# Uniform generation of infinite concurrent runs

## The case of trace monoids


Samy Abbes[1] & Vincent Jugé[2]

1: Université Paris Diderot (IRIF) — 2: Université Paris-Est Marne-la-Vallée (LIGM)


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# Contents

# Heaps of pieces and trace monoids

## Heap of pieces

- Pieces:


## Trace monoid

- Alphabet:

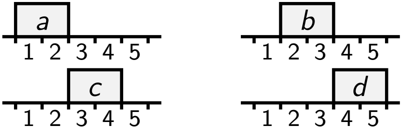
$$\Sigma = \{a, b, c, d\}$$

# Heaps of pieces and trace monoids

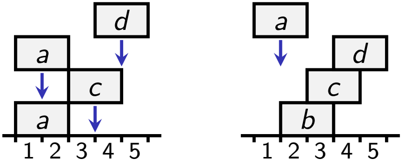
## Heap of pieces

- Pieces:


- Horizontal layout:




- Vertical heaps:

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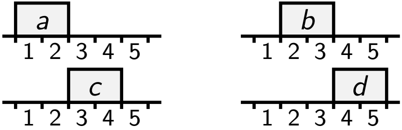


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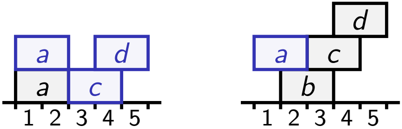


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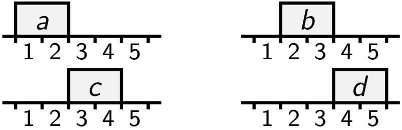


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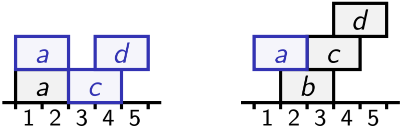



## Trace monoid

- Alphabet:

$$\Sigma = \{a, b, c, d\}$$

- Dependence relation:

$$\mathbf{D}=\{\{a,b\},\{b,c\},\{c,d\}\}$$

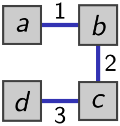
- Trace monoid:

$$\mathcal{M}=\langle a,b,c,d \,|\, ac=ca, ad=da, bd=db\rangle^+$$


Dependency graph

# Heaps of pieces and dependency graph

## Dependency graph

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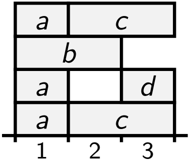
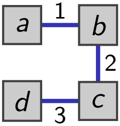
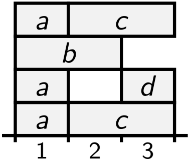
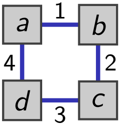


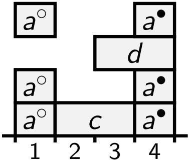
# Heaps of pieces and dependency graph

**Dependency graph** | **Heap of** (disconnected) **pieces**



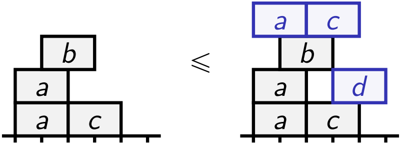



# Heaps of pieces and left divisibility

## Heap of pieces



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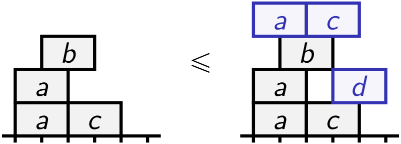
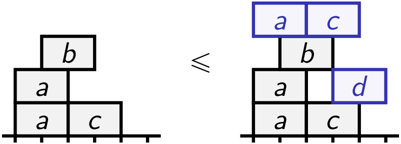



Can you pick one **infinite** heap **uniformly at random**?

1. Define your preferred notion of heap length
2. Study uniform distributions on traces of length $k$: what if $k \to \infty$?

# Heaps of pieces and left divisibility

**Heap of pieces**




Can you pick one **infinite** heap **uniformly at random**?

1. Heap length = #pieces in the heap
2. **Weak convergence** of distributions:

$$\mu_k \to \nu \Leftrightarrow (\forall x \in \mathcal{M}^+, \mathbb{P}_{\mu_k}[x \leqslant \xi] \to \mathbb{P}_\nu[x \leqslant \xi])$$

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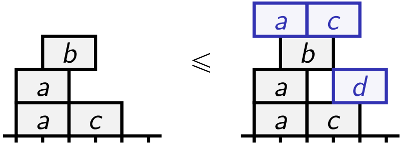


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**Theorem (Abbes & Mairesse 2015)**

If $\mu_k$ is the unifom measure on $\mathcal{M}_k = \{\zeta \in \mathcal{M} : |\zeta| = k\}$, $\nu$ **exists** and is the **critical Bernoulli** distribution of $\mathcal{M}$.

