Walks with large steps in the quadrant

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ArXiv: soon!
Counting quadrant walks

Let $S$ be a finite subset of $\mathbb{Z}^2$ (set of steps).
We look at walks starting at $(0,0)$ and formed of steps of $S$.

Example. $S = \{10, \bar{1}0, 1\bar{1}, \bar{1}1\}$
Counting quadrant walks

Let $S$ be a finite subset of $\mathbb{Z}^2$ (set of steps).
We look at walks starting at $(0,0)$ and formed of steps of $S$.

- What is the number $q(n)$ of $n$-step walks contained in $\mathbb{N}^2$?
- For $(i,j) \in \mathbb{N}^2$, what is the number $q(i,j; n)$ of such walks that end at $(i,j)$?

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The associated generating function:

$$Q(x, y; t) = \sum_{n \geq 0} \sum_{(i, j) \in \mathbb{N}^2} q(i, j; n) x^i y^j t^n$$

What is the nature of this series?
A hierarchy of formal power series

- **Rational series**
  \[ A(t) = \frac{P(t)}{Q(t)} \]

- **Algebraic series**
  \[ \text{Pol}(t, A(t)) = 0 \]

- **Differentially finite series (D-finite)**
  \[ \sum_{i=0}^{d} P_i(t) A^{(i)}(t) = 0 \]

- **D-algebraic series**
  \[ \text{Pol}(t, A(t), A'(t), \ldots, A^{(d)}(t)) = 0 \]
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Multi-variate series: one DE per variable
Classification of quadrant walks with small steps

quadrant models with small steps: 79

$|G|<\infty$: 23

D-finite

OS=0: 4  OS\neq0: 19

algebraic  DF transc.

$G|\infty$: 56

Not D-finite

decoupled: 9  not decoupled: 47

D-alg.  not D-alg. (in $x$)

- $G$ the group of the model
- OS the orbit sum

[mbm-Mishna 10] [Bostan-Kauers 10] [Mishna-Rechnitzer 07]
 Mellczer-Mishna 13] [Kurkova-Raschel 12] [Bostan-Raschel-Salvy 14]
 [Bernardi-mbm-Raschel 17(a)] [Dreyfus-Hardouin-Roques-Singer 17(a)]
Quadrant walks with arbitrary steps

- A mathematical challenge: the small step condition seems crucial in all approaches (apart from computer algebra)
- Large steps occur in “real life”: simple walk models,
Quadrant walks with arbitrary steps

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- Large steps occur in “real life”: simple walk models, queuing theory, bipolar orientations ([Kenyon et al. 15(a)])
A general approach for quadrant walks...

which solves some cases.

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“The simple branch”

[mbm-Mishna 10]
A four step approach

1. Write a functional equation for the tri-variate series \( Q(x, y; t) \). It involves bi-variate series \( Q(x, 0; t), Q(0, y; t), \ldots \) (called sections)

2. Compute the “orbit” of \((x, y)\)

3. Combine the main equation and the orbit to find a functional equation \textit{free from sections}

4. Extract from it \( Q(x, y; t) \)
Step 1: Write a functional equation

Example: \( S = \{01, \bar{1}0, 1\bar{1}\} \) (bipolar triangulations)

\[
Q(x, y; t) \equiv Q(x, y) = 1 + t(y + \bar{x} + x\bar{y})Q(x, y) - t\bar{x}Q(0, y) - tx\bar{y}Q(x, 0)
\]

with \( \bar{x} = 1/x \) and \( \bar{y} = 1/y \).
Step 1: Write a functional equation

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or

\[ (1 - t(y + \overline{x} + x\overline{y}))Q(x, y) = 1 - t\overline{x}Q(0, y) - tx\overline{y}Q(x, 0) \]

- The polynomial \( 1 - t(y + \overline{x} + x\overline{y}) \) is the kernel of this equation
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\]

- The polynomial \( 1 - t(y + \bar{x} + x\bar{y}) \) is the kernel of this equation
- The series \( Q(0, y) \) and \( Q(x, 0) \) are the sections.
Step 1: Write a functional equation

Example: $S = \{01, \bar{1}0, 1\bar{1}\}$ (bipolar triangulations)

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or

$$(1 - t(y + \bar{x} + x\bar{y}))Q(x, y) = 1 - t\bar{x}Q(0, y) - tx\bar{y}Q(x, 0)$$

Example: Bipolar quadrangulations

$$(1 - t(x\bar{y} + \bar{x}^2 + \bar{x}y + y^2))Q(x, y) = 1 - tx\bar{y}Q(x, 0)$$

$$-t\bar{x}^2 (Q_0(y) + xQ_1(y)) - t\bar{x}yQ_0(y),$$

where $Q_i(y)$ counts quadrant walks ending at abscissa $i$.

