Exact Enumeration of Planar Eulerian Orientations

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Joint work with Mireille Bousquet-Mélou

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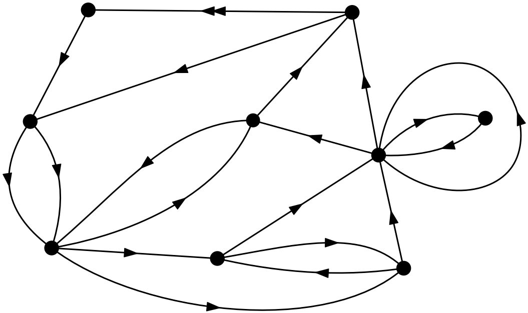
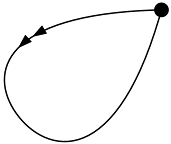
Rooted planar Eulerian orientations



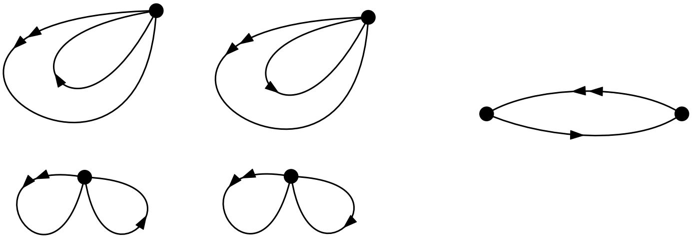
Each vertex has equally many incoming as outgoing edges.

ONE EDGE ROOTED PLANAR EULERIAN ORIENTATIONS



There is 1 planar rooted Eulerian orientations with one edge.

Two edge rooted planar Eulerian orientations



There are 5 planar rooted Eulerian orientations with two edges.

COUNTING ROOTED PLANAR EULERIAN ORIENTATIONS

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- $g_1 = 1$.
- $g_2 = 5$.
- Aim: Find a formula for g_n.

BACKGROUND ON THE PROBLEM

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- In 2016, Bonichon, Bousquet-Mélou, Dorbec and Pennarun posed the problem of enumerating planar rooted Eulerian orientations with a given number of edges.
- They computed the number g_n of these orientations for $n \leq 15$.
- They also proved that the growth rate

$$\mu = \lim_{n\to\infty} \sqrt[n]{g_n}$$

exists and lies in the interval $(11.56, 13.005)$

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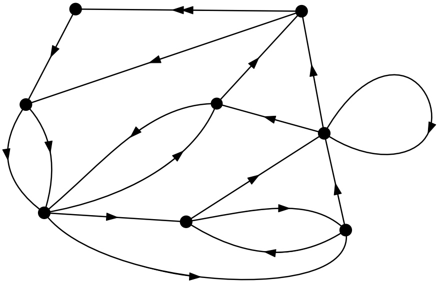
- Bonichon et al. also posed the problem of enumerating these.
- In physics, this is equivalent to the ice type model on a random lattice studied by Zinn-Justin and Kostov.
- Also, $6q_n$ is the number of properly three coloured quadrangulations with n faces.

Bijection to labelled maps



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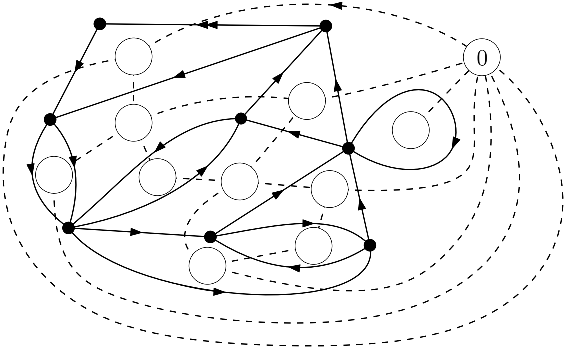

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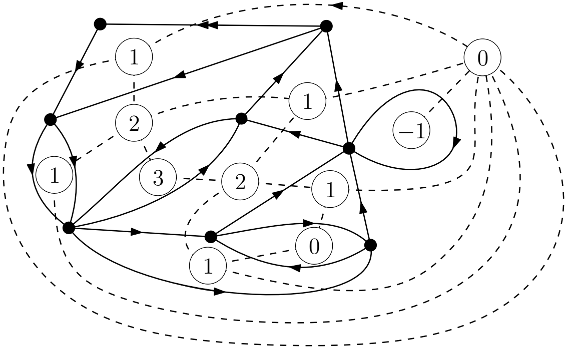

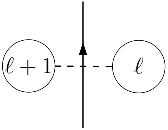

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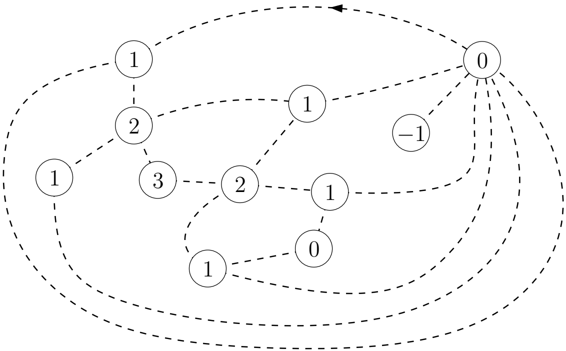

