Mean-field diffusions in stochastic spatial death-birth models

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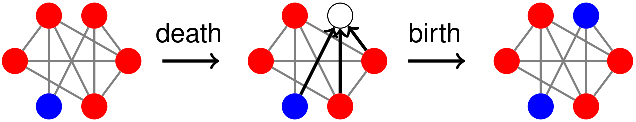
Voter model

Updating rule

1. Individuals die at rate 1.
2. $\mathbb{P}(\text{child of } y \text{ occupies } x) = \dfrac{1}{\deg(x)}, \quad \forall\, y : y \sim x.$

- **Complete graphs:** Moran process.
- **Large discrete tori:** Cox (1989).
- **Rescaled integer lattices:** Cox, Durrett and Perkins (2000), Cox, Bramson and Le Gall (2001).
- **General finite graphs:** Chapter 14 in Aldous–Fill book.



Pair approximation

Matsuda, Ogita, Sasaki and Satō (1992) for the Lotka–Volterra model.

Also,

- Voter models [Sood and Redner (2005) *Physics Review Letters*...].
- Evolutionary games [Ohtsuki, Hauert, Lieberman and Nowak (2006) *Nature*; Szabó and Fáth (2007) *Physics Reports*;...].
- Epidemics (Keeling–Rohani book, Ellner–Guckenheimer book,...).

Quantitative method for ODE or SDE approximations

Very few math proofss

Evolutionary game with death-birth updating

Updating rule

1. Individuals die at rate 1.
2. $$\mathbb{P}(\text{child of } y \text{ occupies } x) = \frac{\beta(y)}{\sum_{z:z\sim x}\beta(z)}, \quad \forall\, y : y \sim x.$$

For $\xi(y) = \sigma \in \{B, R\}$,

$$\beta(y) = (1-w) + w\frac{\sum_{\tau\in\{B,R\}}\Pi(\sigma,\tau)\#\{\tau\text{-neighbor of } y\}}{\deg(y)},$$

$\Pi =$ payoff,

$w =$ selection strength.

Absorbing probabilities on finite graphs

1. **Fundamental difficulties:**
   - **Frequency-dependent at 2 levels, nonlinear birth probabilities:** With the adjacency matrix A and a configuration $\{\xi(z)\}_{z\in G}$,

$$\mathbb{P}(B \text{ occupies } x) = \frac{\sum_y A(x,y)\mathbb{1}_B(\xi(y))\beta(y)}{\sum_y A(x,y)\beta(y)}.$$

   - **Asymmetric landscape of birth rates.**

2. **Exact solutions:** possible ONLY if reducible to 1D problems.

3. **NP-hard:** Ibsen-Jensen, Chatterjee and Nowak (2015) *PNAS.*

Diffusion approximation by pair approximation

1. **Basic inputs:**

$$
\begin{aligned}
G &= \text{random } k\text{-regular graph with } N \gg 1 \text{ vertices} \\
&\quad \text{(locally tree-like)}, \\
(\xi_t) &= \text{evolutionary game under “} w \ll 1\text{”}, \\
p_B(\xi_t) &= \frac{\#\{y \in G; \xi_t(y) = B\}}{N}.
\end{aligned}
$$

2. **Wright–Fisher diffusion (prediction, Ohtsuki et al. 2006):**

$$
\begin{aligned}
p_B(\xi_t) \;\simeq\; & p_B(\xi_0) + w \int_0^t \frac{(k-2)}{k^2(k-1)} p_B(\xi_s)[1 - p_B(\xi_s)][\alpha p_B(\xi_s) + \beta] ds \\
& + \int_0^t \left(\frac{2(k-2)}{N(k-1)} p_B(\xi_s)[1 - p_B(\xi_s)] \right)^{1/2} dW_s.
\end{aligned}
$$

Diffusion approximation by pair approximation

$$p_B(\xi_t) \simeq p_B(\xi_0) + w\int_0^t \frac{(k-2)}{k^2(k-1)} p_B(\xi_s)[1-p_B(\xi_s)][\alpha p_B(\xi_s)+\beta]ds$$
$$+ \int_0^t \left(\frac{2(k-2)}{N(k-1)} p_B(\xi_s)[1-p_B(\xi_s)]\right)^{1/2} dW_s.$$

$$\alpha = (k+1)(k-2)[\Pi(B,B) - \Pi(B,R) - \Pi(R,B) + \Pi(R,R)]$$
$$\beta = (k+1)\Pi(B,B) + (k^2-k-1)\Pi(B,R) - \Pi(R,B) - (k^2-1)\Pi(R,R).$$

Key features:

- Not mean-field in the usual sense.
- w and Π only in the drift coefficient.
- Implied scaling: $t = \mathcal{O}(N)$ and $w = \mathcal{O}(1/N)$.

