# Population persistence & diversity maintenance in fluctuating environments

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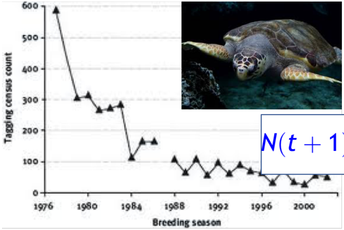
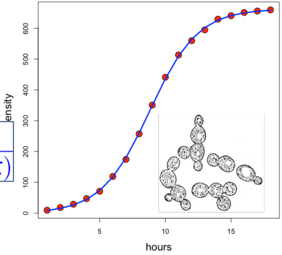
Under what abiotic and biotic conditions does a population persist for a long time? When does it tend toward extinction?




$$N(t+1) = N(t)\frac{\lambda}{1+aN(t)}$$



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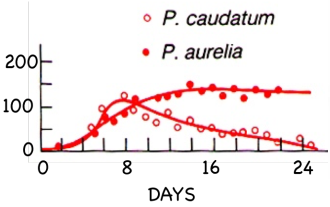
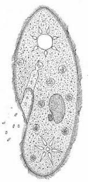
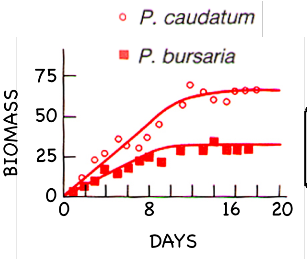
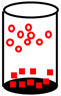
When are interacting populations likely to coexist for extended periods of time? When does exclusion occur and why?

$$\frac{dN_1}{dt} = r_1 N_1 (1 - a_{11} N_1 - a_{12} N_2)$$
$$\frac{dN_2}{dt} = r_2 N_2 (1 - a_{21} N_1 - a_{22} N_2)$$





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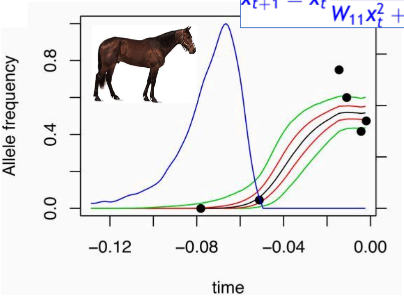
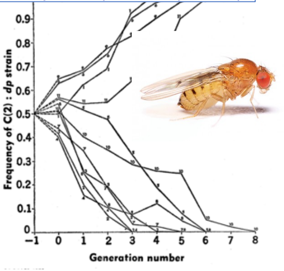
$$x_{t+1} = x_t \frac{W_{11}x_t + W_{12}(1-x_t)}{W_{11}x_t^2 + 2W_{12}x_t(1-x_t) + W_{22}(1-x_t)^2}$$



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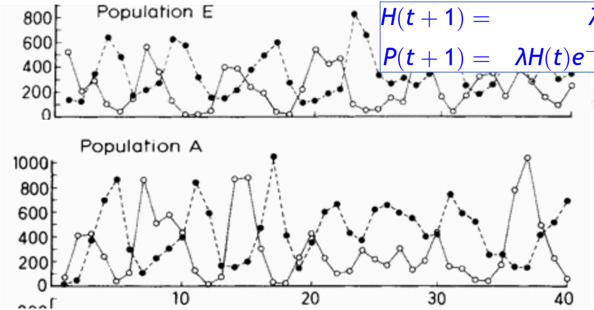


$$H(t+1) = \lambda H(t) e^{-H(t) - aP(t)/H(t)}$$
$$P(t+1) = \lambda H(t) e^{-H(t)} \left(1 - e^{-aP(t)/H(t)}\right)$$

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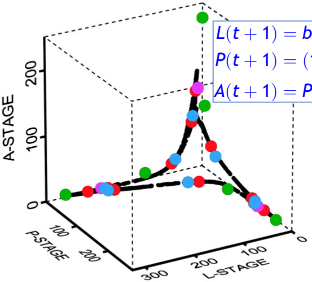
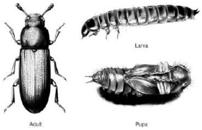
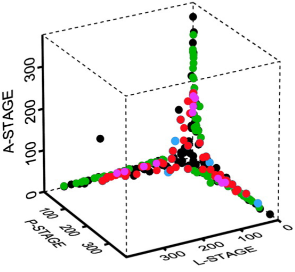



$$L(t+1) = bA(t)e^{-c_{e\ell}L(t)-c_{ea}A(t)}$$
$$P(t+1) = (1-\mu_\ell)L(t)$$
$$A(t+1) = P(t)e^{-c_{\ell a}A(t)} + (1-\mu_a)A(t)$$