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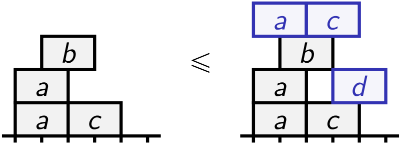


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**Theorem (Abbes & Mairesse 2015) — not constructive!**

If $\mu_k$ is the unifom measure on $\mathcal{M}_k = \{\zeta \in \mathcal{M} : |\zeta| = k\}$, $\nu$ **exists** and is the **critical Bernoulli** distribution of $\mathcal{M}$.

# Bernoulli distributions

## Definition

A probability measure $\mu$ on $\mathcal{M}$ is:

- **Bernoulli** if $\forall x \in \mathcal{M}, \forall \sigma \in \Sigma, \mathbb{P}_\mu[x\,\sigma \leqslant \zeta \mid x \leqslant \zeta] = \mathbb{P}_\mu[\sigma \leqslant \zeta]$
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Uniform Bernoulli $\Rightarrow \mathbb{P}_\mu[x \leqslant \zeta] = p^{|x|} \Rightarrow \mathbb{P}_\mu[x = \zeta] \propto p^{|x|}$.

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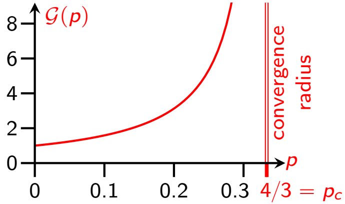
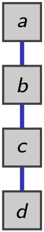
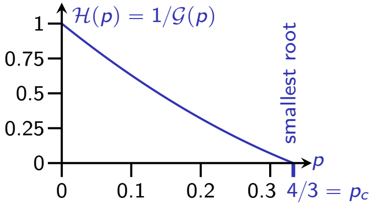


Generating function $\mathcal{G}(p)$



Möbius polynomial $\mathcal{H}(p)$

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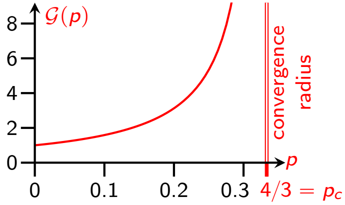
$$\text{Uniform Bernoulli} \Rightarrow \mathbb{P}_\mu[x \leqslant \zeta] = p^{|x|} \Rightarrow \mathbb{P}_\mu[x = \zeta] \propto p^{|x|} \stackrel{?}{=} p^{|x|}\mathcal{H}(p).$$



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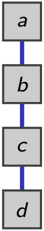


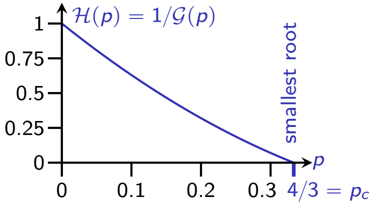

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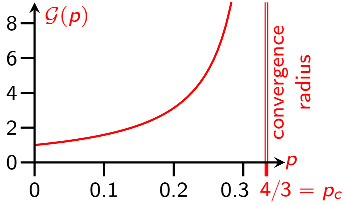


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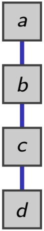


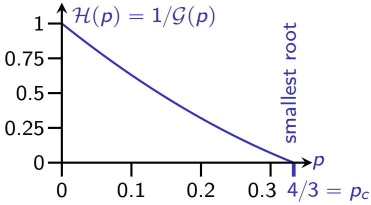


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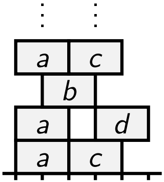
# Infinite heaps

## Heap of pieces



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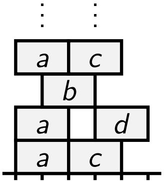


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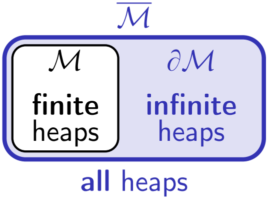


$\overline{\mathcal{M}}$ is the **compactification** of $\mathcal{M}$ for the topology induced by $\Uparrow \mathbf{x} = \{\zeta : \mathbf{x} \leqslant \zeta\}$