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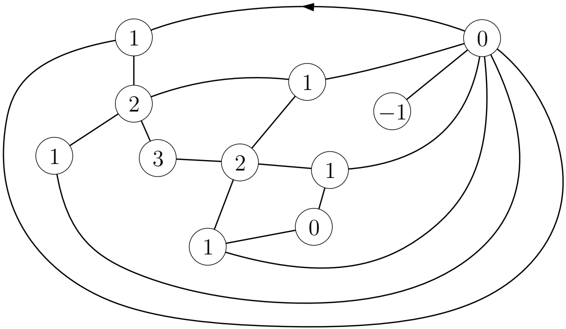
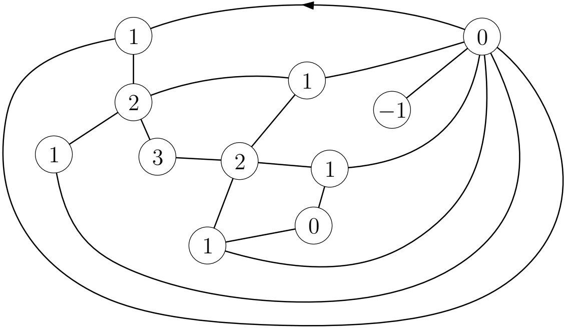


Labelled maps

Labelled maps are rooted planar maps with labelled vertices such that:



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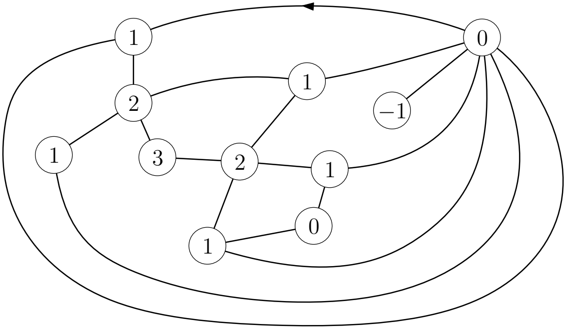

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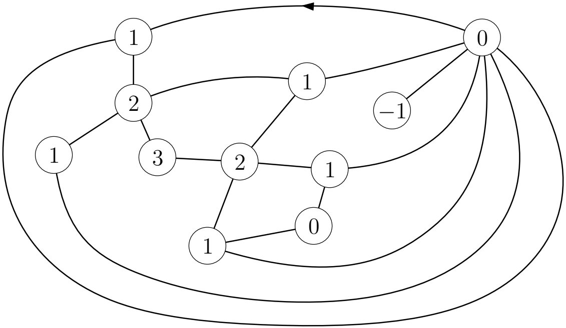

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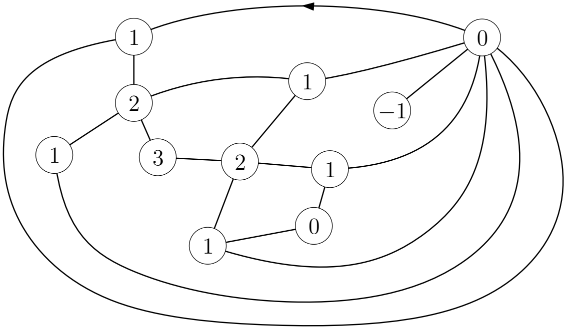
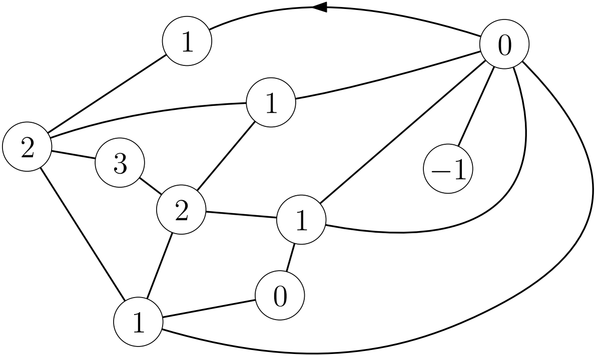


Labelled quadrangulations

By our bijection, q_n (the number of quartic eulerian orientations with n vertices) is the number of labelled quadrangulations with n faces.