Historically studied in ecology and epidemiology using

- differential equations: $\frac{dx}{dt} = f(x)$
- difference equations: $x_{t+1} = f(x_t)$
- on the non-negative orthant $\mathbb{R}^k_+ = [0, \infty)^k$
- persistence $=$ a positive attractor

Lotka 1925, Volterra 1926, Nicholson & Bailey 1935, Kermack & Mckendrick 1927

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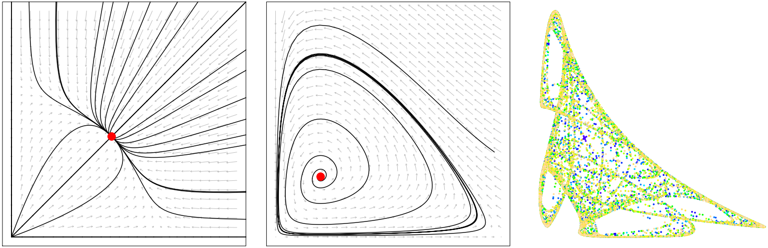

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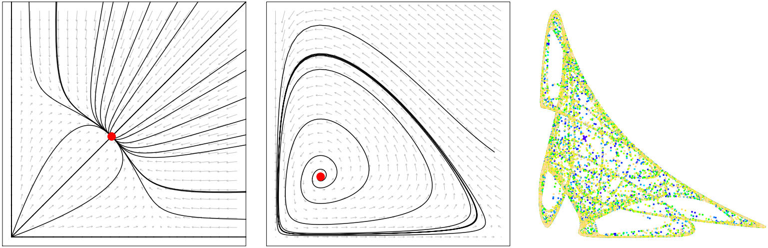

Population geneticists (Wright, Fisher, Haldane) were wiser from the beginning...

Lest men suspect your tale untrue, Keep probability in view. –John Gay

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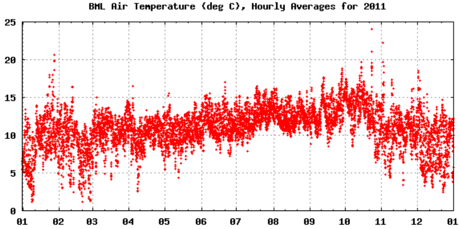

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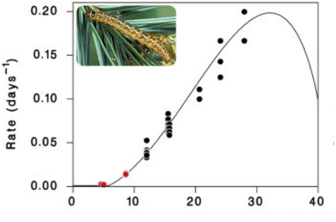

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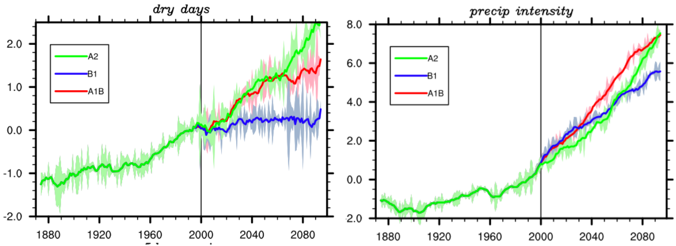

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Main question: How does environmental stochasticity influence population persistence and the maintenance of genetic diversity?

# Single population

$$X_{t+1} = X_t f(X_t, Y_t, \xi_{t+1})$$ population density $[0, \infty)$

$$Y_{t+1} = g(X_t, Y_t, \xi_{t+1})$$ feedback variable $\mathbb{R}^n$

$$\xi_1, \xi_2, \xi_3, \ldots$$ i.i.d.

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- Environmental forcing by a stationary process e.g. $Y_{t+1} = AY_t + \xi_{t+1}$
- Internal feedbacks due to evolving traits (e.g. Breeder's equation) or population structure (e.g. spatial distribution)

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Consider the linearized process

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$$\limsup_{t\to\infty} \frac{\#\{1 \leq s \leq t : X_t \in [0, \varepsilon)\}}{t} \leq \alpha\varepsilon^\beta \text{ a.s. for } Z_0 \in M_+$$

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$$X_{t+1} = X_t f(X_t, Y_t, \xi_{t+1}) \quad \text{population density}$$

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$$\xi_1, \xi_2, \xi_3, \ldots \quad \text{i.i.d.}$$

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Moreover, if $M_0$ is “accessible” then $\lim_{t\to\infty} X_t = 0$ a.s. for all $Z_0$.