$\Rightarrow$ More sections, kernel of higher degree
Step 2: the group of the model (and its orbit)

- The step polynomial:

\[ S(x, y) = \bar{x} + y + x\bar{y} \]
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Observation: \( S(x, y) \) is unchanged by the rational transformations

\[ \Phi: (x, y) \mapsto (\bar{x}y, y) \quad \text{and} \quad \Psi: (x, y) \mapsto (x, x\bar{y}). \]
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- \( \Phi \) and \( \Psi \) are involutions
- They generate a (dihedral) group
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- The pairs \((\bar{x}y, y)\) and \((x, x\bar{y})\) are “adjacent” to \((x, y)\):
  - they have one coordinate in common
  - they give the same value to the step polynomial \( S \)
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  – they have one coordinate in common
  – they give the same value to the step polynomial \( S \)

• Let \( \sim \) be the transitive closure of the adjacency relation. The orbit of \((x, y)\) is its equivalence class.

\[
\begin{array}{c}
\sim \quad (\bar{x}y, y) \quad \sim \quad (\bar{x}y, \bar{x}) \quad \sim \\
\sim \quad (x, y) \quad \sim \quad (x, x\bar{y}) \quad \sim \\
\sim \quad (\bar{y}, \bar{x}) \quad \sim \quad (\bar{y}, x\bar{y}) \quad \sim \\
\end{array}
\]
Step 2: the orbit of the model

**Example:** Bipolar quadrangulations

\[ S(x, y) = \bar{x}^2 + \bar{x}y + y^2 + x\bar{y} \]

The equation (in \( x' \)) \( S(x, y) = S(x', y) \), has 3 solutions, namely \( x \) and \( x_{1,2} = \frac{xy^2 + y \pm \sqrt{y (x^2 y^3 + 4x^3 + 2xy^2 + y)}}{2x^2} \).
Step 2: the orbit of the model

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\[ x_{1,2} = \frac{xy^2 + y \pm \sqrt{y \left( x^2y^3 + 4x^3 + 2xy^2 + y \right)}}{2x^2} \]

- the pairs \((x_1, y)\) and \((x_2, y)\) are adjacent to \((x, y)\)
Step 2: the orbit of the model

Example: Bipolar quadrangulations

\[ S(x, y) = \bar{x}^2 + \bar{x}y + y^2 + xy \]

The equation (in \(x')\) \(S(x, y) = S(x', y)\), has 3 solutions, namely \(x\) and

\[ x_{1,2} = \frac{xy^2 + y \pm \sqrt{y (x^2 y^3 + 4 x^3 + 2 xy^2 + y)}}{2x^2} \]

- the pairs \((x_1, y)\) and \((x_2, y)\) are adjacent to \((x, y)\)
- there are also two pairs \((x, y')\) that are adjacent to \((x, y)\), which happen to be \((x, \bar{x}_1)\) and \((x, \bar{x}_2)\)
Step 2: the orbit of the model

Example: Bipolar quadrangulations

\[ S(x, y) = \tilde{x}^2 + \tilde{x}y + y^2 + x\tilde{y} \]

The equation (in \( x' \)) \( S(x, y) = S(x', y) \), has 3 solutions, namely \( x \) and

\[ x_{1,2} = \frac{xy^2 + y \pm \sqrt{y \left(x^2 y^3 + 4x^3 + 2xy^2 + y\right)}}{2x^2}. \]

- the pairs \((x_1, y)\) and \((x_2, y)\) are adjacent to \((x, y)\)
- there are also two pairs \((x, y')\) that are adjacent to \((x, y)\), which happen to be \((x, \tilde{x}_1)\) and \((x, \tilde{x}_2)\)
- which pairs \((x_1, y')\) are adjacent to \((x_1, y)\)?
Step 2: the orbit of the model

Example: Bipolar quadrangulations

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The equation (in \(x'\)) \( S(x, y) = S(x', y) \), has 3 solutions, namely \(x\) and

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- which pairs \((x_1, y')\) are adjacent to \((x_1, y)\)?
- and so on.
Step 2: the orbit of the model

Example: Bipolar quadrangulations

\[ S(x, y) = \bar{x}^2 + \bar{x}y + y^2 + x\bar{y} \]

The equation (in \( x' \)) \( S(x, y) = S(x', y) \), has 3 solutions, namely \( x \) and

\[ x_{1,2} = \frac{xy^2 + y \pm \sqrt{y \left( x^2y^3 + 4x^3 + 2xy^2 + y \right)}}{2x^2} \]