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$$
\begin{aligned}
q_n &= [yt^n]\mathsf{P}(t,y)\\
\mathsf{P}(t,y) &= \frac{1}{y}[x^1]\mathsf{C}(t,x,y)\\
\mathsf{D}(t,x,y) &= \frac{1}{1-\mathsf{C}\left(t,\frac{1}{1-x},y\right)}\\
\mathsf{D}(t,x,y) &= 1+y\mathsf{D}(t,x,y)[y^1]\mathsf{D}(t,x,y)+y[x^{\geq 0}]\frac{1}{x}\mathsf{P}\left(t,\frac{1}{x}\right)\mathsf{D}(t,x,y)\\
[y^1]\mathsf{D}(t,x,y) &= \frac{1}{1-x}(1+2t[y^2]\mathsf{D}(t,x,y)-t([y^1]\mathsf{D}(t,x,y))^2).
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$$\mathsf{D}(t,x,y) = \frac{1}{1-\mathsf{C}\left(t,\frac{1}{1-x},y\right)}$$

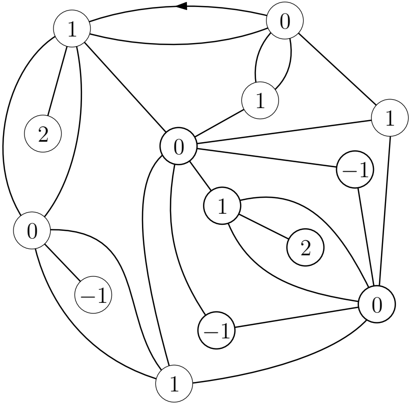
$$\mathsf{D}(t,x,y) = 1 + y\mathsf{D}(t,x,y)[y^1]\mathsf{D}(t,x,y) + y[x^{\geq 0}]\frac{1}{x}\mathsf{P}\left(t,\frac{1}{x}\right)\mathsf{D}(t,x,y)$$

$$[y^1]\mathsf{D}(t,x,y) = \frac{1}{1-x}(1+2t[y^2]\mathsf{D}(t,x,y) - t([y^1]\mathsf{D}(t,x,y))^2).$$

I will show one element of the proof.

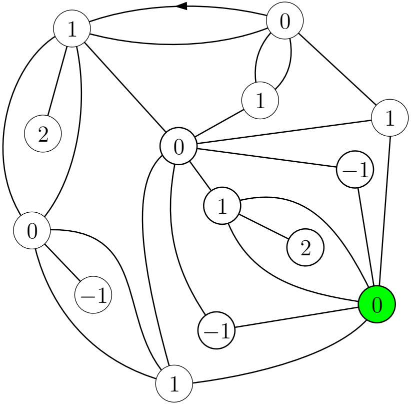
D-PATCHES

D-patch: Digons are allowed next to the root vertex and the outer face may have any degree.



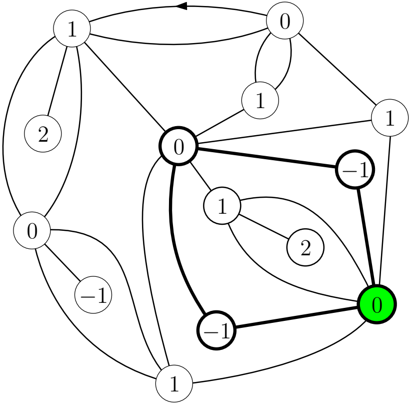
DECOMPOSITION OF D-PATCHES

Colour the vertex two places clockwise from the root vertex around the outer face.



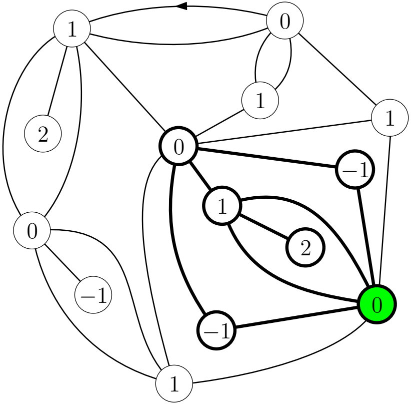
Decomposition of D-patches

Highlight the maximal connected subgraph of nonpositive labels, containing the coloured vertex.



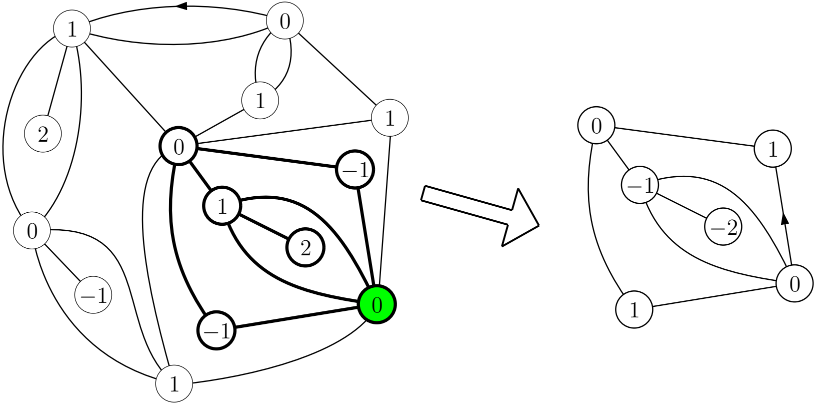
Decomposition of D-patches

Add to the subgraph all vertices and edges contained in its inner face(s).



