Walks with large steps in the quadrant

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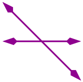
ArXiv: soon !

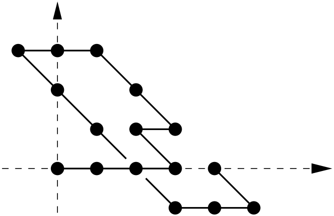
Counting quadrant walks

Let $\mathcal{S}$ be a finite subset of $\mathbb{Z}^2$ (set of steps).

We look at walks starting at $(0,0)$ and formed of steps of $\mathcal{S}$.

Example. $\mathcal{S} = \{10, \bar{1}0, 1\bar{1}, \bar{1}1\}$





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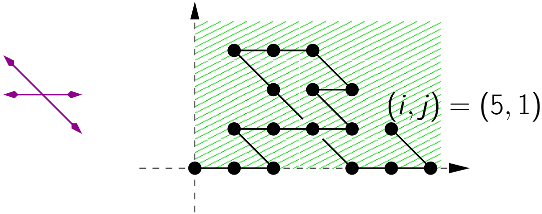

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The associated generating function:

$$Q(x,y;t) = \sum_{n \geq 0} \sum_{(i,j) \in \mathbb{N}^2} q(i,j;n) x^i y^j t^n$$

What is the nature of this series?

A hierarchy of formal power series

- Rational series

$$A(t) = \frac{P(t)}{Q(t)}$$

- Algebraic series

$$\mathrm{Pol}(t, A(t)) = 0$$

- Differentially finite series (D-finite)

$$\sum_{i=0}^{d} P_i(t) A^{(i)}(t) = 0$$

- D-algebraic series

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Multi-variate series: one DE per variable

Classification of quadrant walks with small steps

- quadrant models with small steps: 79
  - $|G|<\infty$: 23
    - **D-finite**
      - OS$=0$: 4
        - **algebraic**
      - OS$\neq 0$: 19
        - **DF transc.**
  - $|G|=\infty$: 56
    - **Not D-finite**
      - decoupled: 9
        - **D-alg.**
      - not decoupled: 47
        - **not D-alg.** (in x)

- G the group of the model
- OS the orbit sum

[mbm-Mishna 10] [Bostan-Kauers 10] [Mishna-Rechnitzer 07] [Melczer-Mishna 13] [Kurkova-Raschel 12] [Bostan-Raschel-Salvy 14] [Bernardi-mbm-Raschel 17(a)] [Dreyfus-Hardouin-Roques-Singer 17(a)]

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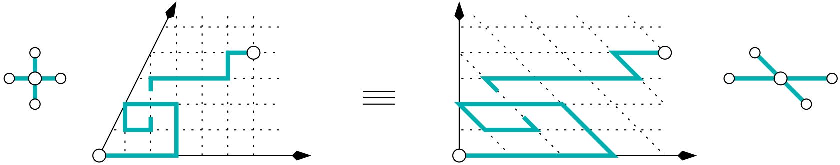

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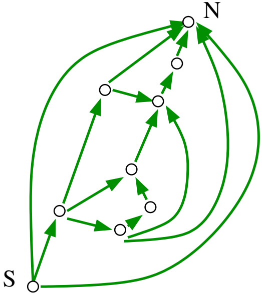
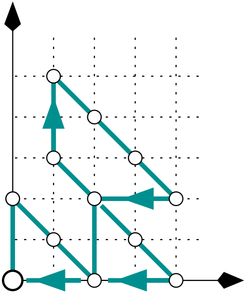
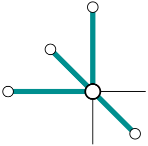
- Large steps occur in "real life": simple walk models, queing theory, bipolar orientations ([Kenyon et al. 15(a)])

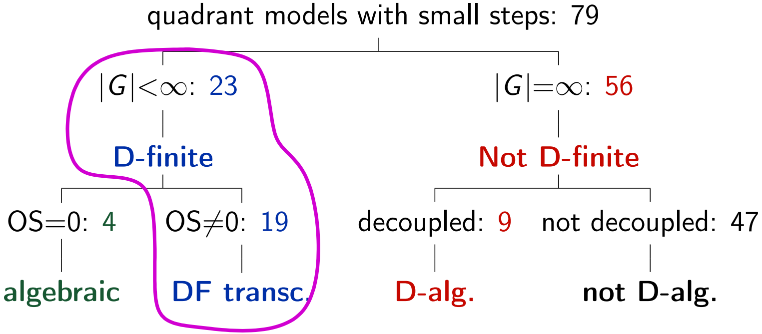






A general approach for quadrant walks...

which solves *some* cases.