- Orbit
Step 3: Find a functional equation free from sections

- The equation reads (with $K(x, y) = 1 - tS(x, y)$):

  $$K(x, y)xyQ(x, y) = xy - tx^2 Q(x, 0) - tyQ(0, y)$$

- The orbit of $(x, y)$ is

  $$(x, y) \approx (\bar{x}y, y) \approx (\bar{x}y, \bar{x}) \approx (\bar{y}, \bar{x}) \approx (\bar{y}, x\bar{y}) \approx (x, x\bar{y})$$
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- The value of $S(x, y)$ (and $K(x, y)$) is the same over the orbit. Hence

$$K(x, y) \ xyQ(x, y) = xy - tx^2Q(x, 0) - tyQ(0, y)$$

$$K(x, y) \ \bar{x}y^2Q(\bar{x}y, y) = \bar{x}y^2 - t\bar{x}^2y^2Q(\bar{x}y, 0) - tyQ(0, y)$$

$$K(x, y) \ \bar{x}^2yQ(\bar{x}y, \bar{x}) = \bar{x}^2y - t\bar{x}^2y^2Q(\bar{x}y, 0) - t\bar{x}Q(0, \bar{x})$$

$$\ldots = \ldots$$

$$K(x, y) \ x^2\bar{y}Q(x, x\bar{y}) = x^2\bar{y} - tx^2Q(x, 0) - tx\bar{y}Q(0, x\bar{y}).$$
Step 3: Find a functional equation free from sections

⇒ Form the alternating sum of the equation over all elements of the orbit:

\[
K(x, y) \left( \sum \right) = xyQ(x, y) - \bar{x}y^2 Q(\bar{x}y, y) + \bar{x}^2 yQ(\bar{x}y, \bar{x}) - \bar{x}\bar{y} Q(\bar{y}, \bar{x}) + x\bar{y}^2 Q(x, \bar{y}) - x^2 \bar{y} Q(x, x\bar{y}) = xy - \bar{x}y^2 + \bar{x}^2 y - \bar{x}\bar{y} + x\bar{y}^2 - x^2 \bar{y}
\]

(the orbit sum).
Step 3: Find a functional equation free from sections

⇒ Form the alternating sum of the equation over all elements of the orbit:

\[ xyQ(x, y) - \bar{x}y^2 Q(\bar{x}y, y) + \bar{x}^2 yQ(\bar{x}y, \bar{x}) \]
\[ - \bar{x}\bar{y} Q(\bar{y}, \bar{x}) + x\bar{y}^2 Q(\bar{y}, x\bar{y}) - x^2 \bar{y} Q(x, x\bar{y}) = \]

\[ \frac{xy - \bar{x}y^2 + \bar{x}^2 y - \bar{x}\bar{y} + x\bar{y}^2 - x^2 \bar{y}}{1 - t(y + \bar{x} + x\bar{y})} \]
Step 4: Extract $Q(x, y)$

⇒ Form the alternating sum of the equation over all elements of the orbit:

$$xyQ(x, y) - \bar{x}y^2Q(\bar{x}y, y) + \bar{x}^2yQ(\bar{x}y, \bar{x})$$

$$- \bar{x}\bar{y}Q(\bar{y}, \bar{x}) + x\bar{y}^2Q(\bar{y}, x\bar{y}) - x^2\bar{y}Q(x, x\bar{y}) =$$

$$\frac{xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})}$$

• Both sides are power series in $t$, with coefficients in $\mathbb{Q}[x, \bar{x}, y, \bar{y}]$. 

[Lipshitz 88]
Step 4: Extract \( Q(x, y) \)

⇒ Form the alternating sum of the equation over all elements of the orbit:

\[
xyQ(x, y) - \bar{x}y^2Q(\bar{x}y, y) + \bar{x}^2yQ(\bar{x}y, \bar{x}) - \bar{x}\bar{y}Q(\bar{y}, \bar{x}) + x\bar{y}^2Q(\bar{y}, x\bar{y}) - x^2\bar{y}Q(x, x\bar{y}) = \frac{xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})}
\]

- Both sides are power series in \( t \), with coefficients in \( \mathbb{Q}[x, \bar{x}, y, \bar{y}] \).

- Extract the part with positive powers of \( x \) and \( y \):

\[
xyQ(x, y) = [x^0 y^0] \frac{xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})}
\]

is a D-finite series.

[Lipshitz 88]
What can go wrong?

