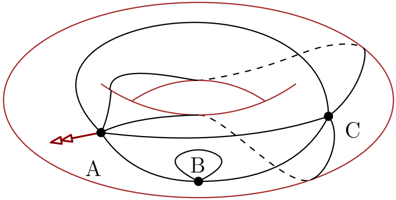
Bijection for higher-genus maps, rationality of their generating function.

Mathias Lepoutre

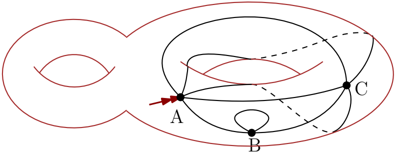
LIX, École polytechnique

March 14, 2018

Maps

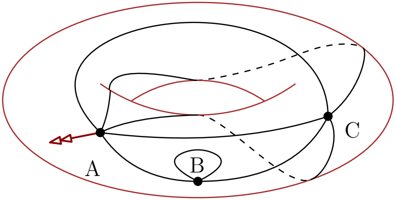


A rooted map on the 1-torus.

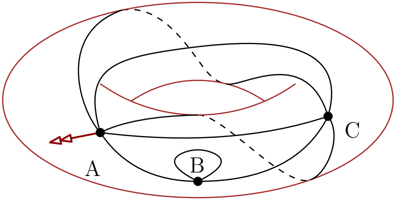


Not a map

Maps

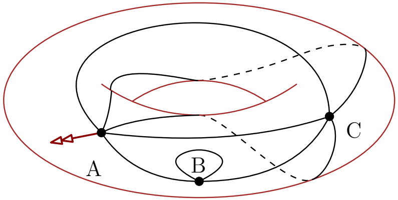


A rooted map

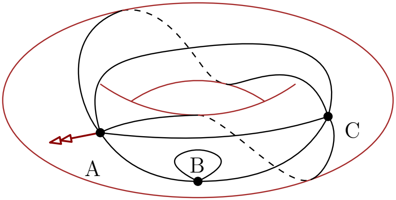


The same map

Maps



A rooted map

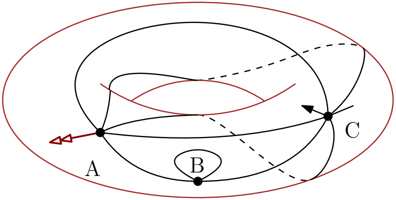


The same map

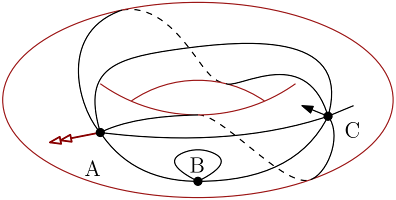
Definition

unicellular: *only one face*

Maps



A rooted map



The same map

Definition

unicellular: *only one face*

blossoming: *with some (ingoing or outgoing) stems on corners*

Introduction

Theorem (Tutte 60's for $g = 0$, Bender Canfield 91 for $g > 0$)

For any $g \geq 0$, the generating series $M_g(z)$ of maps of genus g enumerated by edges is a rational function of z and $\sqrt{1-12z}$.

Introduction

Theorem (Tutte 60's for $g = 0$, Bender Canfield 91 for $g > 0$)

For any $g \geq 0$, the generating series $M_g(z)$ of maps of genus g enumerated by edges is a rational function of z and $\sqrt{1-12z}$.

Labeled maps (mobiles):

- in the plane: Cori Vauquelin 81, Schaeffer 98, Bouttier Di Francesco Guitter 04...
- in higher genus: Chapuy Marcus Schaeffer 09...
- on non-orientable surfaces: Bettinelli 16, Chapuy Dolega 17...

Introduction

Theorem (Tutte 60's for $g = 0$, Bender Canfield 91 for $g > 0$)

For any $g \geq 0$, the generating series $M_g(z)$ of maps of genus g enumerated by edges is a rational function of z and $\sqrt{1 - 12z}$.

Labeled maps (mobiles):

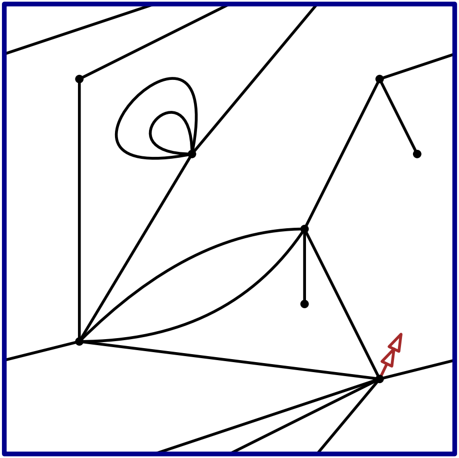
- in the plane: Cori Vauquelin 81, Schaeffer 98, Bouttier Di Francesco Guitter 04...
- in higher genus: Chapuy Marcus Schaeffer 09...
- on non-orientable surfaces: Bettinelli 16, Chapuy Dolega 17...

Blossoming maps:

- in the plane: Schaeffer 97, Bouttier Di Francesco Guitter 02, Poulhalon Schaeffer 06, Bernardi 07...
- in higher genus: Despres Gonçalves Leveque 17, Bernardi Chapuy 11...

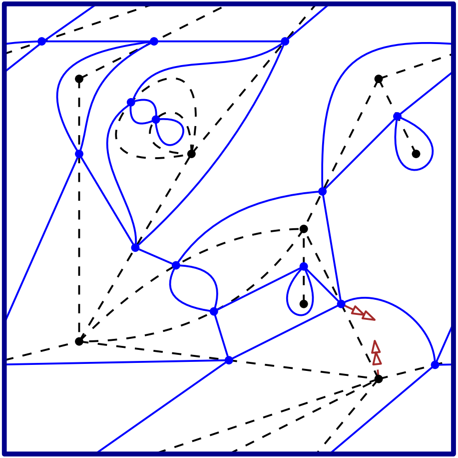
Sommaire

The radial map



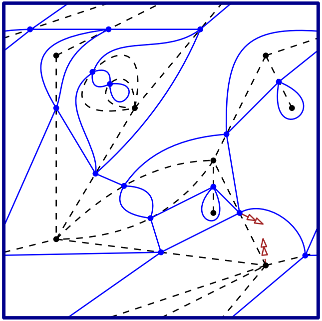
A classical representation of a toroidal map