"The simple branch"
[mbm-Mishna 10]

A four step approach

1. Write a functional equation for the tri-variate series $Q(x, y; t)$.
   It involves bi-variate series $Q(x, 0; t)$, $Q(0, y; t)$, ... (called sections)

2. Compute the "orbit" of (x, y)

3. Combine the main equation and the orbit to find a functional equation free from sections

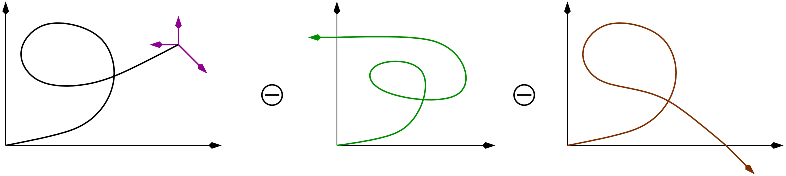
4. Extract from it $Q(x, y; t)$

Step 1: Write a functional equation

Example: $\mathcal{S} = \{01, \bar{1}0, 1\bar{1}\}$ (bipolar triangulations)

$$Q(x, y; t) \equiv Q(x, y) = 1 + t(y + \bar{x} + x\bar{y})Q(x, y) - t\bar{x}Q(0, y) - tx\bar{y}Q(x, 0)$$

with $\bar{x} = 1/x$ and $\bar{y} = 1/y$.



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or

$$\big(1 - t(y + \bar{x} + x\bar{y})\big)Q(x, y) = 1 - t\bar{x}Q(0, y) - tx\bar{y}Q(x, 0)$$

- The polynomial $1 - t(y + \bar{x} + x\bar{y})$ is the kernel of this equation

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- The polynomial $1 - t(y + \bar{x} + x\bar{y})$ is the kernel of this equation
- The series $Q(0, y)$ and $Q(x, 0)$ are the sections.

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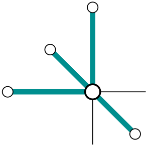

Example: Bipolar quadrangulations

$$(1 - t(x\bar{y} + \bar{x}^2 + \bar{x}y + y^2))Q(x,y) = 1 - tx\bar{y}Q(x,0)$$
$$-t\bar{x}^2\left(Q_0(y) + xQ_1(y)\right) - t\bar{x}yQ_0(y),$$

where $Q_i(y)$ counts quadrant walks ending at abscissa i.

$\Rightarrow$ More sections, kernel of higher degree

Step 2: the group of the model (and its orbit)

- The step polynomial:

$$S(x, y) = \bar{x} + y + x\bar{y}$$

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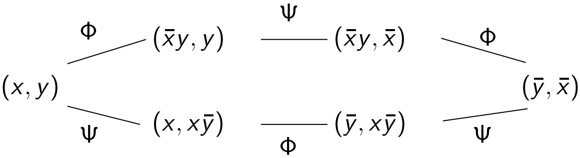
- Φ and Ψ are involutions
- They generate a (dihedral) group



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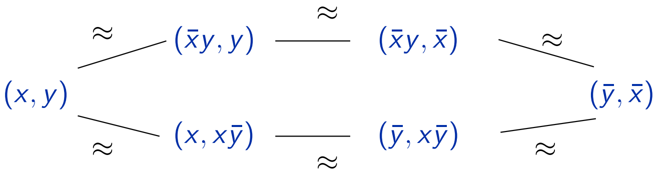
- Let $\sim$ be the transitive closure of the adjacency relation. The orbit of (x,y) is its equivalence class.



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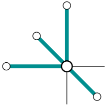
Example: Bipolar quadrangulations



$$S(x,y) = \bar{x}^2 + \bar{x}y + y^2 + x\bar{y}$$

The equation (in x') $S(x,y) = S(x',y)$, has 3 solutions, namely x and

$$x_{1,2} = \frac{xy^2 + y \pm \sqrt{y\left(x^2y^3 + 4\,x^3 + 2\,xy^2 + y\right)}}{2x^2}.$$

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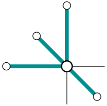

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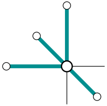

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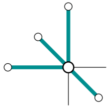

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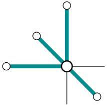

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- which pairs (x_1, y') are adjacent to (x_1, y)?
- and so on.