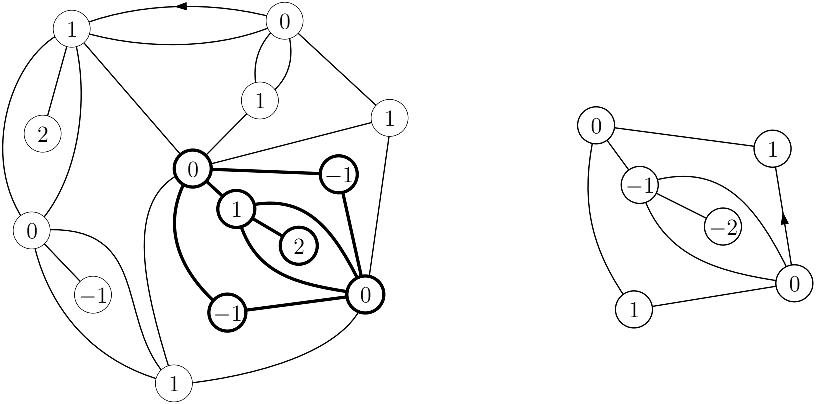
Decomposition of D-patches

Record the subgraph with inverted labels.



DECOMPOSITION OF D-PATCHES

Contract the highlighted map to a single vertex (labelled 0).



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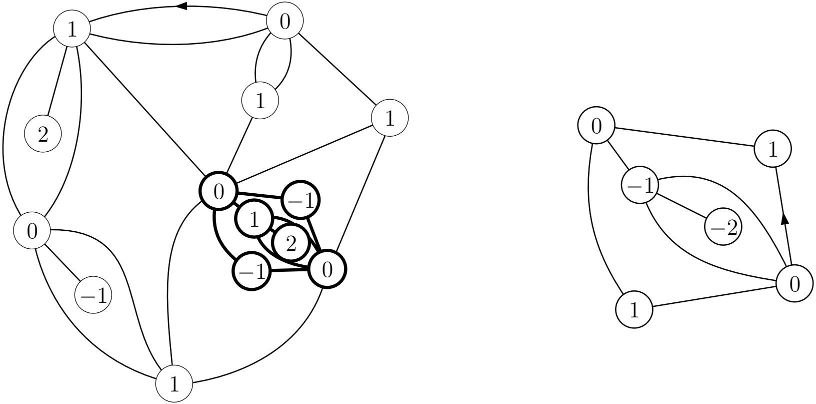

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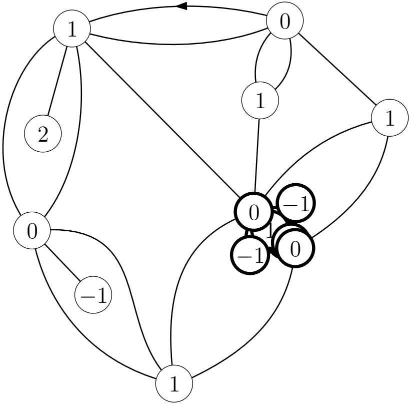

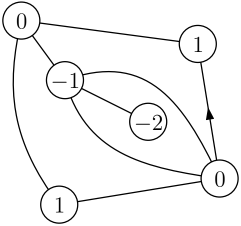

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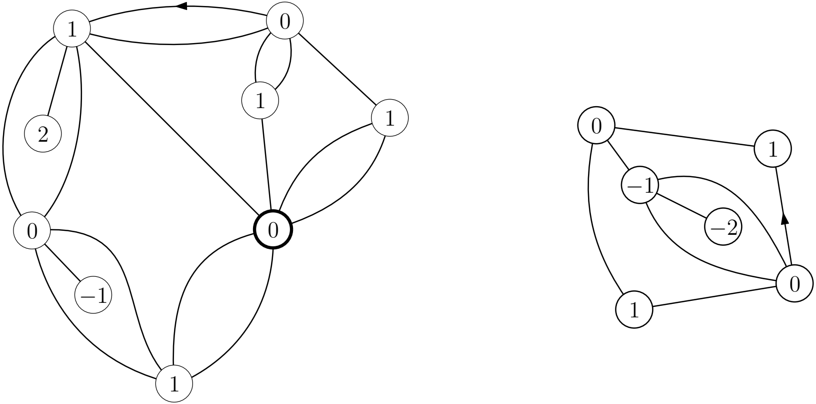