2. **Wright–Fisher diffusion (prediction, Ohtsuki et al. 2006):**

$$p_B(\xi_t) \simeq p_B(\xi_0) + w\int_0^t \frac{(k-2)}{k^2(k-1)} p_B(\xi_s)[1-p_B(\xi_s)][\alpha p_B(\xi_s)+\beta]ds$$
$$+ \int_0^t \left(\frac{2(k-2)}{N(k-1)} p_B(\xi_s)[1-p_B(\xi_s)]\right)^{1/2} dW_s.$$

3. **Absorbing probabilities (Ohtsuki et al. 2006):**

$$\mathbb{P}^w(B\text{'s take over})$$
$$\simeq \underbrace{p_B(\xi_0)}_{=\mathbb{P}^0(B\text{'s take over})} + w\frac{N}{6k^2} p_B(\xi_0)[1-p_B(\xi_0)][(\alpha+3\beta)+\alpha p_B(\xi_0)].$$

Math remark: Beyond the implication of Skorokhod's topology.

Diffusion approximation by pair approximation

4. **The "$b/c/k$" (Ohtsuki et al. 2006):** under a donation evolutionary game

$$\Pi = \begin{array}{cc} & \begin{array}{cc} B & R \end{array} \\ \begin{array}{c} B \\ R \end{array} & \begin{pmatrix} b-c & -c \\ b & 0 \end{pmatrix} \end{array},$$

$$\mathbb{P}^w(B\text{'s take over})$$
$$\simeq \mathbb{P}^0(B\text{'s take over}) + w\frac{N}{6k^2}p_B(\xi_0)[1-p_B(\xi_0)][3k(b-ck)].$$

$$\left(\frac{b}{c}\right)^\star \simeq k.$$

First-derivative test: Ohtsuki and Nowak (2006), Tarnita, Ohtsuki, Antal, Fu and Nowak (2009), C. (2013), Allen and Nowak (2014), C., McAvoy and Nowak (2016).

Integer lattices: Cox, Durrett and Perkins (2013).

$b/c/k$ on finite graphs

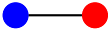
Theorem (C., McAvoy and Nowak, 2016)

On a connected k-regular graph w/o self-loops and multiple edges,

$$\frac{\partial}{\partial w}\mathbb{P}^{w}_{\xi_0}(B\text{'s take over})\Big|_{w=0} = [Np_B(1-p_B) - kp_{BR} - kp_{B\bullet R}]b - [Np_B(1-p_B) - p_{BR}]ck.$$

$$p_B(\xi_0) = \frac{\#\{x;\xi_0(x)=B\}}{N},$$

$$p_{BR}(\xi_0) = \frac{\#\{(x,y);x\sim y,\xi_0(x)=B,\xi_0(y)=R\}}{Nk},$$



$$p_{B\bullet R}(\xi_0) = \frac{\#\{(x,y,z);x\sim y\sim z,\xi_0(x)=B,\xi_0(z)=R\}}{Nk^2}.$$

$b/c/k$: selection strength

Theorem (C. 2018+)

Assume there is **"no hub"***: maximal degree* $= \mathcal{O}(1)$.

Under any general payoff matrix, the comparison of fixation probabilities by

$$\text{sign of } \frac{\partial}{\partial w}\mathbb{P}^w_\xi(B\text{'s take over})\Big|_{w=0}$$

is valid for all $w \leq \mathcal{O}(1/N)$.