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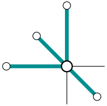

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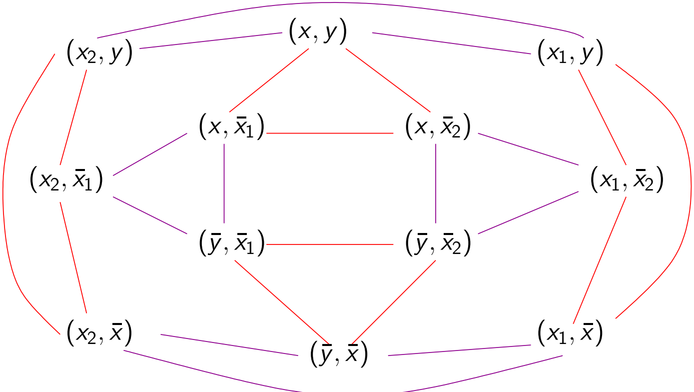
- Orbit



Step 3: Find a functional equation free from sections

- The equation reads (with $K(x,y) = 1 - tS(x,y)$):

$$K(x,y)xyQ(x,y) = xy - tx^2Q(x,0) - tyQ(0,y)$$

- The orbit of (x,y) is

$$(x,y) \approx (\bar{x}y, y) \approx (\bar{x}y, \bar{x}) \approx (\bar{y}, \bar{x}) \approx (\bar{y}, x\bar{y}) \approx (x, x\bar{y})$$

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- The value of $S(x,y)$ (and $K(x,y)$) is the same over the orbit. Hence

$$\begin{array}{rcccccc}
K(x,y)\ xyQ(x,y) & = & xy & - & tx^2Q(x,0) & - & tyQ(0,y) \\
K(x,y)\ \bar{x}y^2Q(\bar{x}y,y) & = & \bar{x}y^2 & - & t\bar{x}^2y^2Q(\bar{x}y,0) & - & tyQ(0,y) \\
K(x,y)\ \bar{x}^2yQ(\bar{x}y,\bar{x}) & = & \bar{x}^2y & - & t\bar{x}^2y^2Q(\bar{x}y,0) & - & t\bar{x}Q(0,\bar{x}) \\
\cdots & = & & & \cdots & & \\
K(x,y)\ x^2\bar{y}Q(x,x\bar{y}) & = & x^2\bar{y} & - & tx^2Q(x,0) & - & tx\bar{y}Q(0,x\bar{y}).
\end{array}$$

Step 3: Find a functional equation free from sections

⇒ Form the alternating sum of the equation over all elements of the orbit:

$$K(x,y)\Big(xyQ(x,y) - \bar{x}y^2Q(\bar{x}y,y) + \bar{x}^2yQ(\bar{x}y,\bar{x}) - \bar{x}\bar{y}Q(\bar{y},\bar{x}) + x\bar{y}^2Q(\bar{y},x\bar{y}) - x^2\bar{y}Q(x,x\bar{y})\Big) = xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}$$

(the orbit sum).

Step 3: Find a functional equation free from sections

$\Rightarrow$ Form the alternating sum of the equation over all elements of the orbit:

$$xyQ(x,y) - \bar{x}y^2Q(\bar{x}y,y) + \bar{x}^2yQ(\bar{x}y,\bar{x}) - \bar{x}\bar{y}Q(\bar{y},\bar{x}) + x\bar{y}^2Q(\bar{y},x\bar{y}) - x^2\bar{y}Q(x,x\bar{y}) = \frac{xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})}$$

Step 4: Extract $Q(x, y)$

$\Rightarrow$ Form the alternating sum of the equation over all elements of the orbit:

$$
\begin{aligned}
xyQ(x,y) - \bar{x}y^2 Q(\bar{x}y, y) + \bar{x}^2 y Q(\bar{x}y, \bar{x}) \\
- \bar{x}\bar{y} Q(\bar{y}, \bar{x}) + x\bar{y}^2 Q(\bar{y}, x\bar{y}) - x^2\bar{y} Q(x, x\bar{y}) = \\
\frac{xy - \bar{x}y^2 + \bar{x}^2 y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})}
\end{aligned}
$$

- Both sides are power series in t, with coefficients in $\mathbb{Q}[x, \bar{x}, y, \bar{y}]$.