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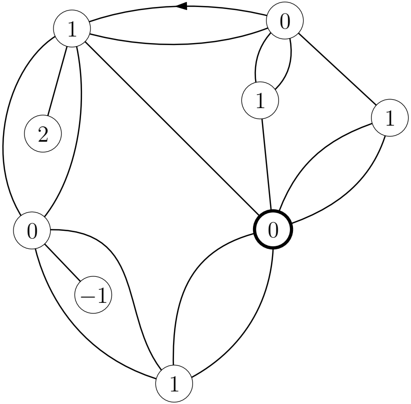
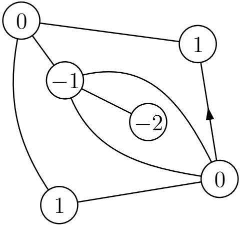
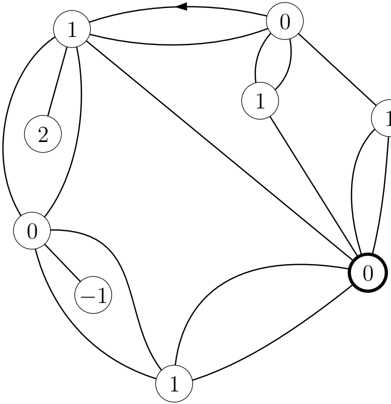
Contract the highlighted map to a single vertex (labelled 0). The new vertex may be adjacent to digons.

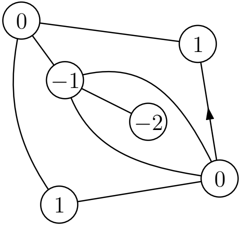




Decomposition of D-patches

Merge the new vertex with the root vertex.





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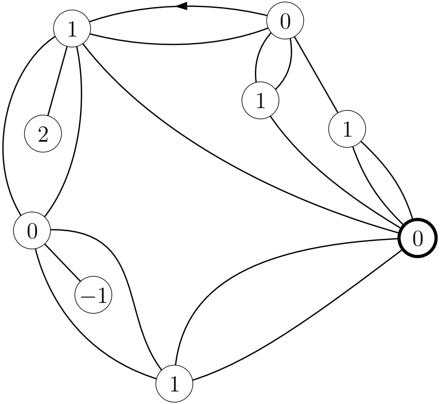

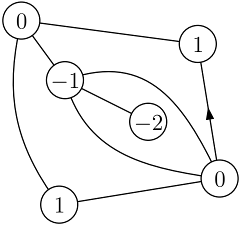

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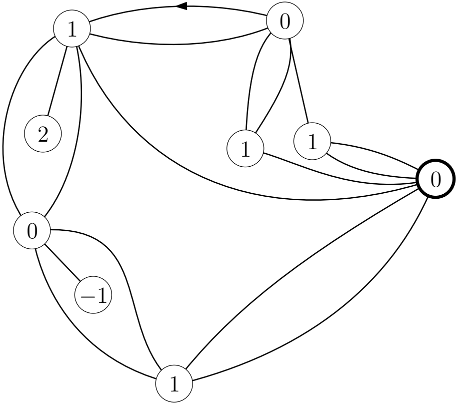

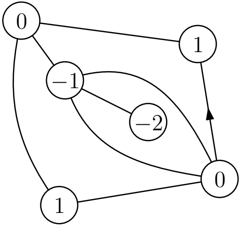

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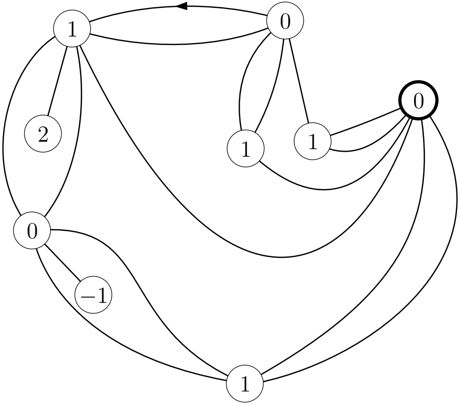

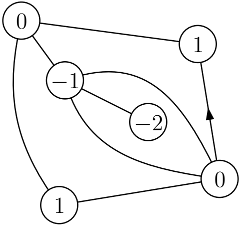

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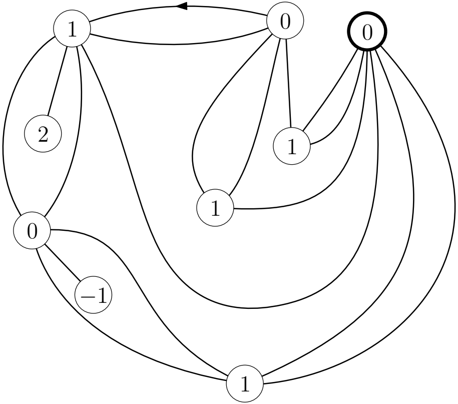