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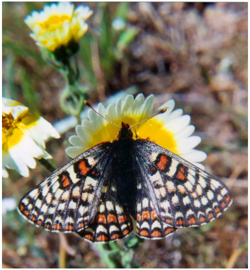
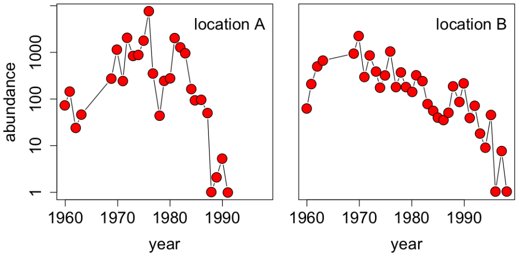


Bay checkerspot

2 checkerspot populations went extinct in 1990s



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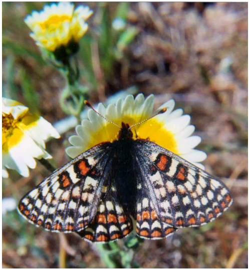

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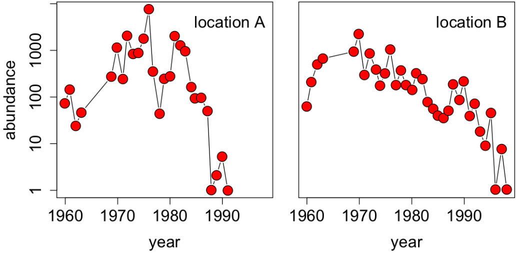


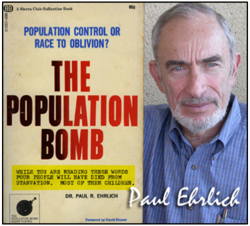


Paul Ehrlich

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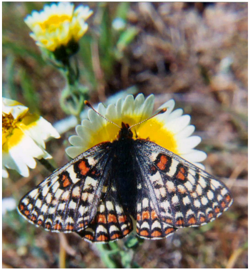

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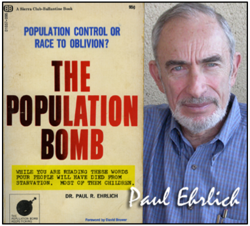


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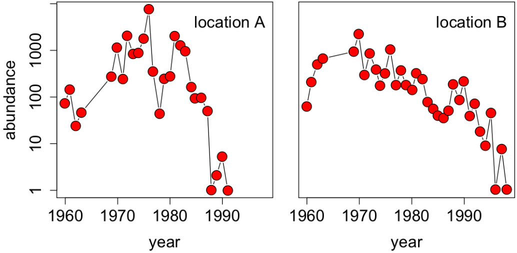


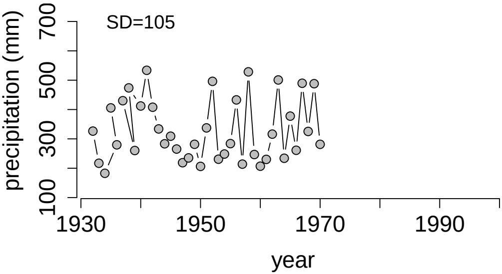

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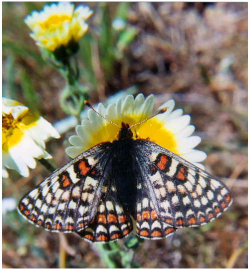

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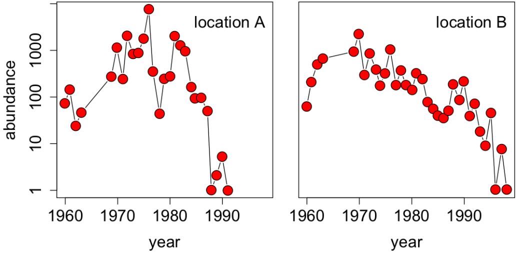