The radial map



The radial construction

The radial map



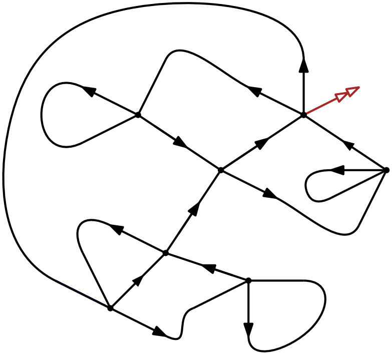
The radial construction

Proposition

There is a bijection between:

- *general maps of genus g with n edges, and*
- *4-valent bicolorable maps of genus g with n vertices.*

Bicolorable orientations



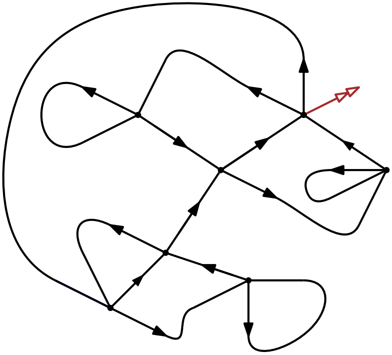
A 4-valent map with an Eulerian orientation

Definition

Eulerian map: all vertices have even degree

Eulerian orientation: all vertices have equal out- and in-degrees

Bicolorable orientations

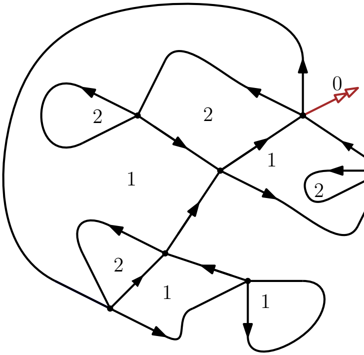


This orientation has no clockwise face...

Theorem

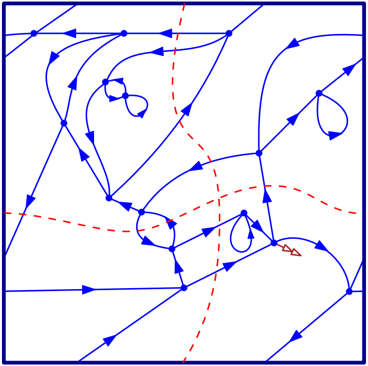
An Eulerian map has a unique Eulerian orientation with no clockwise face.

Bicolorable orientations



... It is the dual-geodesic orientation

Bicolorable orientations



A 4-valent bicolorable toroidal map along with its dual-geodesic orientation

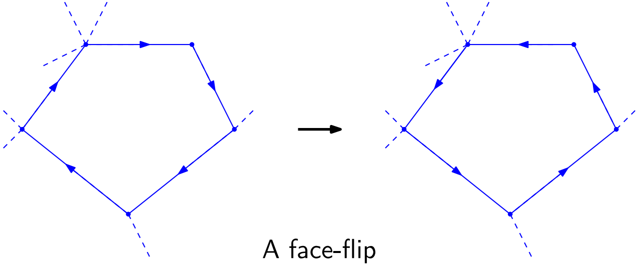
Definition

Bicolorable orientation: any dual cycle has as many edges going to the left and to the right.

Bicolorable orientations

Theorem (Propp 93)

The set of bicolorable orientations of a fixed map with face-flip (except for the root-face) as a cover relation forms a distributive lattice.
Its minimum is the dual-geodesic orientation.



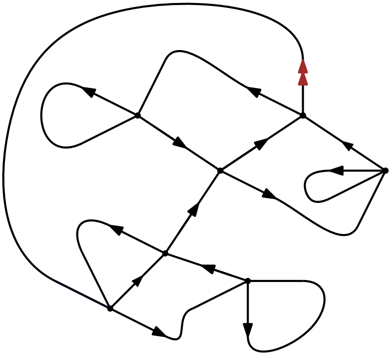
A face-flip

Corollary

The dual-geodesic orientation is the unique bicolorable orientation with no clockwise face.

Sommaire

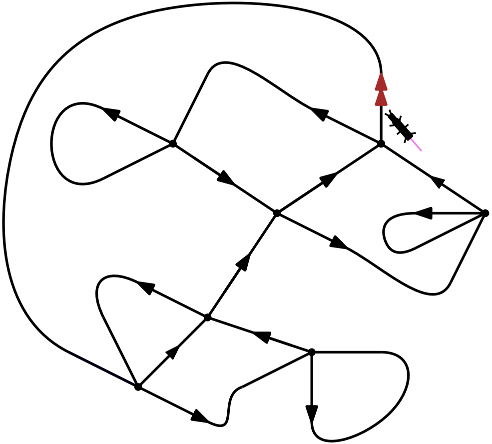
Opening a 4-valent planar map [Schaeffer 97]



A 4-valent (Eulerian) planar map with its dual-geodesic orientation

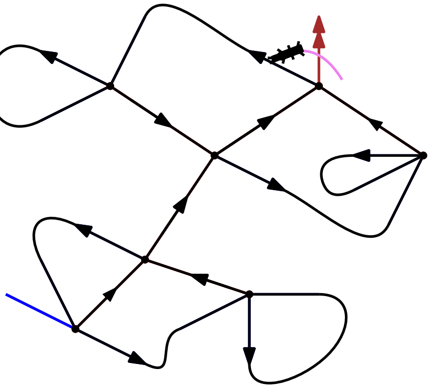
Opening a 4-valent planar map [Schaeffer 97]

- An unvisited outgoing edge is cut
- An unvisited ingoing edge is followed
- A visited leaf is ignored
- A visited outoing edge is followed



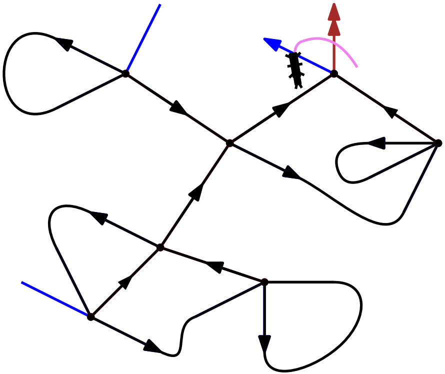
Opening a 4-valent planar map [Schaeffer 97]

- An unvisited ingoing edge is followed
- An unvisited outgoing edge is cut
- A visited leaf is ignored
- A visited outoing edge is followed