Step 4: Extract $Q(x,y)$ 

$\Rightarrow$ Form the alternating sum of the equation over all elements of the orbit:

$$xyQ(x,y) - \bar{x}y^2Q(\bar{x}y,y) + \bar{x}^2yQ(\bar{x}y,\bar{x}) - \bar{x}\bar{y}Q(\bar{y},\bar{x}) + x\bar{y}^2Q(\bar{y},x\bar{y}) - x^2\bar{y}Q(x,x\bar{y}) = \frac{xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})}$$

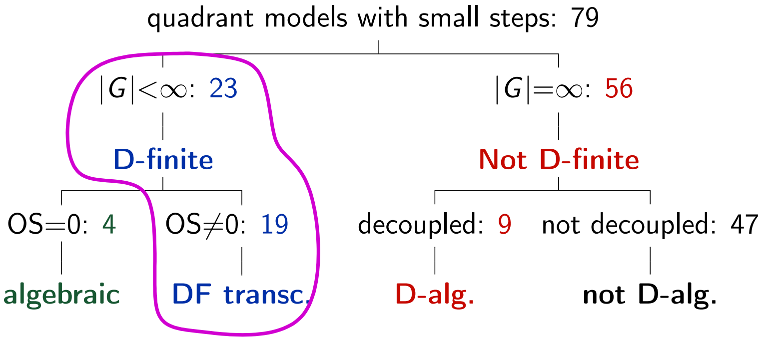
- Both sides are power series in t, with coefficients in $\mathbb{Q}[x,\bar{x},y,\bar{y}]$.

- Extract the part with positive powers of x and y:

$$xyQ(x,y) = [x^{>0}y^{>0}]\,\frac{xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})}$$

is a D-finite series.
[Lipshitz 88]

What can go wrong?



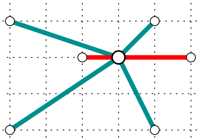
1. Functional equation: OK
2. The "orbit" may be infinite
3. There may be several section-free equations (or none?) [NEW]
4. The extraction may be tricky [NEW], or impossible

Some cases that work

- Hadamard walks:

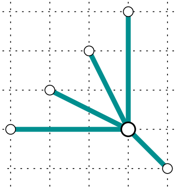
$$S(x,y) = U(x) + V(x)T(y)$$

(Small steps: 16 out of the 19 "simple" models)



- Bipolar maps [mbm, Fusy, Raschel 18]

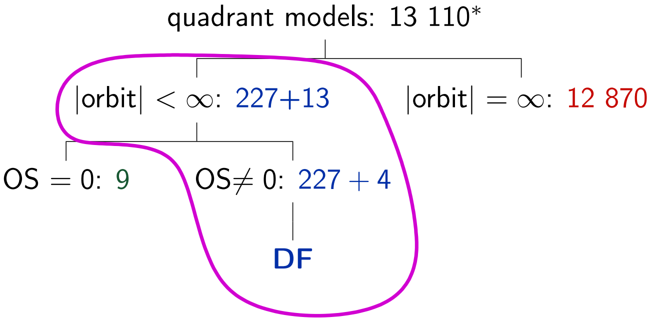
$$\mathcal{S}_p = \{(-p,0), (-p+1,1), \ldots, (0,p), (1,-1)\}$$



In all those cases, the orbit is finite and the series D-finite, expressed as the non-negative part of an algebraic series.

Quadrant walks with steps in $\{-2,-1,0,1\}^2$

- In all cases, a unique section-free equation

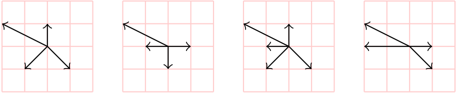


- 227 Hadamard models

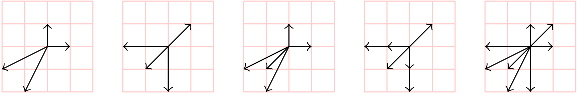
(*) Models with at least one occurrence of -2

Some interesting models

- Non-Hadamard, solvable via our approach (and D-finite):



- Non-Hadamard, orbit sum zero: let's guess!



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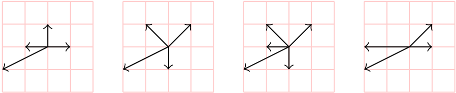
à la Kreweras



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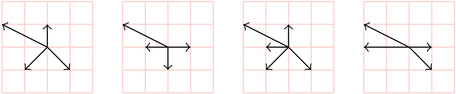
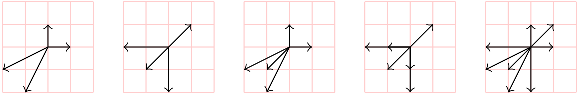
à la Gessel

Some interesting models

- Non-Hadamard, solvable via our approach (and D-finite):



- Non-Hadamard, orbit sum zero: let's guess!