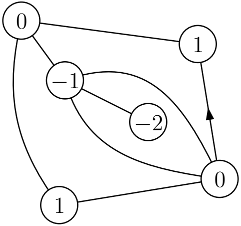

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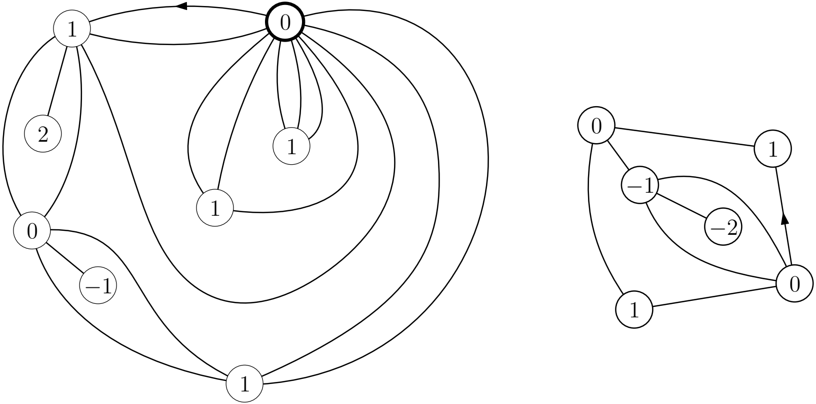
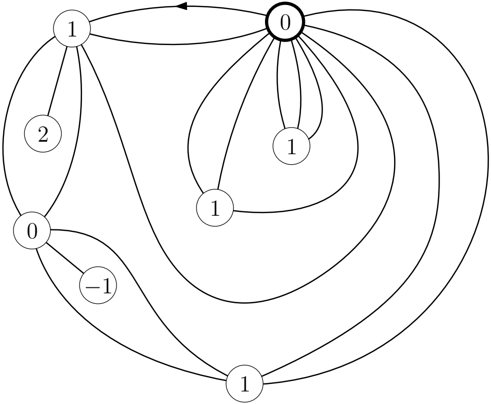
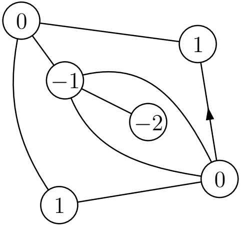


DECOMPOSITION OF D-PATCHES

Merge the new vertex with the root vertex. This new map is a D-patch!





Equations for labelled quadrangulations

$$q_n = [yt^n]\mathsf{P}(t,y)$$

$$\mathsf{P}(t,y) = \frac{1}{y}[x^1]\mathsf{C}(t,x,y)$$

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Solving the equations

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- Bref, we did.

SOLUTION FOR LABELLED QUADRANGULATIONS

$$t\mathsf{P}(t,ty) = \sum_{n\geq 0}\sum_{j=0}^{n} \frac{1}{n+1}\binom{2n-j}{n}\binom{3n-j}{n} y^j \mathsf{R}^{n+1},$$

$$\mathsf{C}(t,x,ty) = 1-\exp\left(-\sum_{n\geq 0}\sum_{j=0}^{n}\sum_{i=0}^{2n-j} \frac{1}{n+1}\binom{2n-j}{n}\binom{3n-i-j}{n} x^{i+1}y^{j+1}\mathsf{R}^{n+1}\right),$$

$$\mathsf{D}(t,x,ty) = \exp\left(\sum_{n\geq 0}\sum_{j=0}^{n}\sum_{i\geq 0} \frac{1}{n+1}\binom{2n-j}{n}\binom{3n+i-j+1}{2n-j} x^{i}y^{j+1}\mathsf{R}^{n+1}\right),$$

where $\mathsf{R}(t)$ satisfies

$$t = \sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n}\binom{3n}{n}\mathsf{R}(t)^{n+1}:$$

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The number of labelled quadrangulations with n faces is then

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$$q_n \sim \kappa \frac{\mu^{n+2}}{n^2(\log n)^2},$$

where $\kappa = 1/18$ and $\mu = 4\sqrt{3}\pi$.

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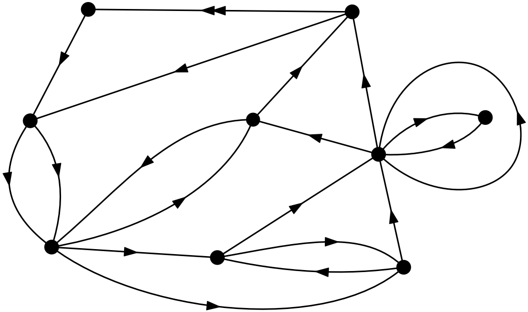
This asymptotic form verifies predictions of Kostov, Zinn-Justin and Guttmann. This appears to be the first exactly solved map problem with this universality class.

General rooted planar Eulerian orientations



ENUMERATING GENERAL ROOTED PLANAR EULERIAN ORIENTATIONS

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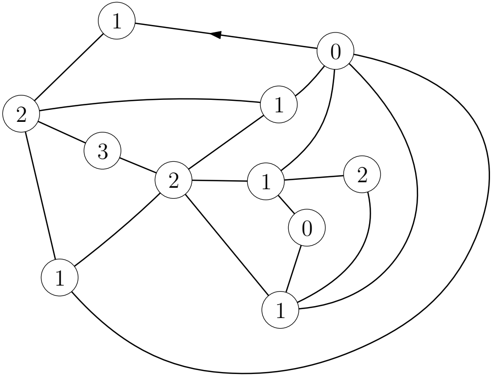
Enumerating general rooted planar Eulerian orientations

- Rooted planar Eulerian are in bijection with labelled maps (both counted by edges).
- I will now describe a bijection to labelled quadrangulations (counted by faces) in which each face has three distinct labels.
- This bijection is based on the mobile construction of Bouttier, Di Francesco and Guitter.