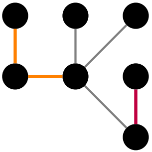
$b/c/k$: exact evaluation of voter models

Theorem (m-point functions)

1. (ξ_t) *is a* **voter model**, *and*
2. $(W^{x_1}, \cdots, W^{x_m})$ *is a system of* **coalescing random walks**, *independent of ξ_0, with W^{x_i} started at site x_i.*

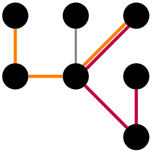
Then

$$(\xi_t(x_1), \cdots, \xi_t(x_m)) \stackrel{(d)}{=} (\xi_0(W_t^{x_1}), \cdots, \xi_0(W_t^{x_m})).$$



before meeting time

W^{x_1} and W^{x_2} are independent



after meeting time

W^{x_1} and W^{x_2} move together

$b/c/k$: exact evaluation of voter models

1. **Donation games:** two-point functions including

$$\int_0^\infty \underbrace{\mathbb{P}^0(\xi_t(U)=B,\xi_t(V)=R)}_{=\mathbb{E}^0[p_{BR}(\xi_t)]=\mathbb{P}(\xi_0(W_t^U)=B,\xi_0(W_t^V)=R)} \quad dt$$

for $U \sim \mathrm{Unif}(G)$ and $V \sim \mathrm{Unif}(\text{neighborhood of } U)$.

Solvability by voter model:

$$p_B(\xi_0)[1-p_B(\xi_0)] - \frac{2}{N}\int_0^T \mathbb{E}^0[p_{BR}(\xi_t)]dt$$
$$=\mathbb{E}^0 p_B(\xi_T)[1-p_B(\xi_T)] \to 0 \quad \text{as } T\to\infty.$$

Solvability of coalescing random walks: Exact equation between first exit and ergodic exit for general Markov chains (Aldous–Fill book).

2. **General games:** no solution for the non-mean-field object

$$\int_0^\infty \mathbb{P}^0(\xi_t(U)=B, \xi_t(V)=R, \xi_t(W)=B)dt, \quad U \sim V \sim W,$$

where $W \sim \mathrm{Unif}(\text{neighborhood of } V)$ if conditioned on (U, V).

Usefulness of pair approximation: **Both** explicit dynamics and explicit fixation probabilities under general payoffs.

Fundamental problem with pair approximation: Types around $U \sim \mathrm{Unif}(G)$ are Bernoulli with parameters defined by p_B and p_{BR}.

Diffusions for voter densities

$$p_B(\xi_t) \simeq p_B(\xi_0) + \int_0^t \left(\frac{2(k-2)}{N(k-1)} p_B(\xi_s)[1 - p_B(\xi_s)] \right)^{1/2} dW_s$$

1. **Method of generators:**

$$\mathbb{E}p_B(\xi_t)^2 = \mathbb{E}p_B(\xi_0)^2 + \frac{2}{N}\mathbb{E}\int_0^t p_{BR}(\xi_s)ds,$$

   $p_{BR}(\xi_t)$ depends on dynamics of neighbors of BR pairs, and so on...

2. **Discrete tori in $d \geq 2$:** Cox (1989) by duality.

Key issue: close $p_B(\xi_t)$ as $N \to \infty$ (reduce $\underline{\infty \text{ eqs}}$ to $\underline{1 \text{ eq}}$), derive coefficients precisely (by spatial structure).

Diffusions for voter densities

$$p_B(\xi_t) \simeq p_B(\xi_0) + \int_0^t \left(\frac{2(k-2)}{N(k-1)} p_B(\xi_s)[1 - p_B(\xi_s)] \right)^{1/2} dW_s$$

Comparison geometry ($w = 0$):
spatial universality

Extension to $w > 0$:
much more complicated duality

Aldous–Fill conjecture

Theorem (Oliveira (2013) *Ann. Probab.*)

Assume mild rapid mixing conditions on general voter models.

1. *In terms of* **full coalescence times**,

   # of active walks in coalescing RWs on time scale γ_N $\xrightarrow[N\to\infty]{}$ *Kingman coalescent.*

2. *In terms of* **fixation times**,

   voter density on time scale γ_N $\xrightarrow[N\to\infty]{}$ *Wright–Fisher (w/o drift).*

(Initial conditions are defined by Bernoulli variables.)

Mode of convergence: L_1-Wasserstein.

