Counting patterns in weighted multigraphs

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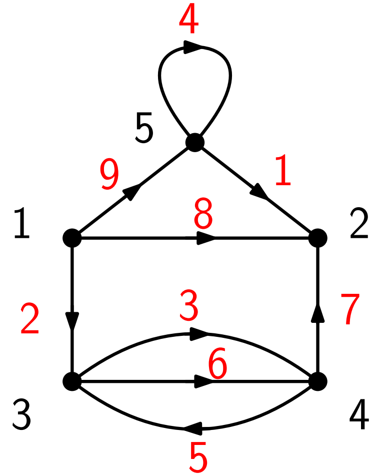
Model and definitions 1: Multigraphs

Multigraphs $\mathcal{MG}$:

labeled vertices,
labeled AND oriented edges,
loops or multiple edges allowed



Number of (n,m)-multigraphs, with n vertices, m edges:

$$MG_{n,m} = n^{2m}$$

Exponential generating series:

$$MG(z,w) = \sum_{n,m} MG_{n,m} \frac{z^n}{n!} \frac{w^m}{2^m m!} = \sum_{n \geq 0} e^{n^2 w/2} \frac{z^n}{n!}$$

Model and definitions 2: Weighted multigraphs

Multigraphs $\mathcal{MG}_\Delta$:

labeled vertices,
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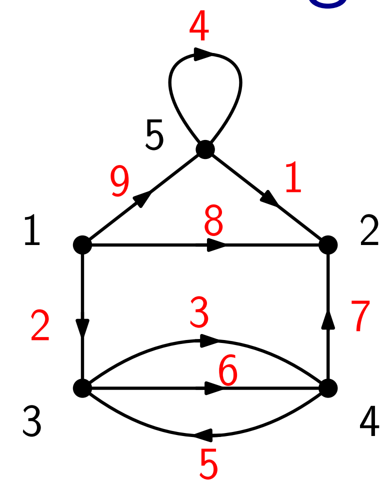
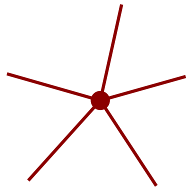
$+$ weights δ_d on vertices of degree d



Exponential GS for a single vertex with pendant half-edges

$$\Delta(x) = \sum_{d \geq 0} \delta_d \frac{x^d}{d!}, \quad x \text{ marks half-edges}$$



Weight of (Δ, n, m)-multigraphs, with n vertices, m edges:

$$MG_{\Delta,n,m} = \sum_{G \in \mathcal{MG}_{\Delta,n,m}} \omega(G) = \sum_{G \in \mathcal{MG}_{\Delta,n,m}} \prod_{v \in G} \delta_{\deg v}$$

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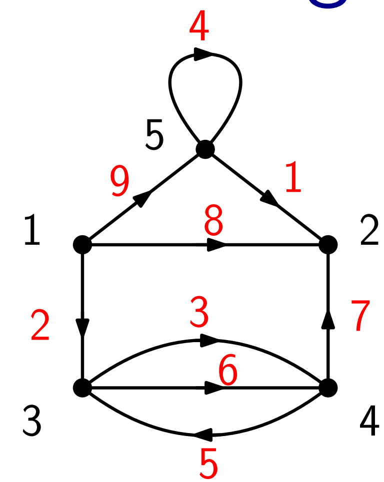

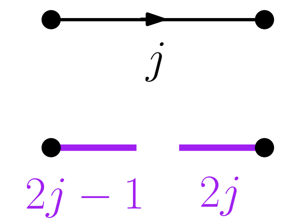

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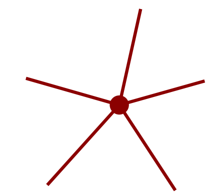

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Weighted multigraph $=$ set of weighted vertices carrying half-edges