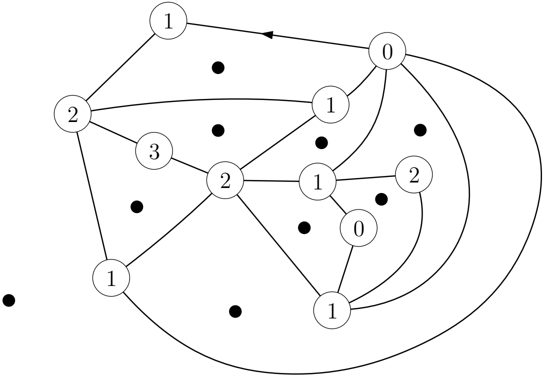
BIJECTION FROM RESTRICTED QUADRANGULATIONS TO LABELLED MAPS.

Start with a quadrangulation in which each face has three labels.



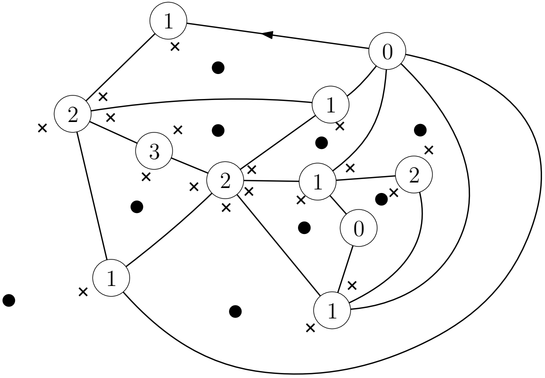
BIJECTION FROM RESTRICTED QUADRANGULATIONS TO LABELLED MAPS.

Add a black vertex in each face.



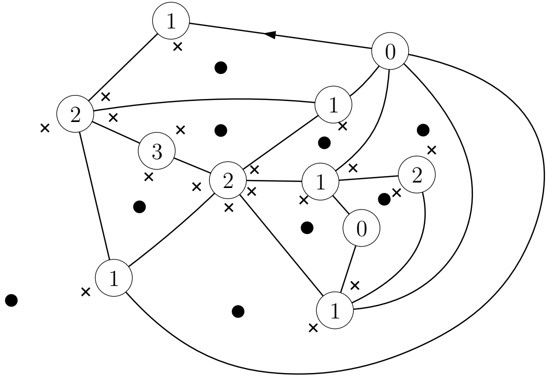
BIJECTION FROM RESTRICTED QUADRANGULATIONS TO LABELLED MAPS.

Around each face, put a cross at each corner whose label is greater than the next label clockwise.



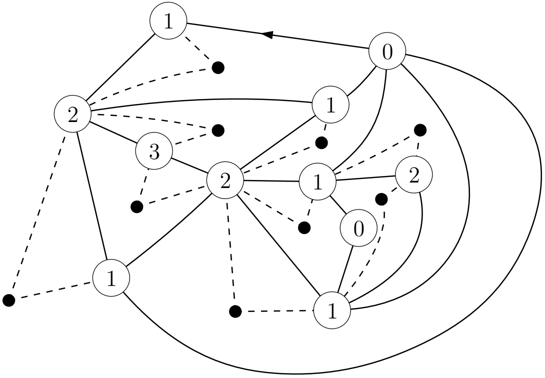
BIJECTION FROM RESTRICTED QUADRANGULATIONS TO LABELLED MAPS.

Draw an edge from each black vertex to each surrounding corner with a cross.



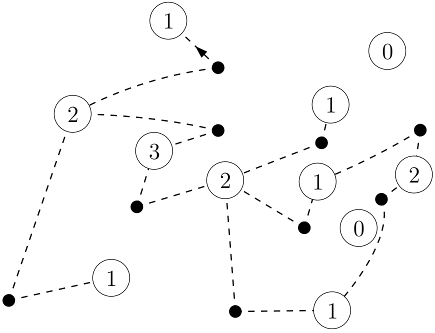
Bijection from restricted quadrangulations to labelled maps.

Draw an edge from each black vertex to each surrounding corner with a cross.



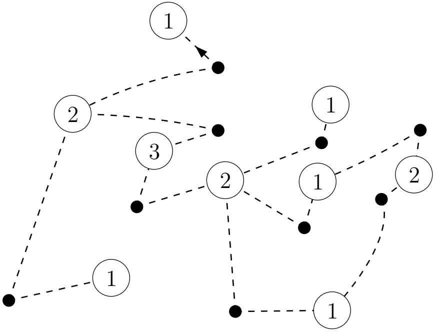
BIJECTION FROM RESTRICTED QUADRANGULATIONS TO LABELLED MAPS.