Beltrán, Chavez and Landim (2018): 1 holds on large discrete tori in $d \geq 2$.

Voting kernel q

Definition

Given an irreducible Q-matrix q, the update rate at x is

$$\sum_{y\in E} \mathbb{1}_{\{\xi(x)\neq\xi(y)\}} q(x,y)$$

- **Example (graph):** $q(x,y) = 1/\deg(x)$ for $y \sim x$,

$$\sum_{y} \mathbb{1}_{\{\xi(x)\neq\xi(y)\}} q(x,y) = \frac{\#\{y \sim x; \xi(y) \neq \xi(x)\}}{\deg(x)}.$$

- **Density:**

$$p_B(\xi) = \sum_x \pi(x) \mathbb{1}_B(\xi(x)) \quad (\pi q = \pi).$$

Time scale

$$\gamma_N = \mathbb{E}[M_{U,U'}] \overset{\text{walk regular}}{=} \frac{\mathbb{E}_U[H_{U'}]}{2}$$

$U, U' \sim \pi$, $M_{x,y}$ = meeting time, H_y = hitting time.

Walk regular: $\mathbb{P}_x(\ell\text{-th step of } q\text{-chain} = x) = \mathrm{Cnst}_\ell, \forall\, \ell, \forall\, x$.

1. Complete graphs: $\gamma_N \sim N/2$,
2. Discrete tori for $d \geq 2$ (Cox '89, partial input from Spitzer):
$$\gamma_N \sim \begin{cases} C_d \cdot N, & d \geq 3, \\ C_d \cdot N \log N, & d = 2. \end{cases}$$
3. Random k-regular graphs (C. 2017+):
$$\gamma_N \sim \frac{N(k-1)}{2(k-2)}, \quad k \geq 3$$
by extended Cox–Spitzer method, Kesten–McKay law for the spectra of regular trees.

Universality of voter densities

Theorem (C., Choi and Cox, 2016; C. and Cox, 2018)

Under mild rapid mixing conditions on the voting kernels,

$$p_B(\xi_{\gamma_N t}) \xrightarrow[N\to\infty]{\text{(d)}} Y_t, \quad dY_t = \sqrt{Y_t(1-Y_t)}dW_t.$$

The convergence extends to multi-type voter models with the Fleming-Viot limit.

1. C., Choi and Cox (2016) compared to Oliveira (2013):
   - general initial conditions,
   - weaker mixing by spectral gaps of voting kernels,
   - partial recovery on fixation times.
2. Ethier–Kurtz atomic convergence (1994) admissible.

Donation games: density

Theorem (C. 2018+, random k-regular graphs)

With $w = w_\infty/N$,

$$p_B(\xi_{\gamma_N t}) \xrightarrow[N\to\infty]{\text{(d)}} Y_t,$$

where

$$dY_t = \frac{w_\infty(b-ck)}{2k} Y_t(1-Y_t)dt + \sqrt{Y_t(1-Y_t)}dW_t.$$

and $\gamma_N \sim [N(k-1)]/[2(k-2)]$.

1. $Y_{\gamma_N^{-1}t}$ and $w_\infty = wN$ validate pair approximation exactly!
2. General games: the martingale part is $\sqrt{Y_t(1-Y_t)}dW_t$.

Donation games: density + Radon–Nikodým

Theorem (C. 2018+, random k-regular graphs)

Under voter models,

$$\left(p_B(\xi_{\gamma_N t}), D^{(N)}_{\gamma_N t}\right) \xrightarrow[N\to\infty]{\text{(d)}} (Y_t, D_t),$$

where

$$D_t^{(N)} = \left.\frac{d\mathrm{Law}(\textbf{donation evolutionary game } \textit{with } w = \frac{w_\infty}{N})}{d\mathrm{Law}(\textit{voter model})}\right|_{\mathcal{F}_t},$$

$$dY_t = \sqrt{Y_t(1-Y_t)}\,dW_t^1,$$

$$dD_t = w_\infty D_t \sqrt{Y_t(1-Y_t)}\left(\frac{b/k-c}{2}dW_t^1 + \frac{b^2/k - 2bc/k + c^2}{2}dW_t^2\right).$$

Key steps: tightness of $D^{(N)}$, closure of $(p_B(\xi_t), D_t^{(N)})$ with precise coefficients.