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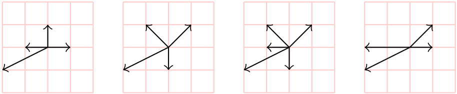

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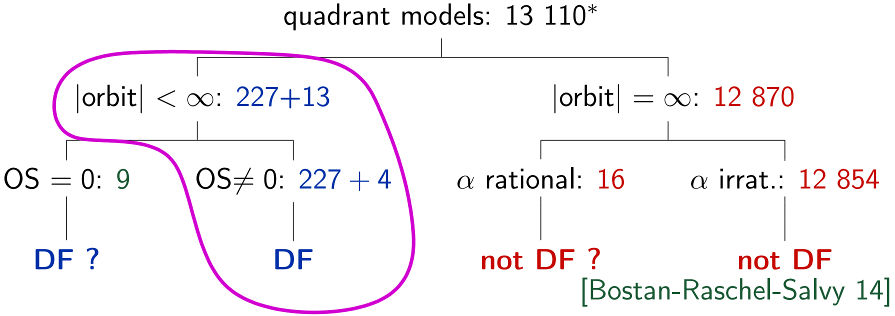
à la Kreweras

à la Gessel

Final comments

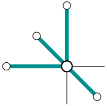
Still a lot to be done...

- Is there a unique section free equation when there are no large forward steps?
- Closer study for tricky examples (the 9 analogues of Kreweras' and Gessel's algebraic models)
- Nature of models where α is rational but the orbit infinite



Step 2: the orbit of the model

Example: Bipolar quadrangulations

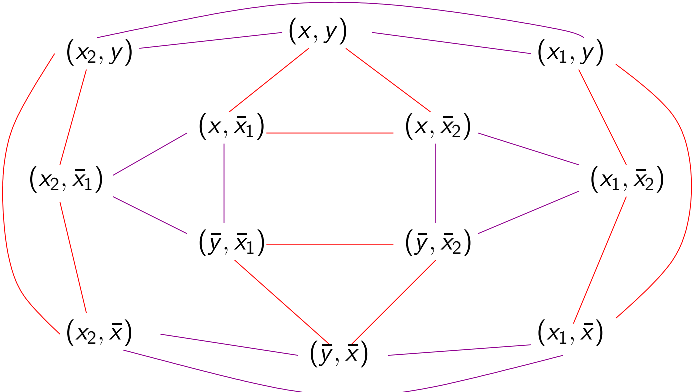


$$S(x,y) = \bar{x}^2 + \bar{x}y + y^2 + x\bar{y}$$

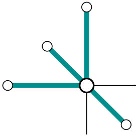
The equation (in x') $S(x,y) = S(x',y)$, has 3 solutions, namely x and

$$x_{1,2} = \frac{xy^2 + y \pm \sqrt{y\left(x^2y^3 + 4\,x^3 + 2\,xy^2 + y\right)}}{2x^2}.$$

- Orbit



Step 3: Find a functional equation free from sections



Example: Bipolar quadrangulations

$$S(x,y) = \bar{x}^2 + \bar{x}y + y^2 + x\bar{y}$$

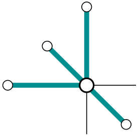
Two section-free linear combinations (+ linear combinations):

$$xy\,Q(x,y) - \bar{x}_1 x\,Q(x,\bar{x}_1) - \frac{{x_1}^2 y\,(x-x_2)\,Q(x_1,y)}{x\,(x_1-x_2)} + \frac{{x_1}^2\,(x-x_2)\,Q(x_1,\bar{x})}{(x_1-x_2)\,x^2}$$
$$+ \frac{{x_2}^2 y\,(x-x_1)\,Q(x_2,y)}{x\,(x_1-x_2)} - \frac{(x_1\,y-1)\,{x_2}^2\,(x-x_2)\,Q(x_2,\bar{x})}{x^2\,(x_1-x_2)\,(x_2\,y-1)}$$
$$+ \frac{(xy-1)\,{x_2}^2\,Q(x_2,\bar{x}_1)}{x_1\,x\,(x_2\,y-1)} + \frac{(x-x_2)\,Q(\bar{y},\bar{x})}{y\,(x_2\,y-1)\,x^2} - \frac{(x-x_2)\,Q(\bar{y},\bar{x}_1)}{yx_1\,x\,(x_2\,y-1)}$$
$$= \frac{(y-\bar{x}_1)(xy-1)(\bar{y}-\bar{x}^2 y - 2\bar{x}^3)}{K(x,y)},$$

and the same equation with x_1 and x_2 exchanged.