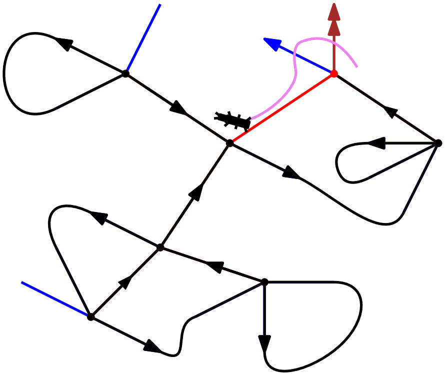
Opening a 4-valent planar map [Schaeffer 97]

- An unvisited ingoing edge is followed
- An unvisited outgoing edge is cut
- A visited leaf is ignored
- A visited outoing edge is followed



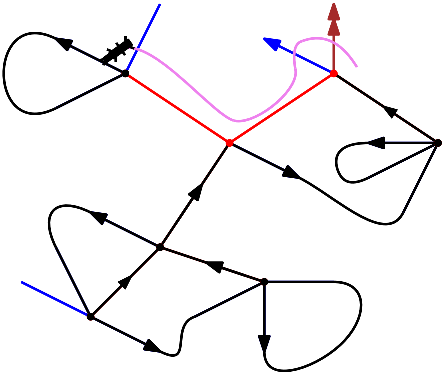
Opening a 4-valent planar map [Schaeffer 97]

- An unvisited ingoing edge is followed
- An unvisited outgoing edge is cut
- A visited leaf is ignored
- A visited outoing edge is followed



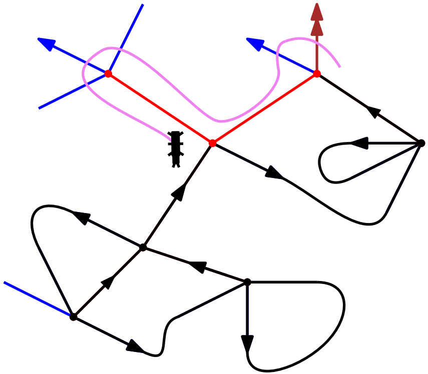
Opening a 4-valent planar map [Schaeffer 97]

- An unvisited ingoing edge is followed
- An unvisited outgoing edge is cut
- A visited leaf is ignored
- A visited outoing edge is followed



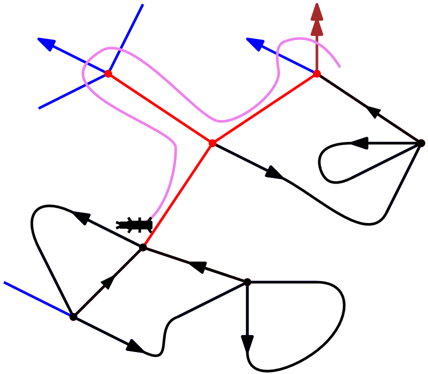
Opening a 4-valent planar map [Schaeffer 97]

- An unvisited ingoing edge is followed
- An unvisited outgoing edge is cut
- A visited leaf is ignored
- A visited outoing edge is followed



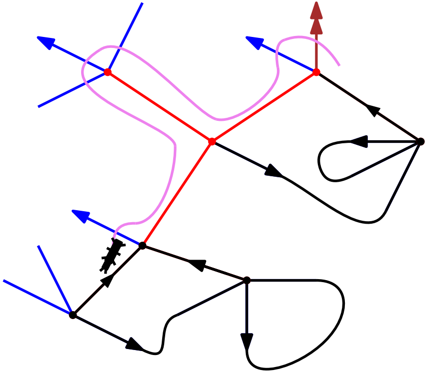
Opening a 4-valent planar map [Schaeffer 97]

- An unvisited ingoing edge is followed
- An unvisited outgoing edge is cut
- A visited leaf is ignored
- A visited outoing edge is followed



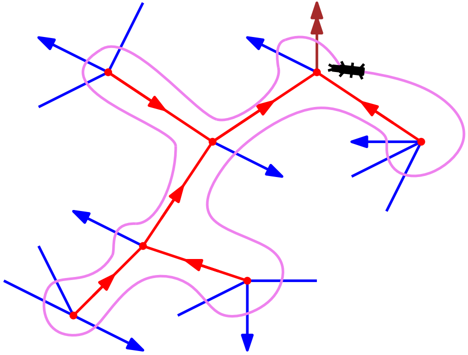
Opening a 4-valent planar map [Schaeffer 97]

- An unvisited ingoing edge is followed
- An unvisited outgoing edge is cut
- A visited leaf is ignored
- A visited outoing edge is followed

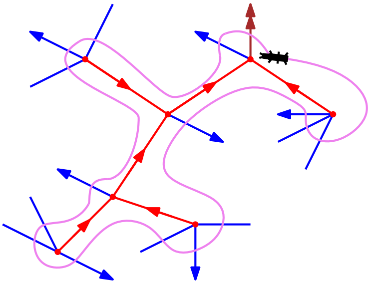


Opening a 4-valent planar map [Schaeffer 97]

- An unvisited ingoing edge is followed
- An unvisited outgoing edge is cut
- A visited leaf is ignored
- A visited outoing edge is followed



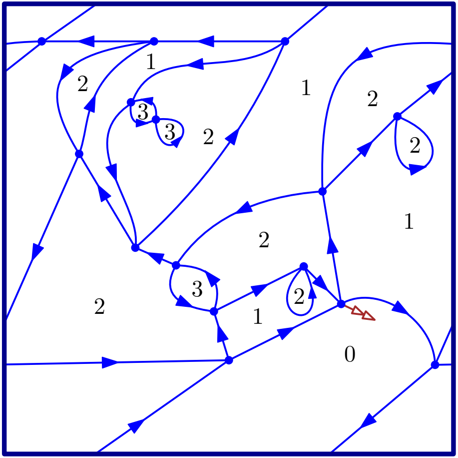
Opening a 4-valent planar map [Schaeffer 97]



Theorem (Schaeffer 97)

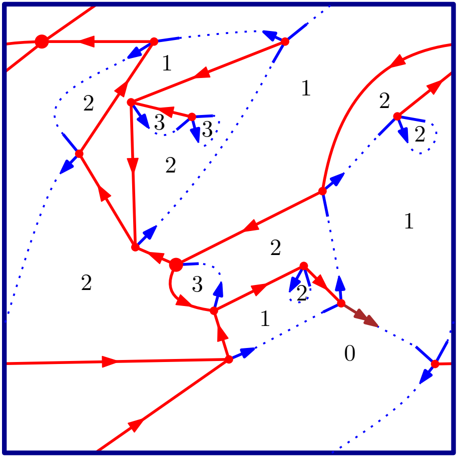
The opening algorithm is a bijection between 4-valent maps and well-rooted Eulerian 4-valent blossoming trees.