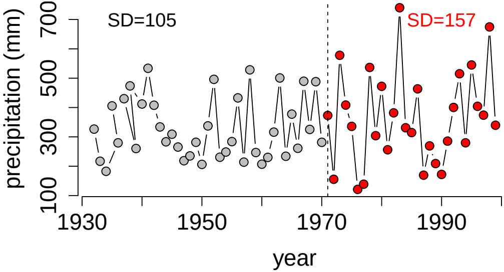


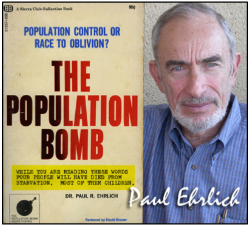


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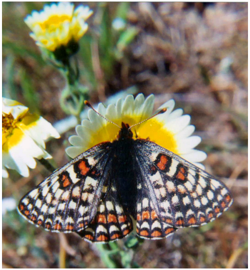

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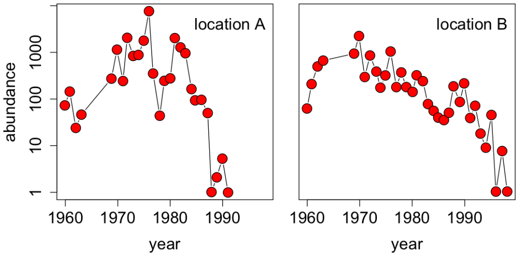


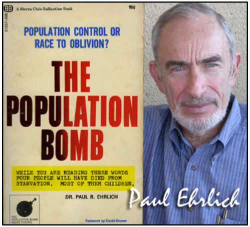


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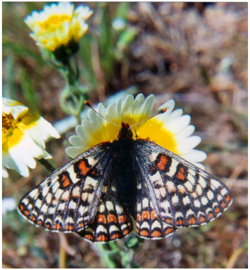

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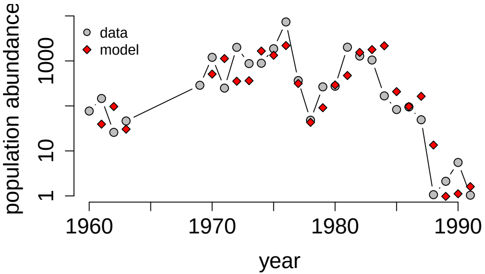

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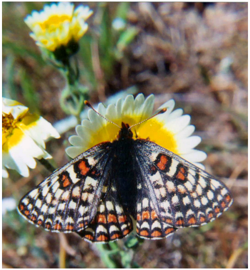

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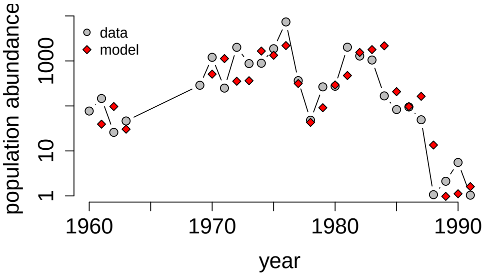

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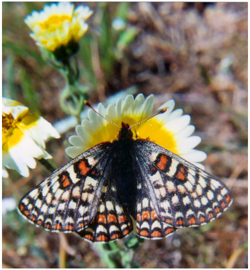

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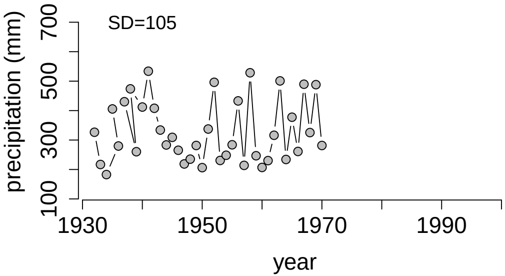


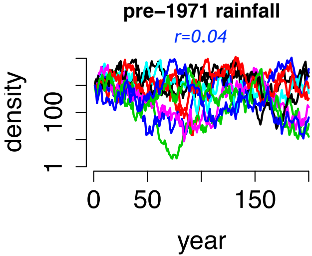

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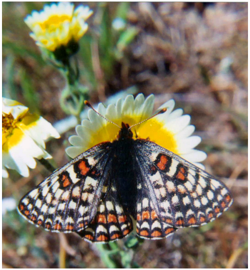

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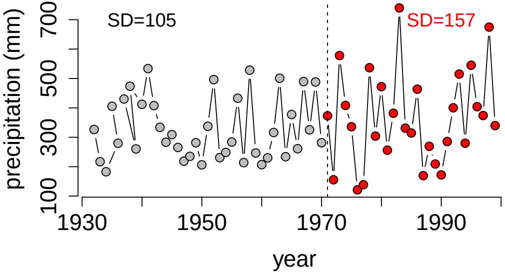