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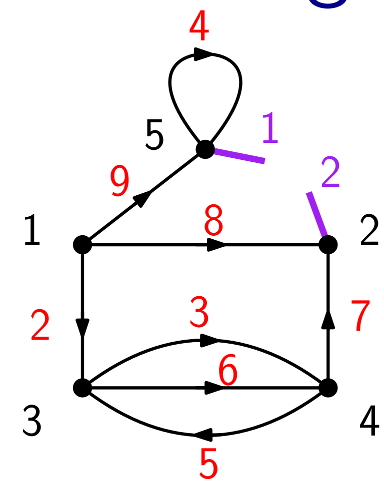

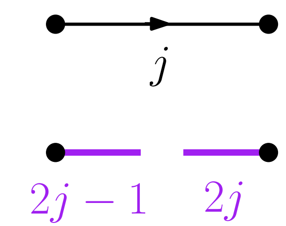

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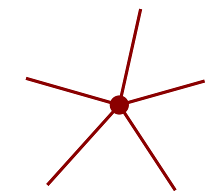

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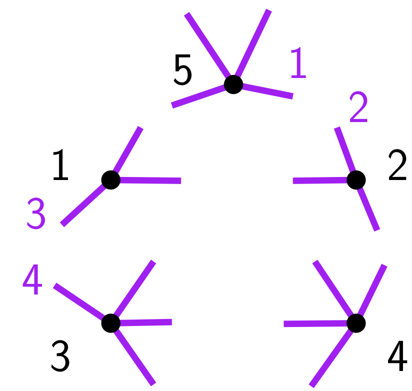

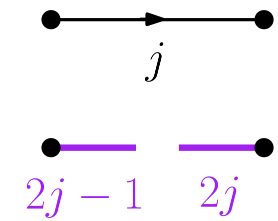

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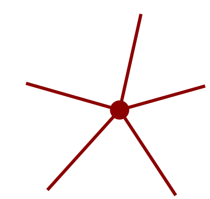

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Weighted multigraph $=$ set of weighted vertices carrying half-edges

Exponential GS: $\qquad MG_\Delta(z,x) = e^{z\Delta(x)}$

$$\Rightarrow MG_{\Delta,n,m} = (2m)![x^{2m}]\Delta(x)^n$$

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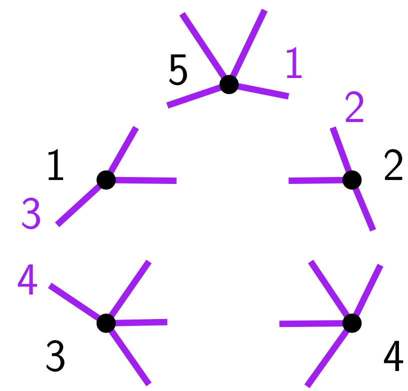

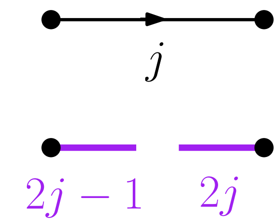

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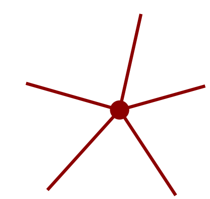


Examples:

- d-regular graphs: $\Delta(x) = \frac{x^d}{d!}$
- Eulerian graphs: $\Delta(x) = \cosh(x)$
- For any nonnegative integer set D, D-graphs: $\delta_d = \mathbf{1}_{d \in D}$
- Any degree allowed, but different distribution: power law graphs

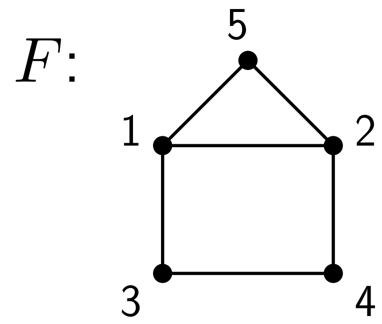
$\mathbf{P}(\deg(v) = d) \propto d^{-\beta}$, i.e. $\delta_d = d^{-\beta} \cdot d!$ and Δ evaluated at $x = 1$

Model and definitions 3: Patterns

Pattern:

- Given a multigraph F, which may appear as a subgraph after relabeling

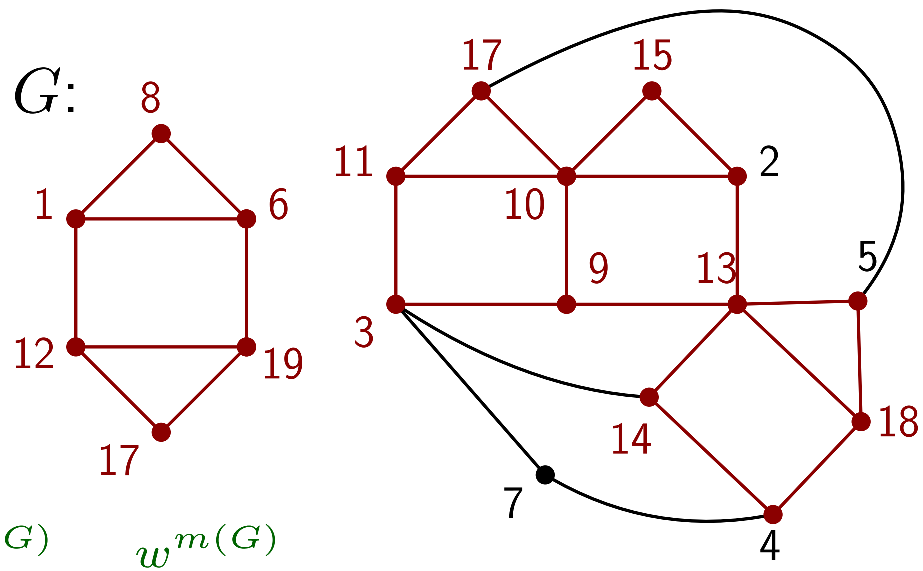
  How to count occurrences of the pattern?



Given a multigraph G:

$G[F] =$ number of occurrences of F in G

ex: here $G[F] = 5$



$$MG_{\Delta}^{F}(z,w,u) = \sum_{G \in \mathcal{MG}_{\Delta}} u^{G[F]} \omega(G) \frac{z^{n(G)}}{n(G)!} \frac{w^{m(G)}}{2^{m(G)} m(G)!}$$

Question: Estimate $[z^n w^m u^t] MG_{\Delta}^{F}(z,w,u)$ **when** $n, m \to \infty$

Marking one occurrence of the pattern

$\mathcal{MG}_{\Delta}^{(F)} :=$ Multigraphs with exactly one distinguished F-subgraph

Theorem [CdPGGV'18]:

Let $F(z, w, \mathbf{y})$ be the EGF of the pattern (y_d marks vertices of degree d),

$$MG_{\Delta,n,m}^{(F)} = n!2^m m![z^n w^m] \sum_{j \geq 0} (2j)![x^{2j}] F(z, w, \bar{\partial}\Delta(x)) e^{z\Delta(x)} \frac{w^j}{2^j j!}$$

where $\bar{\partial}\Delta(x) := (\Delta(x), \Delta'(x), \Delta''(x), \ldots)$.

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Proof (exact):

Construct an element of $\mathcal{MG}_{\Delta}^{(F)}$

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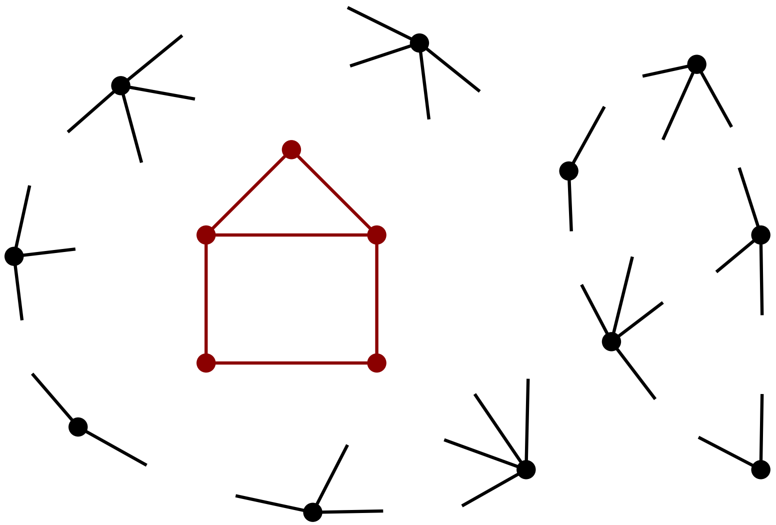
$$F(z, w, \mathbf{y}) e^{z\Delta(x)}$$



