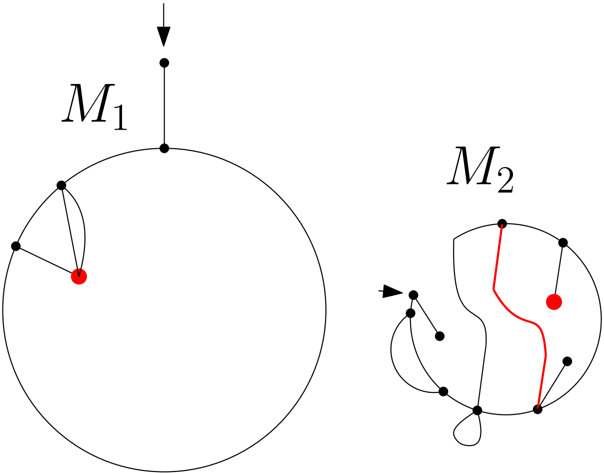
Case 2 : non-disconnecting discovery

Going backwards :

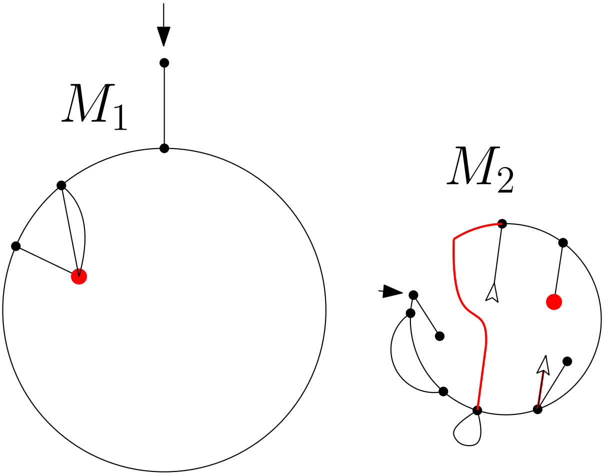


Cut and slide operation

Going backwards :

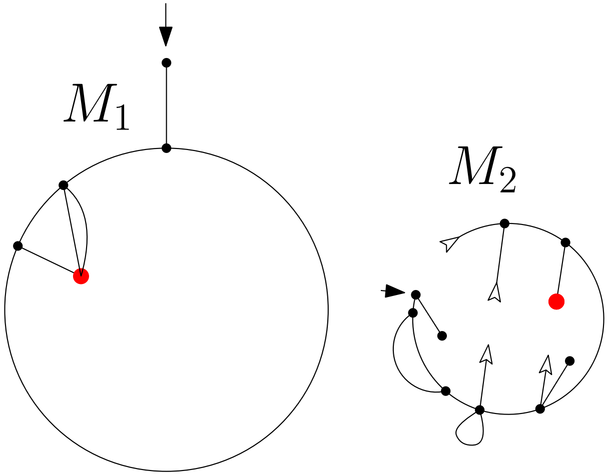


Going backwards :



Cut and slide operation

Going backwards :



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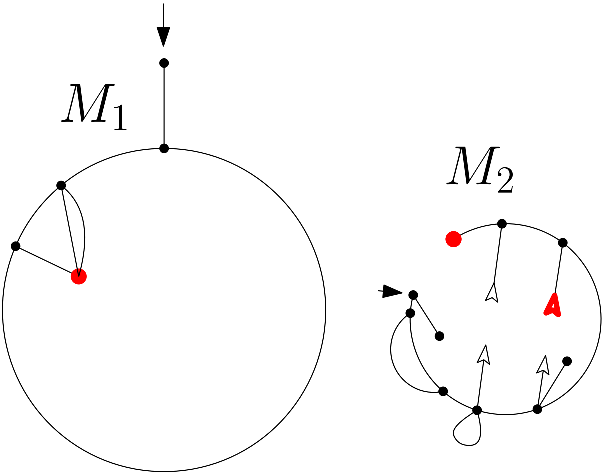
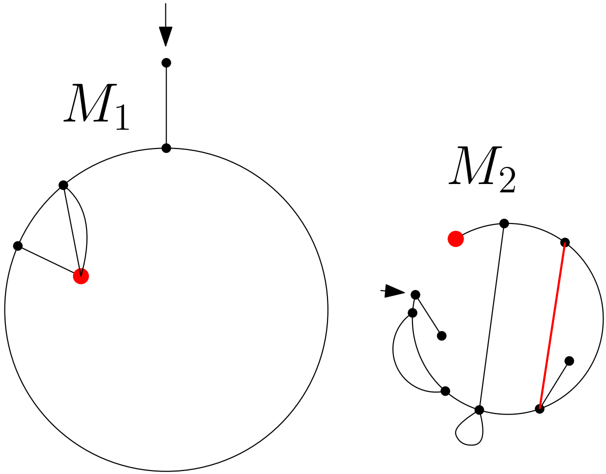