Opening a 4-valent bicolorable map



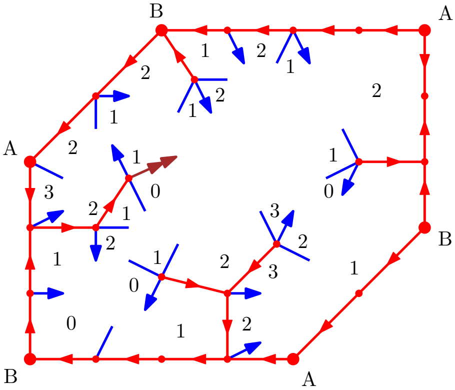
A 4-valent bicolorable map with dual-geodesic orientation

Opening a 4-valent bicolorable map



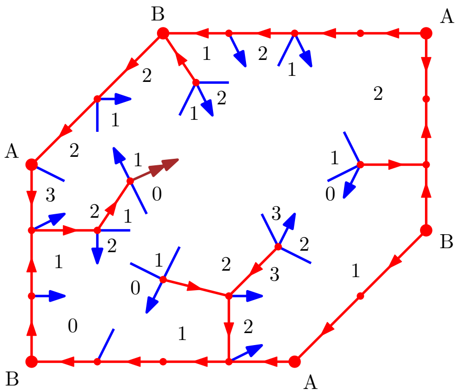
We apply the opening algorithm...

Opening a 4-valent bicolorable map



... And obtain a unicellular map

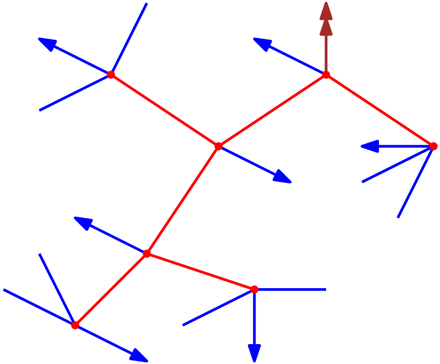
Opening a 4-valent bicolorable map



Theorem (L.)

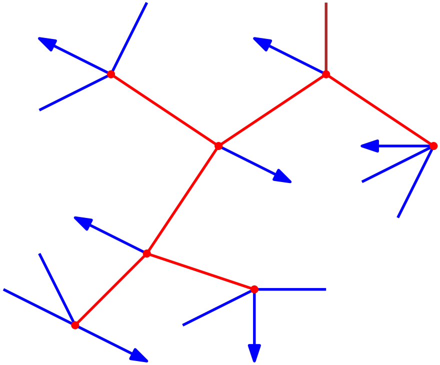
The opening algorithm is a bijection between bicolorable 4-valent map and (so-called) well-rooted well-oriented well-labeled 4-valent unicellular maps.

Closing an Eulerian 4-valent blossoming tree



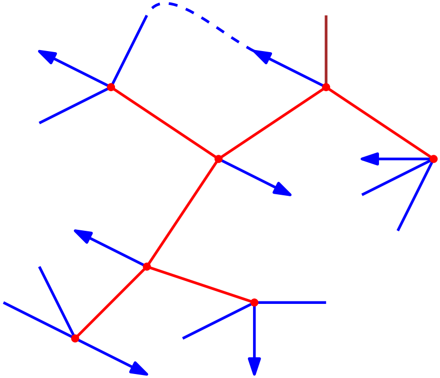
A 4-valent Eulerian rooted tree

Closing an Eulerian 4-valent blossoming tree



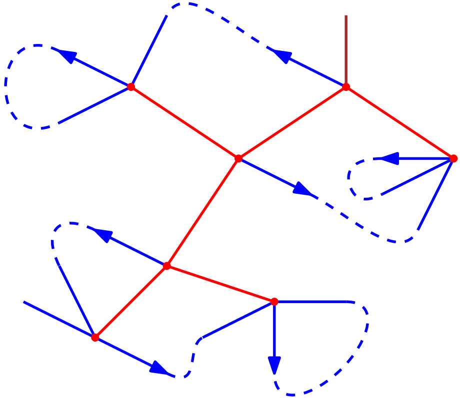
The root is reversed

Closing an Eulerian 4-valent blossoming tree



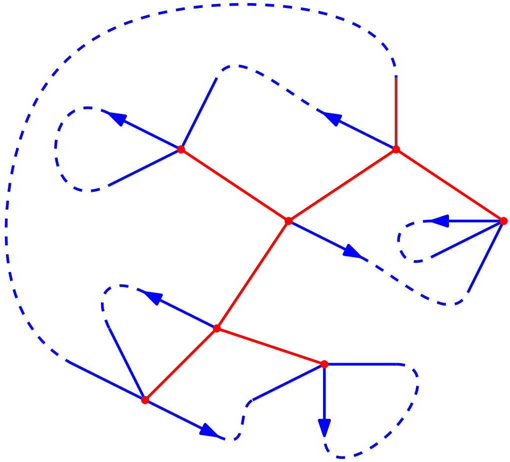
A bud and a leaf following one another are matched

Closing an Eulerian 4-valent blossoming tree



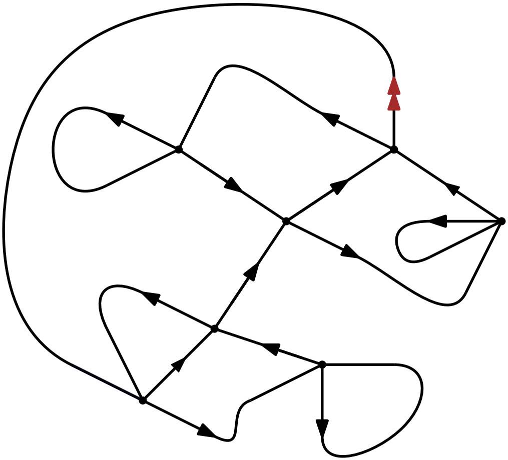
This is repeated until no such pair exists