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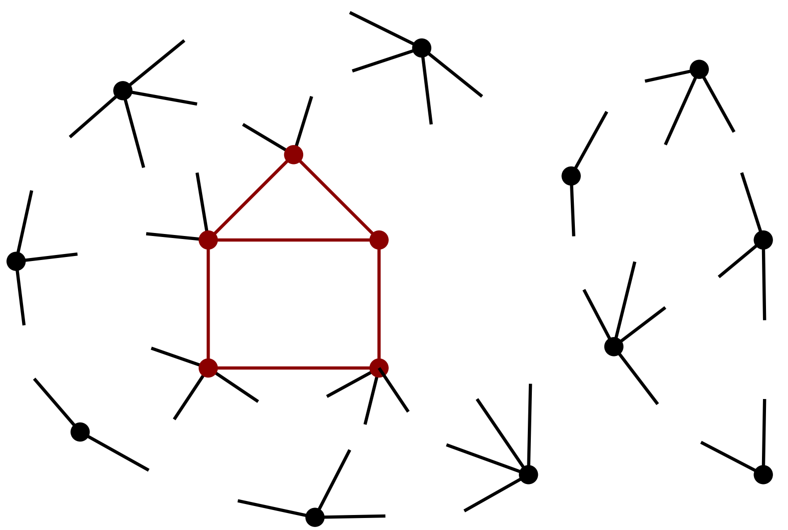
- Extra half-edges on F: DANGER

$$y_d \to \Delta^{(d)}(x)$$



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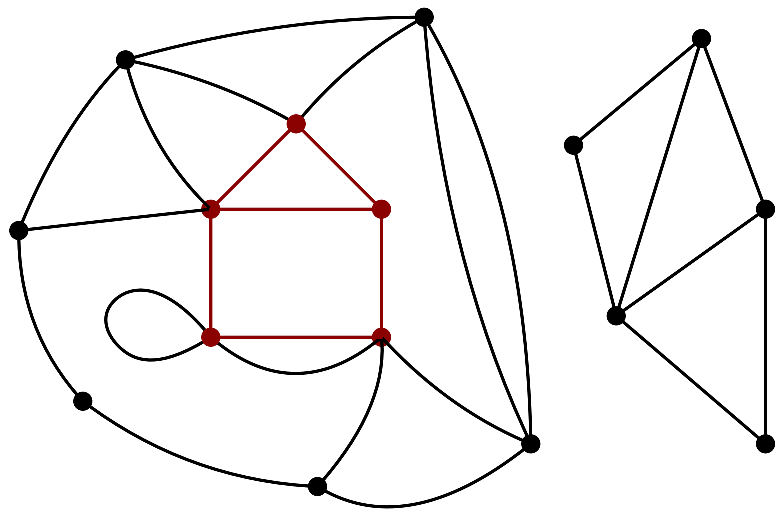
  $$y_d \to \Delta^{(d)}(x)$$
- Connect half-edges $2j-1$ to $2j$ to form edge j



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$\mathcal{MG}_{\Delta}^{(F)}$:= Multigraphs with exactly one distinguished F-subgraph

Theorem [CdPGGV'18]:

Under some regularity conditions on Δ and F, when $\frac{2m}{n} \to_{n\to\infty} \ell$:

(asymptotics) $MG_{\Delta,n,m}^{(F)} \sim_{n\to\infty} F\left(n, \frac{1}{2m}, \left(\frac{\chi^d \Delta^{(d)}(\chi)}{\Delta(\chi)}\right)_{d\geq 0}\right) \cdot MG_{\Delta,n,m}$

where $\chi := \chi_n$ solution of: $\frac{\chi\Delta'(\chi)}{\Delta(\chi)} = \frac{2m}{n}$.