Extensions: general space, mutation, fixation.

Directions to generalize space

Key properties in use for large random *k*-regular graphs:

1. **No hubs.**
2. **$\mathcal{O}(1)$-spectral gap of the transition probability:** Friedman (2008) and Bordenave (2015).
3. **Convergent local geometry:** McKay (1981),

   $$\mathbb{P}_x(\ell\text{-th step of walk} = x) \simeq \mathrm{Cnst}_\ell,$$

   $$\begin{cases} \ell \in \{0, 1, 2\} & \text{(density);} \\ \ell \in \{0, 1, 2, 3\} & \text{(density + Radon–Nikodým)} \end{cases}$$

   for **most** vertices x.

Universality of voter densities: almost exponentiality

1. **Aldous–Brown condition (going back to J. Keilson '79):**

$$H_A = \text{first hitting time of a set } A, \qquad \text{for } \pi(A) \simeq 0$$

by a stationary chain (X_t) with "rapid mixing" satisfies

$$\frac{H_A}{\mathbb{E}_\pi[H_A]} \simeq \text{exponential with mean 1.}$$

2. **Heuristics:**

$$\begin{aligned}
\mathbb{P}_\pi(H_A > t+s | H_A > t) &= \int \mathbb{P}_x(H_A > s)\mathbb{P}_\pi(X_t \in dx | H_A > t) \\
&\simeq \int \mathbb{P}_x(H_A > s)\pi(dx) \\
&= \mathbb{P}_\pi(H_A > s)
\end{aligned}$$

because the tendency to hit the set A "far way" is offset by the rapid mixing.

Suppose that $\xi_0(x)$, $x \in G$, are i.i.d. Bernoulli with $\mathbb{P}(\xi_0(x) = B) = u$.

1.

$$\begin{aligned}
\mathbb{E}[p_B(\xi_t)p_R(\xi_t)] =& \mathbb{P}(\xi_t(U) = B, \xi_t(U') = R) \qquad (U, U' \overset{\text{i.i.d.}}{\sim} \pi) \\
=& \mathbb{P}(\xi_0(W_t^U) = B, \xi_0(W_t^{U'}) = R) \quad ((W^U, W^{U'}) \perp\!\!\!\perp \xi_0) \\
=& u(1-u)\mathbb{P}(W_t^U \neq W_t^{U'}) \\
=& u(1-u)\mathbb{P}(M_{U,U'} > t).
\end{aligned}$$

By Aldous and Brown, if $\gamma_N = \mathbb{E}[M_{U,U'}]$,

$$u(1-u)e^{-t} \simeq u(1-u)\mathbb{P}(M_{U,U'} > \gamma_N t) = \mathbb{E}[p_B(\xi_{\gamma_N t})p_R(\xi_{\gamma_N t})].$$

Universality of voter densities: example

Suppose that $\xi_0(x)$, $x \in G$, are i.i.d. Bernoulli with $\mathbb{P}(\xi_0(x) = B) = u$.

1.

$$
\begin{aligned}
\mathbb{E}[p_B(\xi_t)p_R(\xi_t)] =& \mathbb{P}(\xi_t(U) = B, \xi_t(U') = R) && (U, U' \overset{\text{i.i.d.}}{\sim} \pi) \\
=& \mathbb{P}(\xi_0(W_t^U) = B, \xi_0(W_t^{U'}) = R) && ((W^U, W^{U'}) \perp\!\!\!\perp \xi_0) \\
=& u(1-u)\mathbb{P}(W_t^U \neq W_t^{U'}) \\
=& u(1-u)\mathbb{P}(M_{U,U'} > t).
\end{aligned}
$$

By Aldous and Brown, if $\gamma_N = \mathbb{E}[M_{U,U'}]$,

$$u(1-u)e^{-t} \simeq u(1-u)\mathbb{P}(M_{U,U'} > \gamma_N t) = \mathbb{E}[p_B(\xi_{\gamma_N t})p_R(\xi_{\gamma_N t})].$$

2. If $p_B(\xi_{\gamma_N t}) \simeq Y_t$ for $dY_t = \sqrt{Y_t(1-Y_t)}dW_t$, then

$$\mathbb{E}[p_B(\xi_{\gamma_N t})p_R(\xi_{\gamma_N t})] \simeq \mathbb{E}[Y_t(1-Y_t)] = u(1-u)e^{-t}.$$

Pair approximation: outline

A large random regular graph with degree k.