**Reminder**: $\mu_k \to \nu \Leftrightarrow (\forall x \in \mathcal{M}^+, \mathbb{P}_{\mu_k}[x \leqslant \xi] \to \mathbb{P}_\nu[x \leqslant \xi])$

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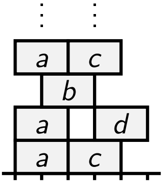


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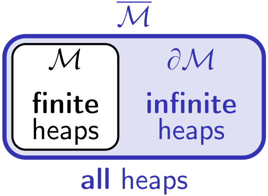


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**Fact #1 (Abbes & Mairesse 2015)**

The limit $\nu$ is a distribution **on the set** $\overline{\mathcal{M}}$ with **support** $\partial\mathcal{M}$.⚠

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# Simulating the limit $\nu$

**Idea #1:** Pick $\zeta_k \sim \mu_k$, pick a piece **x** **wisely** and set $\zeta_{k+1} = \zeta_k \cdot x$

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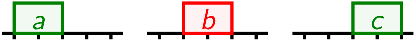
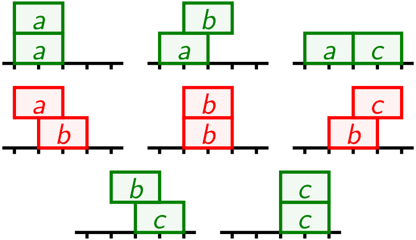
**Problem:** In general, $\zeta_{k+1}$ **cannot** be distributed according to $\mu_{k+1}$

**Example** in the monoid $\langle a, b, c \mid ac = ca \rangle^+$


Dependency graph

$\mu_1(\Uparrow a \cup \Uparrow c) = 2/3 \quad > \quad \mu_2(\Uparrow a \cup \Uparrow c) = 5/8$



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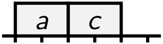

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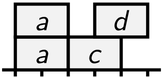

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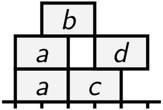

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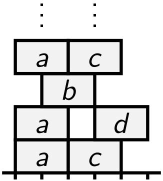

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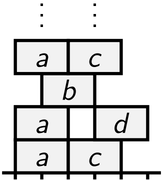


**Fact #2 (Abbes & Mairesse 2015)**

This approach works because $\nu$ is **Bernoulli**!

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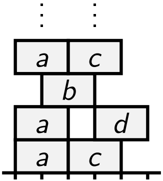


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**Problems:** Huge state space (exponential number of possible floors)
Adding one floor implies synchronising many pieces

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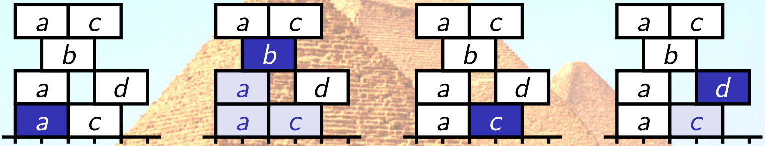


$a$-pyramid $b$-pyramid $c$-pyramid $d$-pyramid

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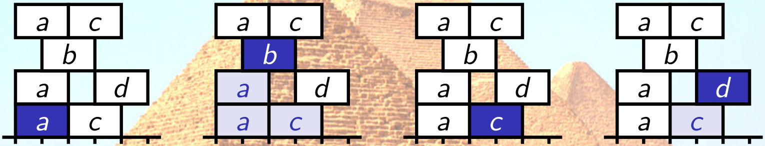


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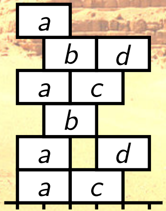
**Example:** Recursive decomposition using *b*-pyramids



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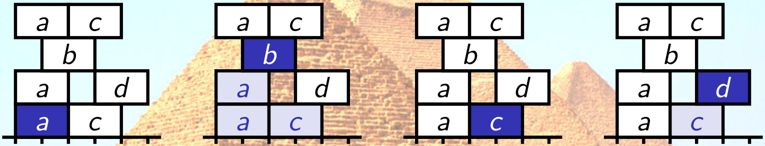


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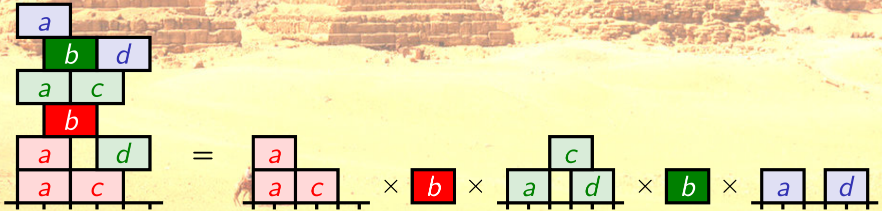