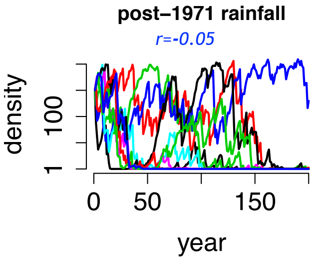

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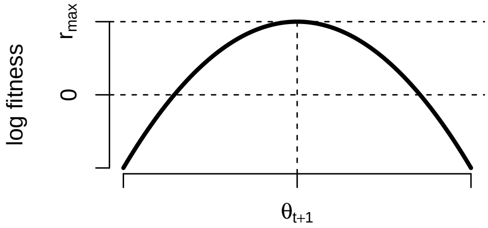


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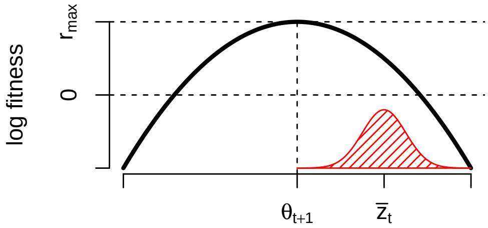


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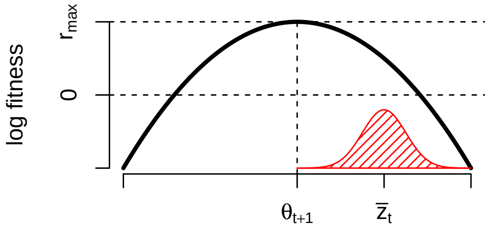


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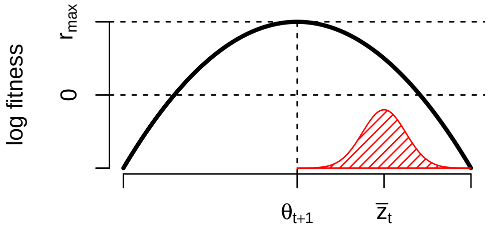


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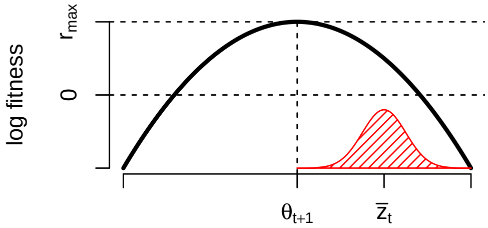


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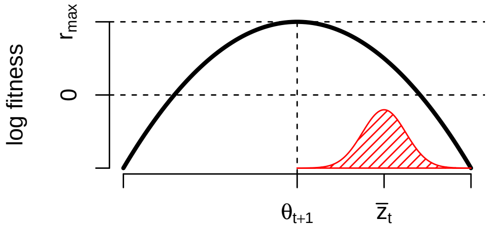


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(Lande 1976; Turelli & Barton 1994)

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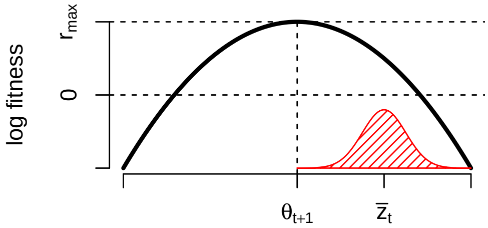


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$\xi_t$ i.i.d. w/ mean 0, variance $\tau^2$

$\rho$ temporal autocorrelation

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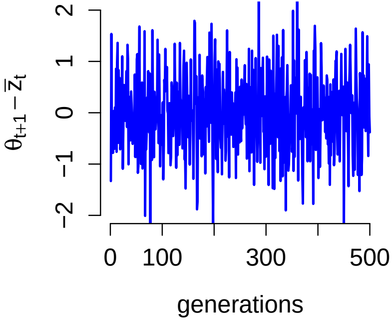