Evolutionary game with $w \ll 1$.

1. **Reducible dynamics:** the following dynamics enough:
   - a focal site $U \sim \mathrm{Unif}(G)$,
   - neighbors of U,
   - neighbors of neighbors of U.
2. **Reduction to low-dimensional density-dependent processes:** frequencies $f_B(y), f_R(y)$ reduced to $p_B(\xi_t)$ and $p_{BR}(\xi_t)$ which form a closed system.

   More below, need two hypotheses.
3. **Assume decoupling:** $p_B(\xi_t)$ has its own dynamics as $N \to \infty$.

Pair approximation

1. **Hypothesis 1:** given ξ_t and $\xi_t(U) = R$, types around $U \sim \mathrm{Unif}(G)$ are i.i.d. **Bernoulli distributed**:

$$\mathbb{P}(\#\{B \text{ neighbors of } U \text{ in } \xi_t\} = k_B)$$
$$\simeq \binom{k}{k_B} p_{B|R}(\xi_t)^{k_B}[1 - p_{B|R}(\xi_t)]^{k-k_B}, \quad 0 \leq k_B \leq k,$$

Pair approximation

1. **Hypothesis 1:** given ξ_t and $\xi_t(U) = R$, types around $U \sim \mathrm{Unif}(G)$ are i.i.d. **Bernoulli distributed**:

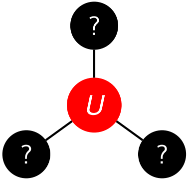
$$\mathbb{P}(\#\{B \text{ neighbors of } U \text{ in } \xi_t\} = k_B) \simeq \binom{k}{k_B} p_{B|R}(\xi_t)^{k_B}[1 - p_{B|R}(\xi_t)]^{k-k_B}, \quad 0 \leq k_B \leq k,$$

where

$$\begin{aligned} p_{B|R}(\xi_t) =& \frac{p_{BR}(\xi_t)}{p_R(\xi_t)} \\ =& \frac{\mathbb{P}(\xi_t(V) = B, \xi_t(U) = R)}{\mathbb{P}(\xi_t(U) = R)} \\ =& \mathbb{P}(\xi_t(V) = B | \xi_t(U) = R) \end{aligned}$$

for $V \sim \mathrm{Unif}(\text{neighborhood of } U)$ by **one** sampling.

Key: "global" $\simeq$ "local", homogeneity of types around U.



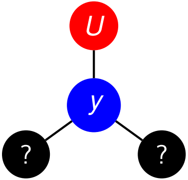
Pair approximation

2. **Hypothesis 2:** given ξ_t and $\xi_t(U) = R$, for $y \sim U$ with $\xi_t(y) = B$,

$$\beta(y) = (1-w) + w\left[\Pi(B,B)\frac{\#\{B\text{-nbd}\}}{k} + \Pi(B,R)\frac{\#\{R\text{-nbd}\}}{k}\right]$$

$$= (1-w) + w\left[\Pi(B,B)\frac{\#\{\textbf{unexplored } B\text{-nbd}\}}{k}\right.$$

$$\left. + \Pi(B,R)\frac{1}{k} + \Pi(B,R)\frac{\#\{\textbf{unexplored } R\text{-nbd}\}}{k}\right]$$



Pair approximation

2. **Hypothesis 2:** given ξ_t and $\xi_t(U) = R$, for $y \sim U$ with $\xi_t(y) = B$,

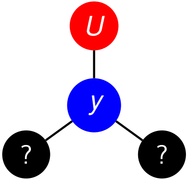
$$\beta(y) \simeq (1-w) + w\left[\Pi(B,B)\frac{(k-1)p_{B|B}(\xi_t)}{k} + \Pi(B,R)\frac{1}{k} + \Pi(B,R)\frac{(k-1)p_{R|B}(\xi_t)}{k}\right] = \beta_B(\xi_t),$$

where, for example,

$$p_{B|B}(\xi_t) = \mathbb{P}(\text{finding } B \text{ at } V' | \text{finding } B \text{ at } U')$$

for $(U', V') \stackrel{\text{(d)}}{=} (U, V)$.