Closing an Eulerian 4-valent blossoming tree



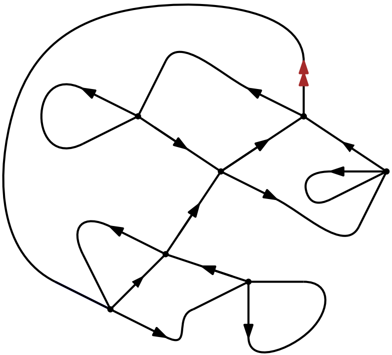
In the end the 2 remaining leaves are merged

Closing an Eulerian 4-valent blossoming tree



We obtain a map

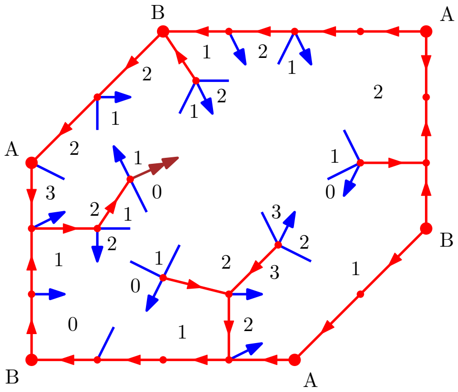
Closing an Eulerian 4-valent blossoming tree



Theorem (Schaeffer 97)

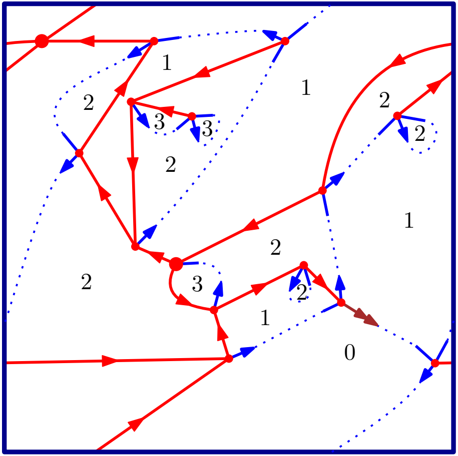
The closing algorithm is the inverse bijection of the opening algorithm.

Closing a well-rooted well-labeled well-oriented 4-valent unicellular blossoming map



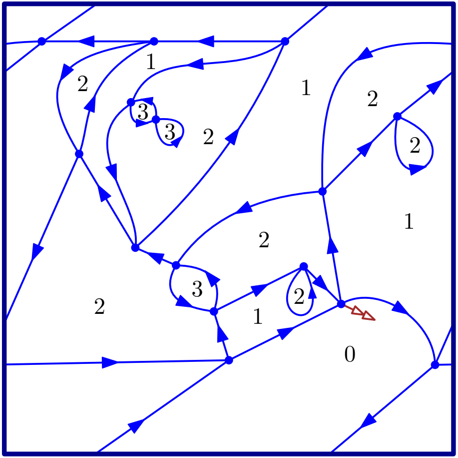
A special map (cf. the never-ending title)

Closing a well-rooted well-labeled well-oriented 4-valent unicellular blossoming map



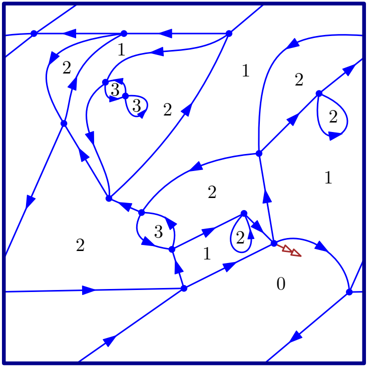
Matching stems

Closing a well-rooted well-labeled well-oriented 4-valent unicellular blossoming map



We again obtain a map, with its dual-geodesic orientation

Closing a well-rooted well-labeled well-oriented 4-valent unicellular blossoming map

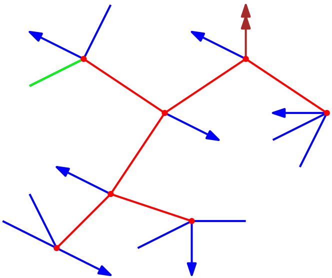


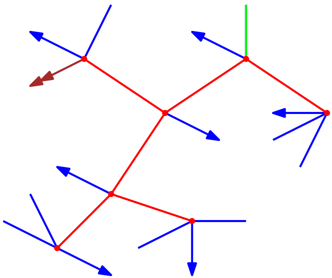
Theorem (L.)

The closing algorithm is the inverse bijection of the opening algorithm.

Sommaire

Well-rooted is an inconveniently global condition



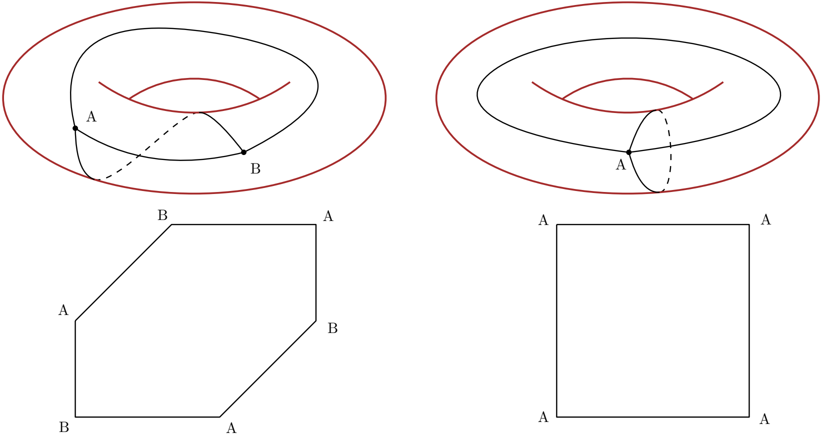


Theorem (Schaeffer 97 for $g = 0$, L. for $g > 0$)

For a fixed interior map m with n leaves, there is a 2-to-$(n+1)$ map from

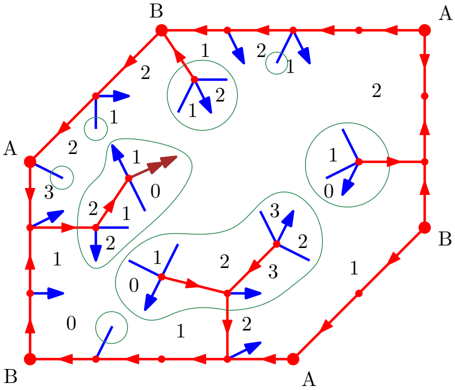
- *well-rooted well-labeled well-oriented 4-regular unicellular map with interior map m, to*
- *rooted well-labeled well-oriented 4-regular unicellular map with interior map m (which has n leaves).*