Remove all of the original edges.



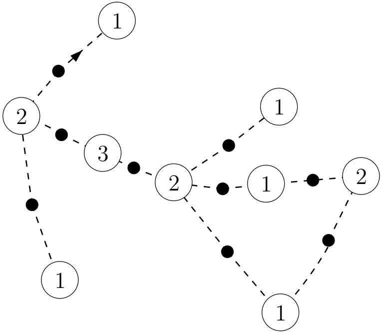
Bijection from restricted quadrangulations to labelled maps.

Remove any isolated vertices.



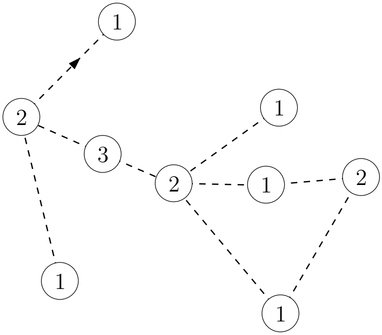
Bijection from restricted quadrangulations to labelled maps.

So far this is identical to the mobile bijection of Bouttier et al. (apart from the initial labelled map)



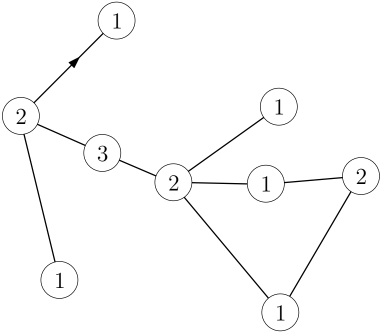
Bijection from restricted quadrangulations to labelled maps.

Remove the black vertices.



Bijection from restricted quadrangulations to labelled maps.

This map will either be a labelled map, or will become one when each label ℓ is changed to $2 - \ell$.



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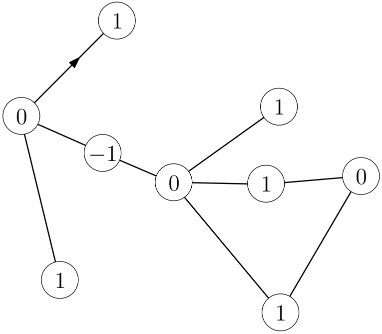

ENUMERATING GENERAL ROOTED PLANAR EULERIAN ORIENTATIONS

- So, $2g_n$ is the number of labelled quadrangulations with n faces in which each face has three distinct labels.

Enumerating general rooted planar Eulerian orientations

- So, $2g_n$ is the number of labelled quadrangulations with n faces in which each face has three distinct labels.
- We solve this problem in a similar way to how we enumerated labelled quadrangulations.

Enumerating rooted planar Eulerian orientations

The number of rooted planar Eulerian orientations with n edges is then

$$g_n = -\frac{1}{4}[t^{n+2}]\mathsf{S}(t),$$

where $\mathsf{S}(t)$ is the unique series with constant term 0 satisfying

$$t = \sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n}^2 \mathsf{S}(t)^{n+1}.$$

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Asymptotically this behaves as

$$g_n \sim \kappa \frac{\mu^{n+2}}{n^2(\log n)^2},$$

where $\kappa = 1/16$ and $\mu = 4\pi$.

FURTHER QUESTIONS

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- Can we generalise these results by counting labelled quadrangulations with a weight ω per face with only two distinct labels? In physics, this corresponds to the six vertex model on a random lattice.
- What about other weights?

Thank you!

Bonus slide: Guessing the solutions

- Using an earlier system of equations, Tony Guttmann and I computed the first 100 values of q_n.

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- By analysing the series, Tony guessed the exact asymptotic form of the series, including the growth rate $4\sqrt{3}\pi$.
- Mireille noticed that this growth rate had appeared before, in enumerating quadrangulations decorated by a spanning forest.
- So, we searched for an algebraic relationship between the problems, and we found one!

BONUS SLIDE: GUESSING THE SOLUTIONS

- We then transformed the series by writing $\mathcal{P}(t, y) = t\mathsf{P}(t, ty)$, $\mathcal{C}(t, x, y) = \mathsf{C}(t, x, ty)$ and $\mathcal{D}(t, x, y) = \mathsf{D}(t, x, ty)$ to remove t from the equations.

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- Next, we wrote $\mathcal{P}(t,y)$, $\mathcal{C}(t,x,y)$ and $\mathcal{D}(t,x,y)$ as series in R, x and y.

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- We noticed that $\mathcal{P}(t, y)$ is a simple hypergeometric function of R and y.
- After looking up some specialisations of $\mathcal{D}(t, x, y)$ in oeis, we guessed that it was an exponential of something simpler.
- Indeed, $\log(\mathcal{D}(t, x, y))$ is a simple hypergeometric function of R, x and y!