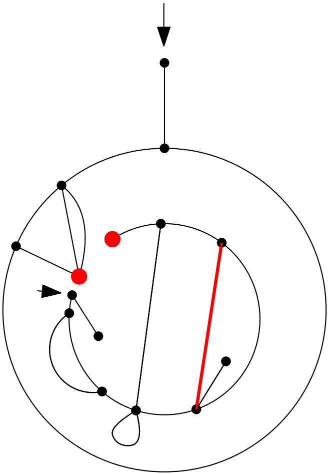
Cut and slide operation

Going backwards :



Cut and slide operation

Going backwards :



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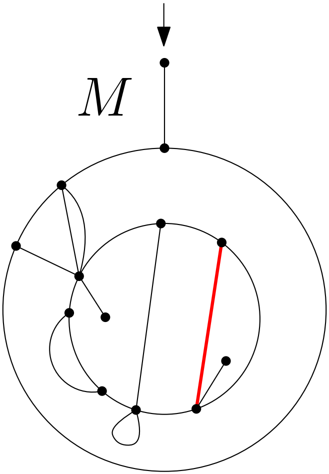
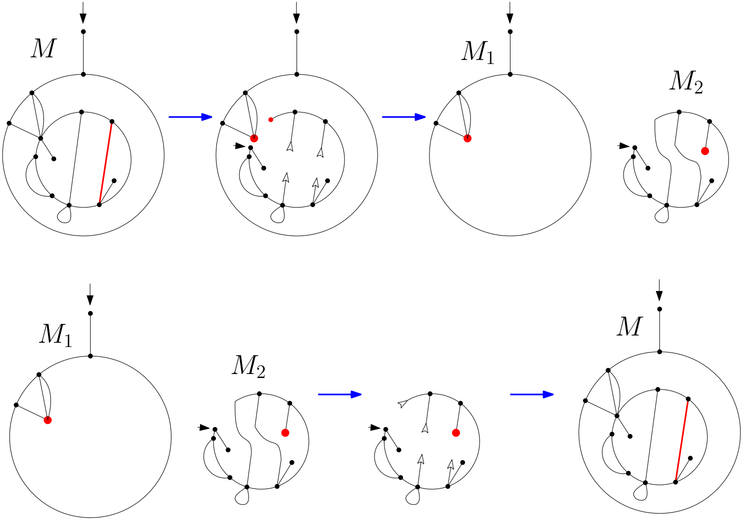


Cut and slide operation

Summary :



Generalized Rémy bijection :

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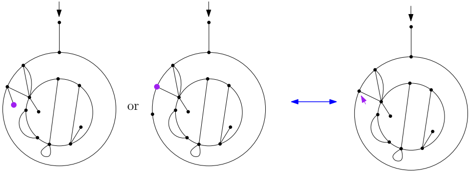
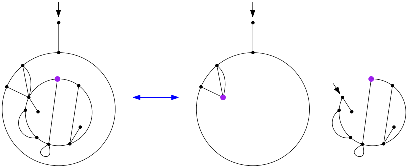
- Mark a vertex
- do Rémy operation if possible
- Otherwise, there is a discovery involved → cut and slide operation





Conclusion

- Summing both formulas + calculations $\rightarrow$ CC planar (general maps)
- Almost all $deg(v)$ remain constant $\rightarrow$ there are versions of the recurrences recording the degrees
- GJ planar (triangulations) is just a special case of the cut-slide bijection

Interesting perspectives :

- cut and slide operations are nice (link with [Bettinelli '14] ?) :
  - although non-local, don't modify the map a lot
  - act directly on the map
- most explorations are geodesic (aka BFS), DFS explorations worth investigating
- big hope : unite work for $f > 1, g = 0$ and $f = 1, g > 0$ to get a general bijective proof of KP-based formulas
  - A very special case : GJ for cubic maps (dual to triangulations) with 2 faces ✓

Thank you !