The structure of unicellular maps



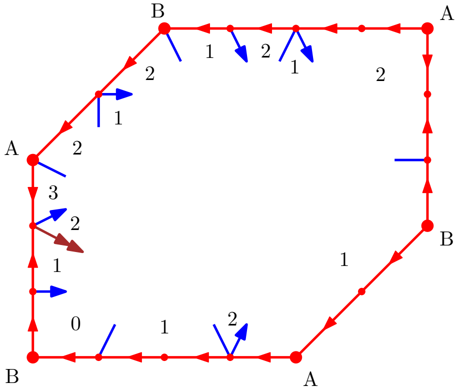
The schemes of genus 1

Pruning the map



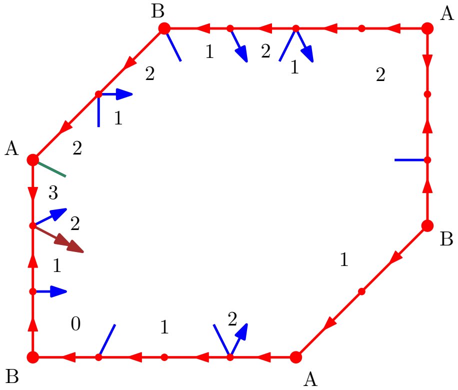
The opened map contains treelike parts

Pruning the map



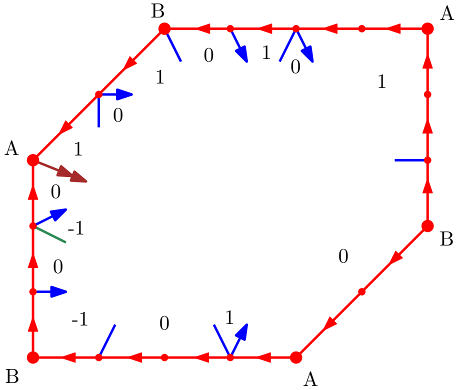
These treelike parts are removed

Rerooting on the scheme



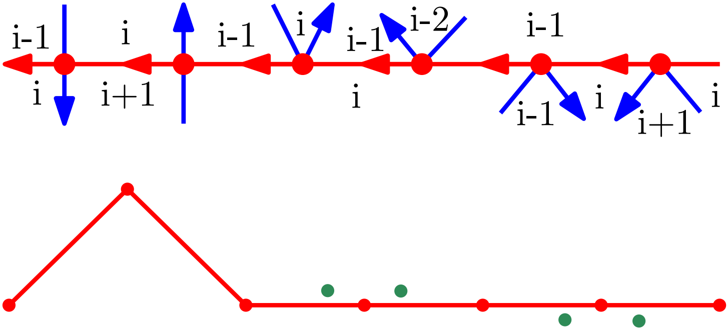
The pruned map...

Rerooting on the scheme



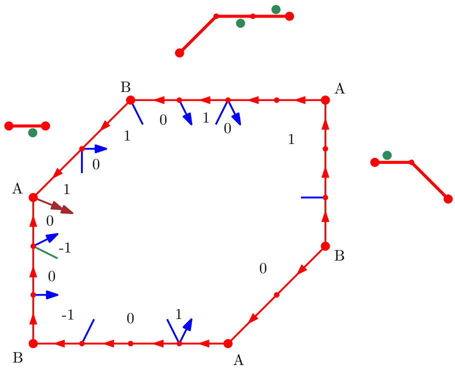
...is rerooted on the scheme

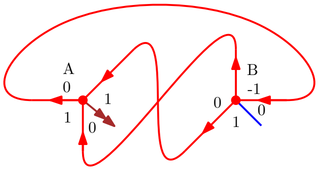
Replace branches by decorated Motzkin paths



There are 6 type of vertices of interior degree 2, each of which can be represented by a decorated Motzkin step

Replace branches by decorated Motzkin paths





What about generating series?

Notation

The series of decorated Motzkin bridges is $B(z)$.
The series of decorated Motzkin positive bridges followed by a downstep is $D(z)$.

What about generating series?

Notation

The series of decorated Motzkin bridges is $B(z)$.
The series of decorated Motzkin positive bridges followed by a downstep is $D(z)$.

The generating series of Motzkin paths from height i to j is: $B \cdot D^{|i-j|}$.

We can hence write a closed formula for $R^s(t)$, the series of scheme-rooted pruned well-labeled well-oriented unicellular blossoming map, with a given unlabeled scheme s.

What about generating series?

We can hence write a closed formula for $R^s(t)$, the series of scheme-rooted pruned well-labeled well-oriented unicellular blossoming map, with a given unlabeled scheme s.

Lemma (L.)

$R^s(z)$ is rational and symmetric in D.

Proof (rough sketch).

We rewrite the expression of $R^s(z)$ as a sum over partial permutations.
An inclusion-exclusion argument allows to prove the symmetry. □

What about generating series?

Lemma (L.)

$R^s(z)$ is rational and symmetric in D.

Notation

The series of decorated Motzkin bridges is $B(z)$.
The series of decorated Motzkin positive bridges followed by a downstep is $D(z)$.

We have $D = z(1 + 4D + D^2)$ and $B = 1 + 4zB + 2zDB$.
Hence $B = \frac{1+4D+D^2}{1-D^2}$ and $z = \frac{1}{D^{-1}+4+D}$.

What about generating series?

Lemma (L.)

$R^s(z)$ is rational and symmetric in D.

We have $D = z(1 + 4D + D^2)$ and $B = 1 + 4zB + 2zDB$.
Hence $B = \frac{1+4D+D^2}{1-D^2}$ and $z = \frac{1}{D^{-1}+4+D}$.

Lemma (Chapuy Marcus Schaeffer 09)

A series is rational in z if and only if it is rational and symmetric in D.

A series F in D is symmetric in D if $F(D^{-1}) = F(D)$.

Let's put everything together

We derive from the previous work that the generating series $M^s(t)$ of all maps (counted by number of edges) with a given (unlabeled) scheme s is:

$$M^s(t) = \frac{2t^{2g-2}}{2g - v_4(s)} \cdot T(t) \cdot R^s(T(t)),$$

where T is the series of trees, and R^s the series of scheme-rooted well-oriented well-labeled pruned unicellular maps with unlabeled scheme s.