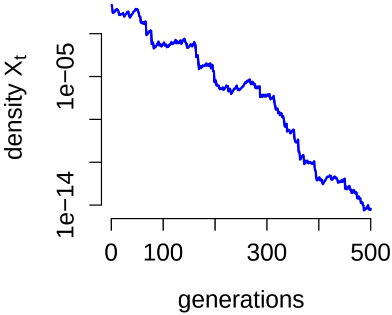

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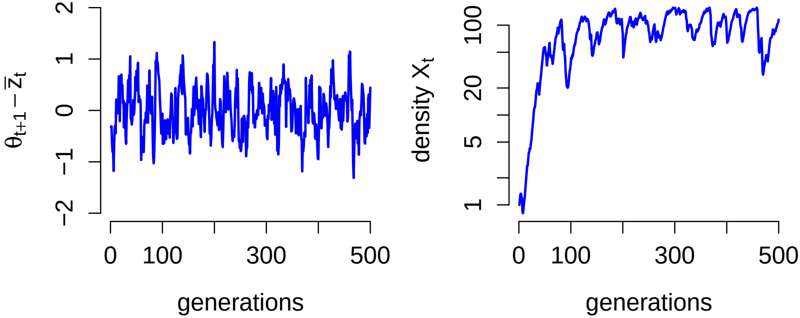

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$$X^i_{t+1} = X^i_t f_i(X_t, Y_t, \xi_{t+1})$$ species/genotype #s w/ $X_t = (X^1_t, \ldots, X^k_t)$

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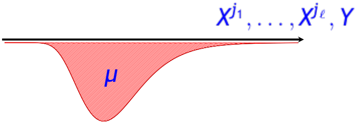

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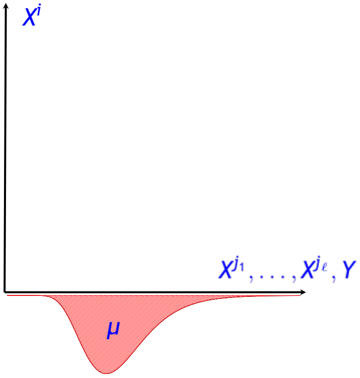

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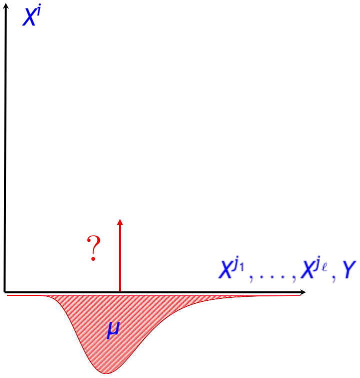

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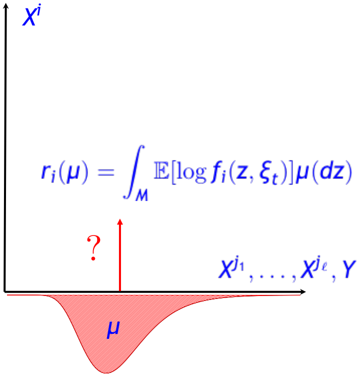

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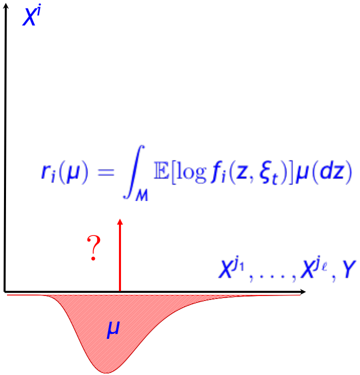


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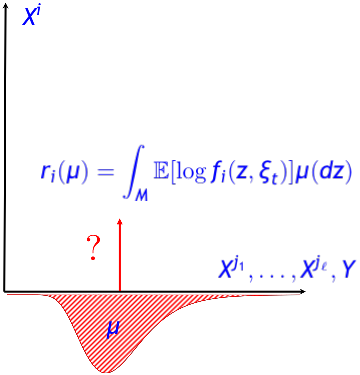


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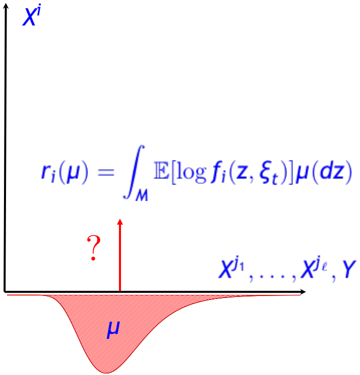


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$M_0$ accessible $\Rightarrow \mathbb{P}\left[\lim_{t\to\infty} \mathrm{dist}(Z_t, M_0) = 0 \middle| Z_0 = z\right] = 1$

members of the same species, being very similar to each other, being machines for preserving genes in the same kind of place, with the same kind of way of life, are particularly direct competitors for all the resources necessary for life. - The Selfish Gene

"And NUH is the letter I use to spell Nutches, Who live in small caves, known as Niches, for hutches. These Nutches have troubles, the biggest of which is the fact there are many more Nutches than Niches. Each Nutch in a Nich knows that some other Nutch Would like to move into his Nich very much. So each Nutch in a Nich has to watch that small Nich or Nutches who haven't got Niches will snitch."