1. Functional equation: OK
2. The “orbit” may be infinite
3. There may be several section-free equations (or none?) [NEW]
4. The extraction may be tricky [NEW], or impossible
Some cases that work

- Hadamard walks:
  \[ S(x, y) = U(x) + V(x) T(y) \]
  (Small steps: 16 out of the 19 “simple” models)

- Bipolar maps [mbm, Fusy, Raschel 18]

\[ S_p = \{(-p, 0), (-p + 1, 1), \ldots, (0, p), (1, -1)\} \]

In all those cases, the orbit is finite and the series D-finite, expressed as the non-negative part of an algebraic series.
Quadrant walks with steps in \([-2, -1, 0, 1]\)^2

- In all cases, a unique section-free equation

  quadrant models: 13 110*

  \[\text{orient} < \infty: \, 227 + 13 \quad \text{orient} = \infty: \, 12 \, 870\]

  \[\text{OS} = 0: \, 9 \quad \text{OS} \neq 0: \, 227 + 4\]

  DF

- 227 Hadamard models

(*) Models with at least one occurrence of \(-2\)
Some interesting models

• Non-Hadamard, solvable via our approach (and D-finite):

  \[
  \begin{array}{cccc}
  \text{DF} & \text{DF} & \text{DF} & \text{DF} \\
  \text{DF} & \text{DF} & \text{DF} & \text{DF} \\
  \text{DF} & \text{Alg} & \text{Alg} & \text{DF} \\
  \end{array}
  \]

• Non-Hadamard, orbit sum zero: let’s guess!

  \[
  \begin{array}{cccc}
  \text{DF} & \text{DF} & \text{DF} & \text{DF} \\
  \text{DF} & \text{DF} & \text{DF} & \text{DF} \\
  \text{DF} & \text{Alg} & \text{Alg} & \text{DF} \\
  \end{array}
  \]

à la Kreweras

à la Gessel
Some interesting models

• Non-Hadamard, solvable via our approach (and D-finite):

• Non-Hadamard, orbit sum zero: let’s guess!

à la Kreweras

DF  DF  DF  DF  DF

à la Gessel

DF  Alg  Alg  DF
Final comments

Still a lot to be done...

- Is there a unique section free equation when there are no large forward steps?
- Closer study for tricky examples (the 9 analogues of Kreweras’ and Gessel’s algebraic models)
- Nature of models where $\alpha$ is rational but the orbit infinite

quadrant models: 13 110

$|\text{orbit}| < \infty$:
- OS = 0: 9
- OS $\neq$ 0: 227 + 4

$|\text{orbit}| = \infty$:
- $\alpha$ rational: 16
- $\alpha$ irrat.: 12 854

DF $?$

DF

not DF $?$

not DF

[Bo\text{-}st\text{-}an-Raschel-Salvy 14]
Step 2: the orbit of the model

Example: Bipolar quadrangulations

\[ S(x, y) = \bar{x}^2 + \bar{x}y + y^2 + x\bar{y} \]

The equation (in \( x' \)) \( S(x, y) = S(x', y) \), has 3 solutions, namely \( x \) and \( x_{1,2} = \frac{xy^2 + y \pm \sqrt{y \left(x^2y^3 + 4x^3 + 2xy^2 + y\right)}}{2x^2} \).

- Orbit
Step 3: Find a functional equation free from sections

Example: Bipolar quadrangulations

\[ S(x, y) = \bar{x}^2 + \bar{x}y + y^2 + x\bar{y} \]

Two section-free linear combinations (+ linear combinations):

\[
xy Q(x, y) - \bar{x}_1 x Q(x, \bar{x}_1) - \frac{x_1^2 y (x - x_2)}{x (x_1 - x_2)} \frac{Q(x_1, y)}{x_1^2 (x - x_2)} + \frac{x_1^2 (x - x_2)}{(x_1 - x_2) x^2} \frac{Q(x_1, \bar{x})}{x_1^2 (x - x_2)}
\]
\[
+ \frac{x_2^2 y (x - x_1)}{x (x_1 - x_2)} \frac{Q(x_2, y)}{x_2^2 (x - x_2)} - \frac{(x_1 y - 1) x_2^2 (x - x_2)}{x_2 y - 1} \frac{Q(x_2, \bar{x})}{x_1^2 (x - x_2)}
\]
\[
+ \frac{(xy - 1) x_2^2 Q(x_2, \bar{x}_1)}{x_1 x (x_2 y - 1)} + \frac{(x - x_2) Q(\bar{y}, \bar{x})}{y (x_2 y - 1) x^2} - \frac{(x - x_2) Q(\bar{y}, \bar{x}_1)}{yx_1 x (x_2 y - 1)}
\]
\[
= \frac{(y - \bar{x}_1)(xy - 1)(\bar{y} - \bar{x}^2 y - 2\bar{x}^3)}{K(x, y)},
\]

and the same equation with \( x_1 \) and \( x_2 \) exchanged.
Step 4: Extract $Q(x, y)$