Proof (asymptotics): Laplace method and saddle-point techniques,
with moving saddle-points.

Corollary:

For G a random multigraph in $\mathcal{MG}_{\Delta,n,m}$, with $\mathbf{P}[G] = \frac{\omega(G)}{MG_{\Delta,n,m}}$,

$$\mathbf{E}[G[F]] \sim_{n\to\infty} F\left(n, \frac{1}{2m}, \left(\frac{\chi^d \Delta^{(d)}(\chi)}{\Delta(\chi)}\right)_{d\geq 0}\right)$$

Marking one occurrence of the pattern

Corollary [Erdős-Rényi'60]:

Given a pattern F, and G a uniform random multigraph from $\mathcal{MG}_{n,m}$

$\mathbf{P}(G[F] > 0) \to 0$ when $n \to \infty$ and $m = o(n^{2-n(F)/m(F)})$

Proof:

Uniform from $\mathcal{MG}_{n,m} \Rightarrow \Delta(x) = e^x \Rightarrow \chi = \frac{2m}{n}$

$$\mathbf{E}(G[F]) = \frac{MG^{(F)}_{n,m}}{MG_{n,m}} \sim F\left(n, \frac{1}{2m}, \left(\left(\frac{2m}{n}\right)^d\right)_{d \geq 0}\right) = \frac{n^{n(F)}}{n(F)!} \frac{(2m)^{m(F)}}{n^{2m(F)} 2^{m(F)} m(F)!}$$

$$= o\left(n^{n(F)-2m(F)+m(F)(2-n(F)/m(F))}\right)$$

Threshold phenomenon:

- $m = o(n^{2-n(F)/m(F)})$: no occurrence of F a.s.
- $m = cn^{2-n(F)/m(F)}$: $G[F]$ follows a Poisson law, for special F
- $m \gg n^{2-n(F)/m(F)}$: many occurrences of F a.s.

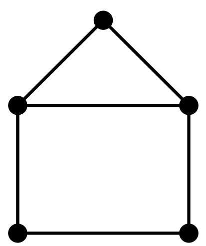
Patchworks

Problem: we can only access expectancy!

Solution: patchworks!

$\mathcal{PW}^F :=$ Set of gluings of copies of the pattern

F:



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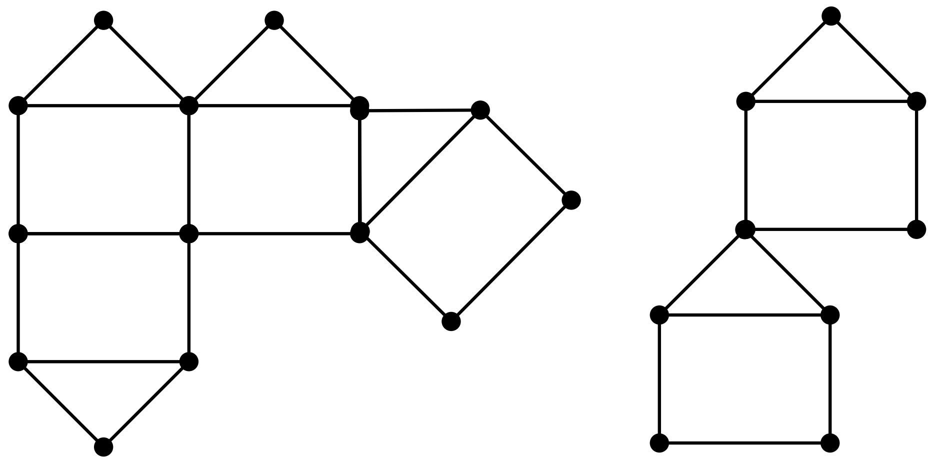