Dr. Seuss

*On Beyond the Zebra*

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State space $\Delta \times [0, \infty)$ w/ $\Delta = \{(x^1, \ldots, x^k) : x^i \ge 0, \sum_i x_i = 1\}$

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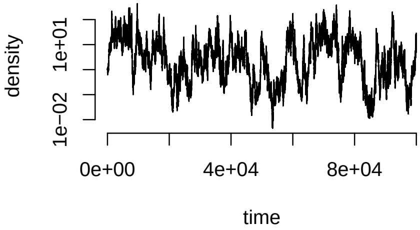


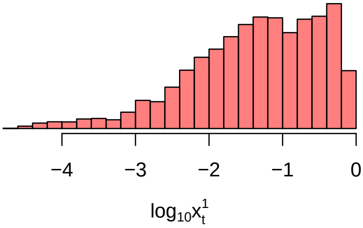

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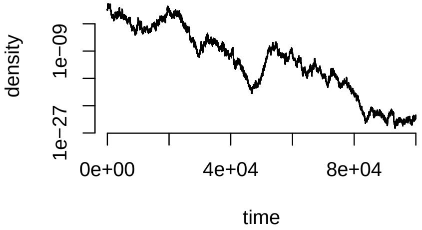


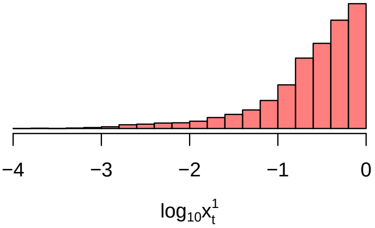

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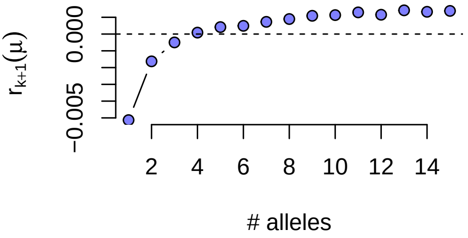

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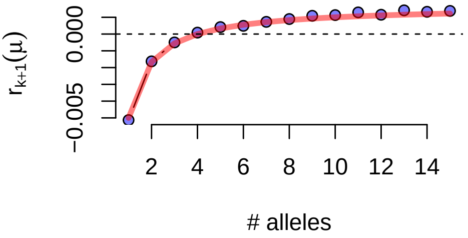
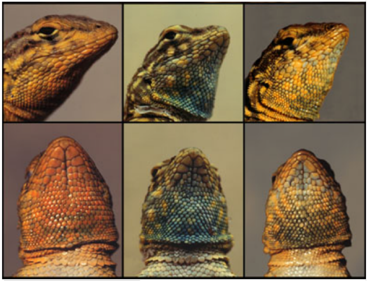
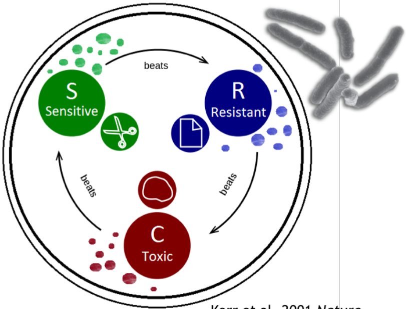
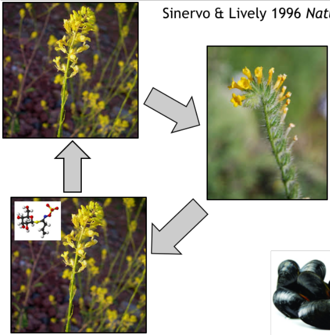