**Example:** Bipolar quadrangulations

$$S(x, y) = \bar{x}^2 + \bar{x}y + y^2 + x\bar{y}$$

- A section-free equation:

$$xyQ(x, y) - \bar{x}_1xQ(x, \bar{x}_1) - \frac{x_1^2y(x - x_2)Q(x_1, y)}{x(x_1 - x_2)} + \ldots$$

$$= \frac{(y - \bar{x}_1)(xy - 1)(\bar{y} - \bar{x}^2y - 2\bar{x}^3)}{K(x, y)}$$

- Then

$$xyQ(x, y) = [x>0, y>0] \frac{(y - \bar{x}_1)(xy - 1)(\bar{y} - \bar{x}^2y - 2\bar{x}^3)}{K(x, y)}$$

provided the RHS is expanded first in $t$, then in $\bar{y}$, and finally in $x$. 
Some interesting models

- Non-Hadamard, solvable via our approach (and D-finite):

For the first model,

\[ Q(x, y) = [x \geq 0 \; y \geq 0] \frac{(x^3 - 2y^2 - x) (y^2 - x) (x^2y^2 - y^2 - 2x)}{x^5y^4 (1 - t(y + x\bar{y} + x\bar{y} + \bar{x}^2y))}. \]

The coefficients are nice: for \( n = 2i + j + 4m \),

\[ q(i, j; n) = \frac{(i + 1)(j + 1)(i + j + 2)n!(n + 2)!}{m!(3m + 2i + j + 2)!(2m + i + 1)!(2m + i + j + 2)!}. \]
Walks on a half-line ($d = 1$)

Let $S \subset \mathbb{Z}$ with $\min S = -m$.

**Proposition [Bostan, mbm, Melczer 18]**

$$Q(x) = [x^{\geq 0}] \frac{\prod_{j=1}^{m} (1 - \bar{x} x_j)}{1 - tS(x)},$$

where the $x_j$ are the roots of $S(x_j) = S(x)$ whose expansion in $\bar{x}$ involves no positive power of $x$.
Walks on a half-line \((d = 1)\)

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where the \(x_j\) are the roots of \(S(x_j) = S(x)\) whose expansion in \(\bar{x}\) involves no positive power of \(x\).

**Classical solution [Gessel 80, mbm-Petkovšek 00, Banderier-Flajolet 02...]**

\[
Q(x) = \frac{\prod_{j=1}^{m} (1 - \bar{x}X_j)}{1 - tS(x)}
\]

where the \(X_j \equiv X_j(t)\) are the roots of \(1 - tS(x)\) whose expansion in \(t\) involves no negative power of \(t\). The series \(Q(x)\) is algebraic.

These solutions are (of course) equivalent.
Hadamard walks in 2D

Assume

\[ S(x, y) = U(x) + V(x) T(y) \]

Proposition [Bostan, mbm, Melczer 18]

The series \( Q(x, y) \) is D-finite, and reads

\[
Q(x, y) = \left[ x \geq y \geq \right] \frac{\prod_{i=1}^{m} (1 - \bar{x} x_i(y)) \prod_{j=1}^{m'} (1 - \bar{y} y_j)}{1 - tS(x, y)}
\]

where

- the \( x_i(y) \) are the roots of \( S(x, y) = S(x', y) \) (solved for \( x' \)), whose expansion in \( \bar{x} \) involves no positive power of \( x \),
- the \( y_j \) are the roots of \( S(x, y) = S(x, y') \), or \( T(y) = T(y') \) (solved for \( y' \)) whose expansion in \( \bar{y} \) involve no positive powers of \( y \).
Bipolar maps

Proposition [mbm, Fusy, Raschel 18]

The generating function of bipolar maps with faces of degree \( p + 2 \) is

\[
Q(x, y) = [x^\geq y^\geq] \frac{(y - \bar{x}_1)(1 - \bar{x}\bar{y}) S_x(x, y)}{1 - tS(x, y)},
\]

where \( S(x, y) \) is the step polynomial:

\[
S(x, y) = x\bar{y} + \bar{x}^p + \bar{x}^{p-1}y + \cdots + \bar{x}y^{p-1} + y^p.
\]

and \( x_1 \) is the only root of \( S(x, y) = S(x', y) \) (solved for \( x' \)) whose expansion in \( \bar{y} \) involves a positive power of \( y \).

It is D-finite.