Idea: replace $F(z, w, \mathbf{y})$ by $PW_u^F(z, w, \mathbf{y})$ in previous equations

$\Rightarrow$ distinguish a subset of copies of F, u marks the number of copies

inclusion-exclusion principle: $u \to u + 1$

$$MG_\Delta^F(z, w, u+1) = \sum_{j \geq 0} (2j)! [x^{2j}] PW_u^F(z, w, \bar{\partial}\Delta(x)) e^{z\Delta(x)} \frac{w^j}{2^j j!}$$

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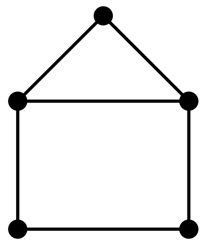

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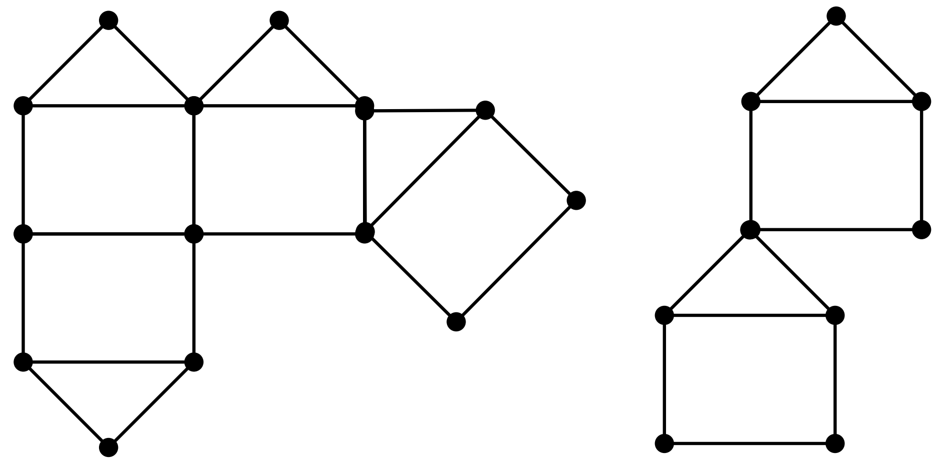

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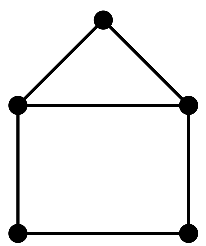

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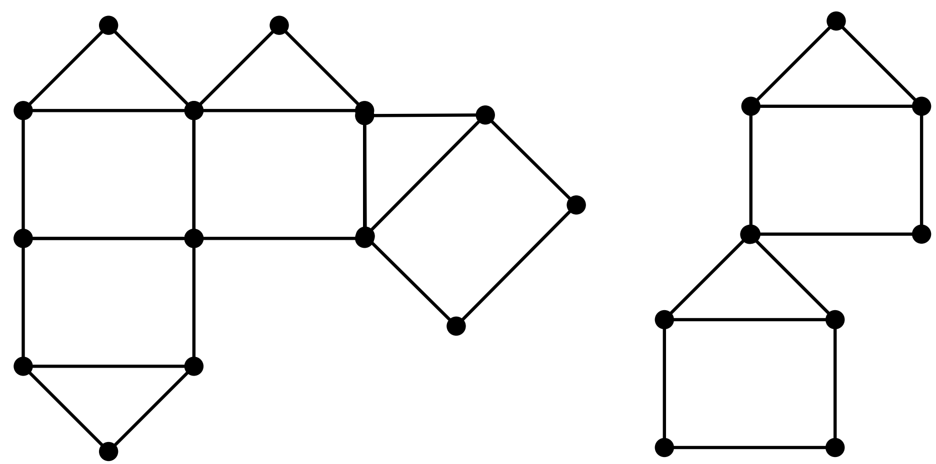

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Strictly balanced patterns

Problem (again): patchworks are messy (in general)!
Solution: strictly balanced graphs!