Step 4: Extract $Q(x,y)$

Example: Bipolar quadrangulations



$$S(x,y) = \bar{x}^2 + \bar{x}y + y^2 + x\bar{y}$$

- A section-free equation:

$$xyQ(x,y) - \bar{x}_1 x Q(x,\bar{x}_1) - \frac{{x_1}^2 y\,(x-x_2)\,Q(x_1,y)}{x\,(x_1-x_2)} + \cdots$$

$$= \frac{(y-\bar{x}_1)(xy-1)(\bar{y}-\bar{x}^2y-2\bar{x}^3)}{K(x,y)}$$

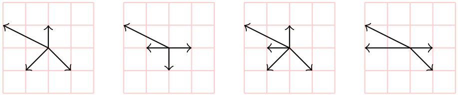
- Then

$$xyQ(x,y) = [x^{>0}y^{>0}]\frac{(y-\bar{x}_1)(xy-1)(\bar{y}-\bar{x}^2y-2\bar{x}^3)}{K(x,y)},$$

provided the RHS is expanded first in t, then in $\bar{y}$, and finally in x.

Some interesting models

- Non-Hadamard, solvable via our approach (and D-finite):



For the first model,

$$Q(x,y) = [x^{\geq 0} y^{\geq 0}] \frac{(x^3 - 2\,y^2 - x)\,(y^2 - x)\,(x^2y^2 - y^2 - 2\,x)}{x^5y^4\,(1 - t(y + x\bar{y} + \bar{x}\bar{y} + \bar{x}^2 y))}.$$

The coefficients are nice: for $n = 2i + j + 4m$,

$$q(i,j;n) = \frac{(i+1)(j+1)(i+j+2)n!(n+2)!}{m!(3m+2i+j+2)!(2m+i+1)!(2m+i+j+2)!}.$$

Walks on a half-line ($d = 1$)

Let $\mathcal{S} \subset \mathbb{Z}$ with $\min \mathcal{S} = -m$.

Proposition [Bostan, mbm, Melczer 18]

$$Q(x) = [x^{\geq 0}] \frac{\prod_{j=1}^{m}(1 - \bar{x}x_j)}{1 - tS(x)},$$

where the x_j are the roots of $S(x_j) = S(x)$ whose expansion in $\bar{x}$ involves no positive power of x.

Walks on a half-line ($d = 1$)

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Classical solution [Gessel 80, mbm-Petkovšek 00, Banderier-Flajolet 02...]

$$Q(x) = \frac{\prod_{j=1}^{m}(1 - \bar{x}X_j)}{1 - tS(x)}$$

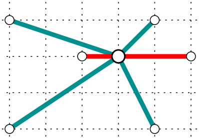
where the $X_j \equiv X_j(t)$ are the roots of $1 - tS(x)$ whose expansion in t involves no negative power of t. The series $Q(x)$ is algebraic.

These solutions are (of course) equivalent.

Hadamard walks in 2D

Assume

$$S(x,y) = U(x) + V(x)T(y)$$



Proposition [Bostan, mbm, Melczer 18]

The series $Q(x,y)$ is D-finite, and reads

$$Q(x,y) = [x^{\geq}y^{\geq}]\frac{\prod_{i=1}^{m}(1-\bar{x}x_i(y))\prod_{j=1}^{m'}(1-\bar{y}y_j)}{1-tS(x,y)},$$

where

- the $x_i(y)$ are the roots of $S(x,y) = S(x',y)$ (solved for x'), whose expansion in $\bar{x}$ involves no positive power of x,
- the y_j are the roots of $S(x,y) = S(x,y')$, or $T(y) = T(y')$ (solved for y') whose expansion in $\bar{y}$ involve no positive powers of y.

Bipolar maps

Proposition [mbm, Fusy, Raschel 18]

The generating function of bipolar maps with faces of degree $p+2$ is

$$Q(x,y) = [x^{\geq} y^{\geq}] \frac{(y - \bar{x}_1)(1 - \bar{x}\bar{y}) S_x(x,y)}{1 - tS(x,y)},$$

where $S(x,y)$ is the step polynomial:

$$S(x,y) = x\bar{y} + \bar{x}^p + \bar{x}^{p-1} y + \cdots + \bar{x} y^{p-1} + y^p.$$

and x_1 is the only root of $S(x,y) = S(x',y)$ (solved for x') whose expansion in $\bar{y}$ involves a positive power of y.
It is D-finite.