Let's put everything together

We derive from the previous work that the generating series $M^s(t)$ of all maps (counted by number of edges) with a given (unlabeled) scheme s is:

$$M^s(t) = \frac{2t^{2g-2}}{2g - v_4(s)} \cdot T(t) \cdot R^s(T(t)),$$

where T is the series of trees, and R^s the series of scheme-rooted well-oriented well-labeled pruned unicellular maps with unlabeled scheme s.

Theorem (L.)

The generating series $M^s(t)$ is a rational function of $T(t)$.

Let's put everything together

Theorem (L.)

The generating series $M^s(t)$ is a rational function of $T(t)$.

Since the set of unlabeled schemes $\mathcal{S}_g$ is finite for any fixed g, it implies that $M_g(t) = \sum_{s \in \mathcal{S}_g} M^s(t)$ is rational in $T(t)$, and proves our main theorem.

Theorem (Tutte 60's for $g = 0$, Bender Canfield 91 for $g > 0$)

For any g, the generating series $M_g(t)$ is a rational function of $T(t)$.

The end

Thanks for your attention!

Obtaining an equation for a given scheme

$$R^{l} \quad = \prod_{(v_i, v_j) \in E_l} B \cdot D^{|h_i - h_j|}$$

- l is a labeled scheme
- E_l is the set of its edges
- h_i is the label of vertex v_i.

Obtaining an equation for a given scheme

$$R^l = \prod_{(v_i,v_j)\in E_l} B \cdot D^{|h_i - h_j|}$$
$$R^s = \sum_{\substack{h_1 \cdots h_{n_v} \in \mathbb{N} \\ \min(h_1,\cdots,h_{n_v})=0}} \prod_{(v_i,v_j)\in E_s} B \cdot D^{|h_i - h_j|}$$

- s is an unlabeled scheme.
- Labels are defined up to translation, so we decide to force the minimal label to be 0.

Obtaining an equation for a given scheme

$$\begin{aligned} R^l &= \prod_{(v_i,v_j)\in E_l} B\cdot D^{|h_i-h_j|} \\ R^s &= \sum_{\substack{h_1\cdots h_{n_v}\in\mathbb{N}\\ \min(h_1,\cdots,h_{n_v})=0}} \prod_{(v_i,v_j)\in E_s} B\cdot D^{|h_i-h_j|} \\ &= B^{|E_l|}\cdot \sum_{S_0\subsetneq S_1\subsetneq\cdots\subsetneq S_k} \prod_{i=1}^{k-1} \frac{D^{Cut(S_i)}}{1-U^{Cut(S_i)}} \end{aligned}$$

- $\emptyset = S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_k = V_s$ is the ordered partition corresponding to the ordering of labels.
- $Cut(S)$ is the number of edges going from S to $\overline{S}$.

Obtaining an equation for a given scheme

$$
\begin{aligned}
R^l &= \prod_{(v_i,v_j)\in E_l} B \cdot D^{|h_i-h_j|} \\
R^s &= \sum_{\substack{h_1\cdots h_{n_v}\in\mathbb{N} \\ \min(h_1,\cdots,h_{n_v})=0}} \prod_{(v_i,v_j)\in E_s} B \cdot D^{|h_i-h_j|} \\
&= B^{|E_l|} \cdot \sum_{S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_k} \prod_{i=1}^{k-1} \frac{D^{Cut(S_i)}}{1-U^{Cut(S_i)}} \\
&= B^{|E_l|} \cdot \sum_{S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_k} \prod_{i=1}^{k-1} \Phi(S_i)
\end{aligned}
$$

- $\Phi(S) = \frac{D^{Cut(S)}}{1-D^{Cut(S)}}$.

Obtaining an equation for a given scheme

$$
\begin{aligned}
R^l &= \prod_{(v_i,v_j)\in E_l} B\cdot D^{|h_i-h_j|} \\
R^s &= \sum_{\substack{h_1\cdots h_{n_V}\in\mathbb{N} \\ \min(h_1,\cdots,h_{n_V})=0}} \prod_{(v_i,v_j)\in E_s} B\cdot D^{|h_i-h_j|} \\
&= B^{|E_l|}\cdot \sum_{S_0\subsetneq S_1\subsetneq\cdots\subsetneq S_k} \prod_{i=1}^{k-1} \frac{D^{Cut(S_i)}}{1-U^{Cut(S_i)}} \\
&= B^{|E_l|}\cdot \sum_{S_0\subsetneq S_1\subsetneq\cdots\subsetneq S_k} \prod_{i=1}^{k-1} \Phi(S_i) \\
&= B^{|E_l|}\cdot \sum_{\pi} \prod_{i=1}^{k-1} \Phi(S_i)
\end{aligned}
$$

- The ordered partition is called π instead of $S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_k$.

Proving that this expression is symmetric

$$R^s = B^{|E_I|} \quad \cdot \sum_{\pi} \prod_{i=1}^{k-1} \Phi(S_i)$$

Proving that this expression is symmetric

$$R^s = B^{|E_I|} \quad \cdot \sum_{\pi} \prod_{i=1}^{k-1} \Phi(S_i)$$

$$\overline{R^s} = (-1)^{|E_I|} \cdot B^{|E_I|} \quad \cdot \sum_{\pi} \left((-1)^{k-1} \cdot \prod_{i=1}^{k-1} (1 + \Phi(S_i)) \right)$$

- B is antisymmetric.
- $\overline{\Phi(S)} = -(1 + \Phi(S))$.

Proving that this expression is symmetric

$$
\begin{aligned}
R^s &= B^{|E_I|} &&\cdot \sum_{\pi} \prod_{i=1}^{k-1} \Phi(S_i) \\
\overline{R^s} &= (-1)^{|E_I|} \cdot B^{|E_I|} &&\cdot \sum_{\pi} \left((-1)^{k-1} \cdot \prod_{i=1}^{k-1} (1 + \Phi(S_i)) \right) \\
&= (-1)^{|E_I|} \cdot B^{|E_I|} &&\cdot \sum_{\pi} \left((-1)^{k(\pi)-1} \cdot \sum_{\mu < \pi} \prod_{i=1}^{k(\mu)-1} \Phi(S_i(\mu)) \right)
\end{aligned}
$$

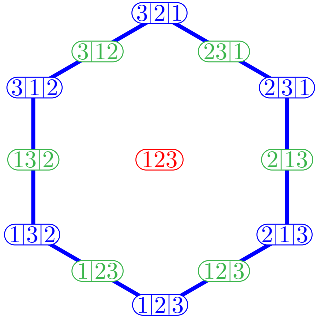
- $\mu < \pi$ means that π refines μ as an ordered partition.
- We apply the inclusion-exclusion principle.