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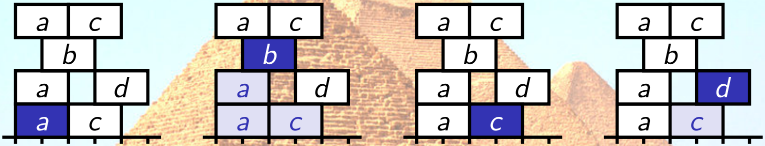
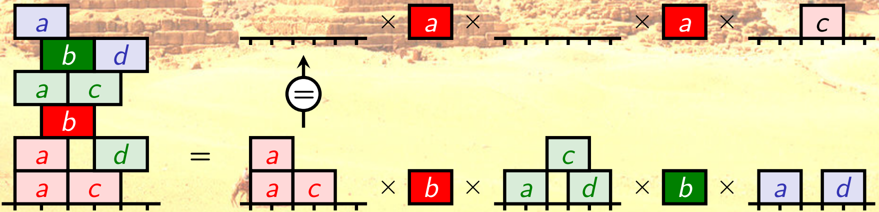


$a$-pyramid $b$-pyramid $c$-pyramid $d$-pyramid

**Example:** Recursive decomposition using $b$-pyramids, then $a$-pyramids...



# Simulating the limit $\nu$ piece by piece⚠

**Idea #3:** Decompose heaps recursively by using **independent pyramids**

**Theorem – continued (Abbes & J. 2017)**

If $\mathbf{a} \in \Sigma$ and $\nu$ is **Bernoulli** on $\overline{\mathcal{M}}(\Sigma)$, then

Heap

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**a**-pyramid

$\times$

Heap

$+$

**a**-free heap

where right-hand side random variables are **independent**.

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Heap = **a**-pyramid × Heap + **a**-free heap

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=

× **A**

**a**-pyramid = **a**-free heap $\xi$ s.t. $\mathbf{S}(\xi) \subseteq \mathbf{D}(\mathbf{a})$ × **A**

- $\mathbf{S}(\xi) = \{\sigma \in \Sigma : \xi \in \mathcal{M}^+ \cdot \sigma\}$
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## Theorem – continued (Abbes & J. 2017)

If $X \subseteq \Sigma$, $\mathbf{a} \in \Sigma$ and $\nu$ is **Bernoulli** on $\overline{\mathcal{M}}(\Sigma)$, then

 $= \sum_{k=1}^{\infty}$  $^{k}$ $\times$  $+$ 

Heap $\xi$ s.t. $\mathbf{S}(\xi) \subseteq X$ — $\mathbf{a}$-pyramid — $\mathbf{a}$-free heap $\xi$ s.t. $\mathbf{S}(\mathbf{a} \cdot \xi) \subseteq X$ — $\mathbf{a}$-free heap $\xi$ s.t. $\mathbf{S}(\xi) \subseteq X$

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# Our algorithm: Generating heaps distributed according to $\nu$

**Remarks:**

- if $\nu$ has support $\partial\mathcal{M}$, $\xi$ = infinite sequence of $\mathbf{a}$-pyramids $\qquad (k \leftarrow \infty)$
- $\mathbf{a}$-pyramid = ( $\mathbf{a}$-free sequence $\xi$ with $\mathbf{S}(\xi) \subseteq \mathbf{D}(\mathbf{a})$ ) $\cdot$ $\mathbf{a}$
- $\mathbf{S}(\mathbf{a}\cdot\zeta) \subseteq X \Rightarrow \mathbf{S}(\zeta) \subseteq X$

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**Variant:** Choose $\mathbf{a} \in X$ and avoid rejection

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**Running time/piece:** $n\,q$ or $n$ $\qquad$ **Read-only memory usage:** $n$ or $2^n$

where $\mathcal{M}' = \mathcal{M}(\Sigma\backslash\{a\})$ and $q = \mathcal{G}_{\mathcal{M}'}(p_c) \leqslant 1/p_c^n$ $\qquad$ ($q = n^{\Theta(n)}$ is possible)

# Contents

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Algorithm based on:

- precomputing and storing **Möbius polynomials** of sub-monoids
- decomposing heaps into **independent** pyramids/heaps in sub-monoids
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Very efficient on graphs with:

- few cycles (small storage/high efficiency for a mix between variants)
- small tree-width (no preprocessing/storage)


Thank you