F strictly balanced iff $\forall H \subset F, H \neq F, d(H) < d(F)$

Property: Any connected patchwork of a strictly balanced graph with at least two copies is strictly denser than the graph itself.

Idea: replace $PW_u^F(z, w, \mathbf{y})$ by set of disjoint copies of $F \rightarrow e^{uF(z,w,\mathbf{y})}$.

Theorem [CdPGGV'18]:

Under some regularity conditions on Δ and F, when $m = \Theta(n^{2-1/d(F)})$:

if F is strictly balanced then $G[F]$ follows a Poisson law

of parameter $\lim_{n\to\infty} F\left(n, \frac{1}{2m}, \left(\frac{\chi^d \Delta^{(d)}(\chi)}{\Delta(\chi)}\right)_{d\geq 0}\right)$

where $\chi := \chi_n$ solution of: $\frac{\chi\Delta'(\chi)}{\Delta(\chi)} = \frac{2m}{n}$.

Applications

Corollary [Cycles]:

Let $m = cn \quad (c > 0)$, if $\lambda := \lim_{n\to\infty} \frac{n^k}{2m^k} \left(\frac{\chi^2 \Delta''(\chi)}{\Delta(\chi)} \right)^k > 0$

The number of cycles of length k in a random (n, m, Δ)-multigraph follows a Poisson law of parameter λ.

Corollary [Complex patterns]:

Any connected graph that is neither a tree nor a unicycle is asymptotically almost surely not a subgraph of G, when $m = cn \quad (c > 0)$.

Beyond

Extensions:

- Many results transfer to simple graphs.
- Main theorem can be extended to special cases:

  counting trees $\to m = o(n)$

  periodic sets D

  power law profile $\to \Delta(x)$ has finite radius of convergence

Theorem [CdPGGV'18]:

Expected number of cycles of length ℓ, when $\delta_d = d^{-\beta} d!, 2 < \beta < 3$

$$\mathbf{E}[G[C_\ell]] \sim_{n \to \infty} \kappa_{\beta,\ell} n^{\ell \frac{3-\beta}{\beta-1}}$$

where $\kappa_{\beta,\ell}$ computable constant.

Further research:

- Obtain laws for non-strictly balanced patterns
- Behaviour around the threshold $m = cn^{2-n(F)/m(F)}$

Thank you

Master Theorem Hypothesis

Let χ be the unique positive solution of $\frac{\chi\Delta'(\chi)}{\Delta(\chi)} = \frac{2m}{n}$

Assume:

(L1) $2m/n \to_{n\to\infty} c > 0$,

(L2) $\frac{F\left(\frac{nz}{\Delta(x\chi)},\frac{(x\chi)^2}{2mt^2},(\bar{\partial}\Delta)(x\chi)\right)}{F\left(\frac{n}{\Delta(\chi)},\frac{\chi^2}{2m},(\bar{\partial}\Delta)(\chi)\right)} \overset{(unif)}{\to} L(z,x,t)$, analytic, as $n \to \infty$

for $|z|, |x| = 1$ and t in any compact sub-interval of $[0, \infty[$.

(L3) $\forall 0 < \epsilon < \frac{\pi}{2}, \int_{-\pi}^{\pi} \frac{\Delta\left(\chi e^{i\theta}\right)^n}{e^{2ni\theta}} d\theta \sim \int_{-\epsilon}^{\epsilon} \frac{\Delta\left(\chi e^{i\theta}\right)^n}{e^{2ni\theta}} d\theta$, as $n \to \infty$.

Then, the total weight of (n, m, Δ)-multigraphs is asymptotically

$$|MG_{n,m,\Delta}^{(F)}| \sim |MG_{n,m,\Delta}| \cdot F\left(\frac{n}{\Delta(\chi)}, \frac{\chi^2}{2m}, (\bar{\partial}\Delta)(\chi)\right)$$

where $|MG_{n,m,\Delta}| = (2m)![x^{2m}]\Delta(x)^n$