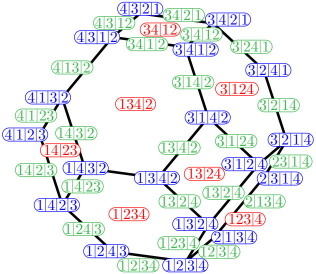
Proving that this expression is symmetric

$$\begin{aligned}
R^s &= B^{|E_I|} &&\cdot \sum_{\pi} \prod_{i=1}^{k-1} \Phi(S_i) \\
\overline{R^s} &= (-1)^{|E_I|} \cdot B^{|E_I|} &&\cdot \sum_{\pi} \left((-1)^{k-1} \cdot \prod_{i=1}^{k-1} (1 + \Phi(S_i)) \right) \\
&= (-1)^{|E_I|} \cdot B^{|E_I|} &&\cdot \sum_{\pi} \left((-1)^{k(\pi)-1} \cdot \sum_{\mu < \pi} \prod_{i=1}^{k(\mu)-1} \Phi(S_i(\mu)) \right) \\
&= (-1)^{|E_I|} \cdot B^{|E_I|} &&\cdot \sum_{\pi} \prod_{i=1}^{k(\pi)-1} \Phi(S_i(\pi)) \cdot \sum_{\pi < \rho} \left((-1)^{k(\rho)-1} \right)
\end{aligned}$$

- We swap the two sumations.

Proving that this expression is symmetric





- ordered partitions are faces of the permutahedron.
- Euler-Poincaré's formula states:

$$0 = \sum_{i=-1}^{d} (-1)^i f_i(P),$$

where P is a polytope of degree d with $f_i(P)$ faces of degree i.

Proving that this expression is symmetric

$$R^s = B^{|E_I|} \cdot \sum_{\pi} \prod_{i=1}^{k-1} \Phi(S_i)$$

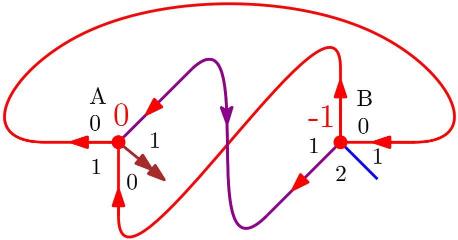
$$\overline{R^s} = (-1)^{|E_I|} \cdot B^{|E_I|} \cdot \sum_{\pi} \prod_{i=1}^{k(\pi)-1} \Phi(S_i(\pi)) \cdot \sum_{\pi<\rho} \left((-1)^{k(\rho)-1}\right)$$

- ordered partitions are faces of the permutahedron.
- Euler-Poincaré's formula states:

$$0 = \sum_{i=-1}^{d} (-1)^i f_i(P),$$

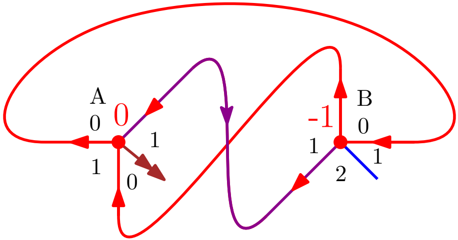
where P is a polytope of degree d with $f_i(P)$ faces of degree i.

The offset graph



An offset edge (purple)

The offset graph



Theorem (L.)

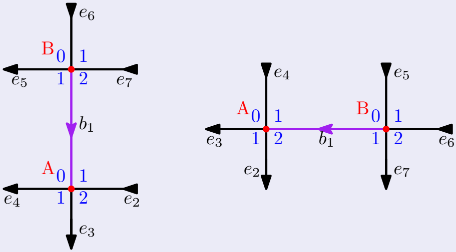
The offset graph of a scheme-rooted well-oriented well-labeled unicellular map is acyclic.

The offset graph

Theorem (L.)

The offset graph of a scheme-rooted well-oriented well-labeled unicellular map is acyclic.

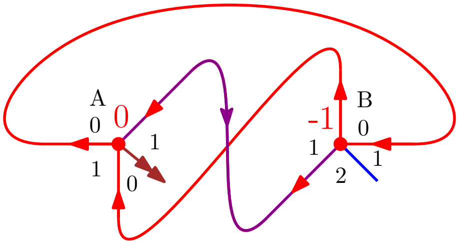
Proof.



The two possible case for the first step of an offset cycle

□

The offset graph



With an offset graph:

$$R^s = B^{|E_l|} \cdot \sum_{S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_k} \left(\prod_{i=1}^{k-1} \Phi(S_i) \right) \cdot D^{n_t + n_a - n_i}.$$

The offset graph

With an offset graph:

$$R^s = B^{|E_l|} \cdot \sum_{S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_k} \left(\prod_{i=1}^{k-1} \Phi(S_i) \right) \cdot D^{n_t + n_a - n_i}.$$

By consequence:

$$\overline{R^s} = (-1)^{|E_l|} \cdot B^{|E_l|} \cdot \sum_{\pi} \prod_{i=1}^{k(\pi)-1} \Phi(S_i(\pi)) \cdot \sum_{\pi < \rho} \left((-1)^{k(\rho)-1} \cdot D^{-n_t(\rho) - n_a(\rho) + n_i(\rho)} \right)$$

The offset graph

With an offset graph:

$$R^s = B^{|E_I|} \cdot \sum_{S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_k} \left(\prod_{i=1}^{k-1} \Phi(S_i) \right) \cdot D^{n_t + n_a - n_i}.$$

By consequence:

$$\overline{R^s} = (-1)^{|E_I|} . B^{|E_I|} . \sum_{\pi} \prod_{i=1}^{k(\pi)-1} \Phi(S_i(\pi)) \cdot \sum_{\pi < \rho} \left((-1)^{k(\rho)-1} \cdot D^{-n_t(\rho) - n_a(\rho) + n_i(\rho)} \right)$$

We need to prove the following lemma:

Lemma (L.)

$$\sum_{\pi^{-1} < \rho} \left((-1)^{k(\rho)-1} \cdot U^{-n_t(\rho) - n_a(\rho) + n_i(\rho)} \right) = (-1)^{|E_s|} \cdot D^{n_t(\pi) + n_a(\pi) - n_i(\pi)}$$