Key: The **reduced** birth rate $\beta_B(\xi_t)$ does **not** depend on the site y.

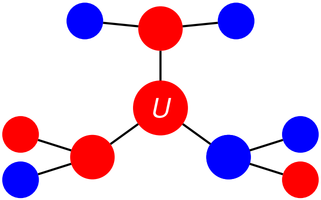


2. **Hypothesis 2:** given ξ_t and $\xi_t(U) = R$,

$$\mathbb{P}(U \text{ replaced by a } B) = \frac{\sum_{y:y\sim U, \xi_t(y)=B} \beta(y)}{\sum_{y:y\sim U} \beta(y)} \simeq \frac{k_B \beta_B}{(k-k_B)\beta_R + k_B\beta_B},$$

where k_B is the number of B-neighbors and $k - k_B$ is the number of R-neighbors.

3. **A large random regular graph is locally tree-like:** no repeated estimates for types of unexplored neighbors of neighbors of U.



Pair approximation

4. **Approximate transition probability:** If t is an update time,

$$\mathbb{P}\left(p_B(\xi_t) - p_B(\xi_{t-}) = \frac{1}{N}\middle|\xi_{t-}\right)$$

Pair approximation

4. **Approximate transition probability:** If t is an update time,

$$\mathbb{P}\left(p_B(\xi_t) - p_B(\xi_{t-}) = \frac{1}{N}\middle|\xi_{t-}\right)$$

$$= \mathbb{P}(\xi_{t-}(U) = R)\mathbb{P}\big(B\text{-nbd's take over } U\big|\xi_{t-}(U) = R\big)$$

Pair approximation

4. **Approximate transition probability:** If t is an update time,

$$\mathbb{P}\left(p_B(\xi_t) - p_B(\xi_{t-}) = \frac{1}{N}\middle|\xi_{t-}\right)$$

$$= \mathbb{P}(\xi_{t-}(U) = R)\mathbb{P}\left(B\text{-nbd's take over } U\middle|\xi_{t-}(U) = R\right)$$

$$\simeq p_R(\xi_{t-}) \sum_{k_B=0}^{k} \underbrace{\binom{k}{k_B} p_{B|R}(\xi_{t-})^{k_B}[1 - p_{B|R}(\xi_{t-})]^{k-k_B}}_{\text{Use Hypothesis 1 to estimate } k_B}$$

$$\times \underbrace{\frac{k_B\beta_B(\xi_{t-})}{(k - k_B)\beta_R(\xi_{t-}) + k_B\beta_B(\xi_{t-})}}_{\text{Use Hypothesis 2 to estimate } \beta(y) \text{ for all } y \sim U},$$

Pair approximation

4. **Approximate transition probability:** If t is an update time,

$$\mathbb{P}\left(p_B(\xi_t) - p_B(\xi_{t-}) = \frac{1}{N}\middle|\xi_{t-}\right)$$

$$\simeq p_R(\xi_{t-}) \sum_{k_B=0}^{k} \binom{k}{k_B} p_{B|R}(\xi_{t-})^{k_B}[1 - p_{B|R}(\xi_{t-})]^{k-k_B}$$

$$\times \frac{k_B \beta_B(\xi_{t-})}{(k-k_B)\beta_R(\xi_{t-}) + k_B\beta_B(\xi_{t-})},$$

which depends only on $p_B(\xi_{t-})$ and $p_{BR}(\xi_{t-})$ after some algebra.

A similar argument shows that the dynamics of $p_{BR}(\xi_t)$ depends only on itself and $p_B(\xi_t)$.

Summary

1. **Voter model perturbations on large finite sets.**

   More models in Cox, Durrett and Perkins (2013).

2. **Method: almost exponentiality (to close dynamics).**

   Aldous (1982) and Brown (1983).

3. **Method: asymptotics of coalescing RW (to compute cnst).**

   Physically doable, but math tools (severely??) lacking.

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Thank you for your attention