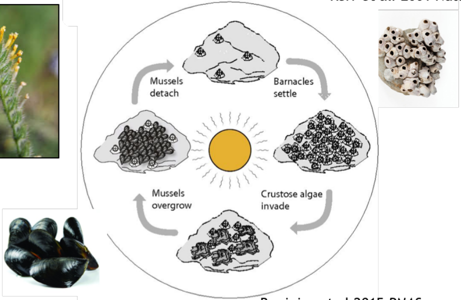
Sinervo & Lively 1996 *Nature*




Kerr et al. 2001 *Nature*



Lankau & Strauss 2007 *Science*




Beninica et al 2015 *PNAS*

# Rock (1), paper (2), scissors (0) mod 3

$W_t^i > 1, 1, L_t^i < 1$ payoff to $i$ when interacts w/ $i-1, i, i+1$

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$X_t^i$ frequency of $i$

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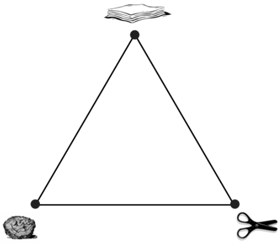

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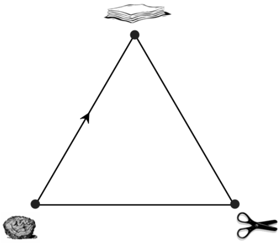

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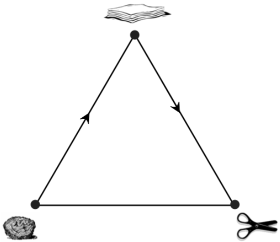

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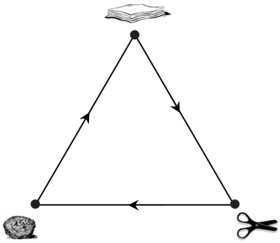

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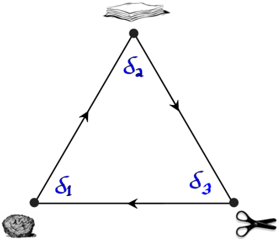


$$r_i(\delta_i) = 0,\ r_i(\delta_{i+1}) = \mathbb{E}[\log L_t^i],\ r_i(\delta_{i-1}) = \mathbb{E}[\log W_t^i]$$

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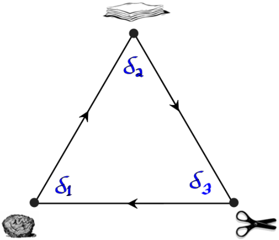


$$\mathbb{E}[\log W_t^i] + \mathbb{E}[\log L_t^i] > 0$$

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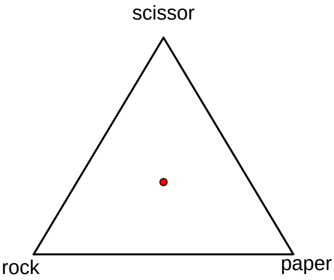


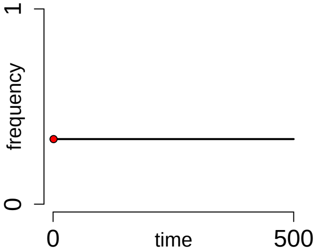


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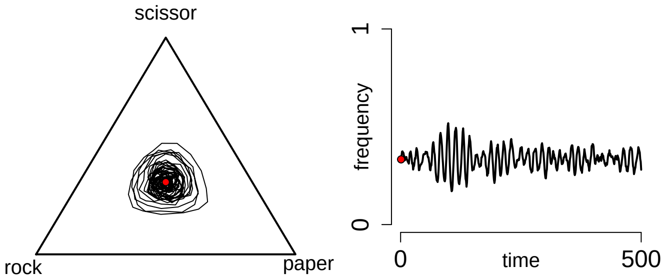


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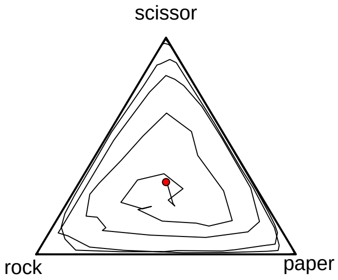


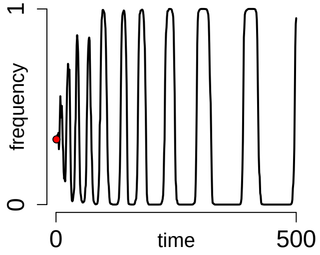


$$\mathbb{E}[\log W_t^i] + \mathbb{E}[\log L_t^i] < 0$$

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Can inhibit or promote population persistence

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Questions?