Positive Opetopes with Contractions form a Test Category

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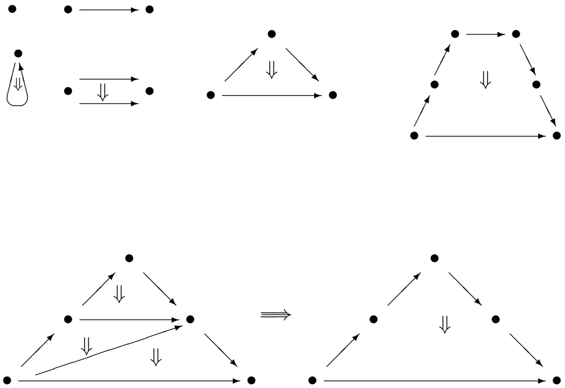
Plan of the talk

1. Opetopes - informal introduction
2. Overview of (definitions) opetopic sets
3. $\mathbf{pOpe}_\iota$ - category of positive opetopic sets with contractions
4. Main Theorem (and what it takes to prove it)
5. Informal description of the product $I \times P$
6. Formal description of $I \times P$ in $\widehat{\mathbf{pOpe}_\iota}$

Opetopes - informal introduction

Opetopes are shapes like these...



Opetopes - informal introduction

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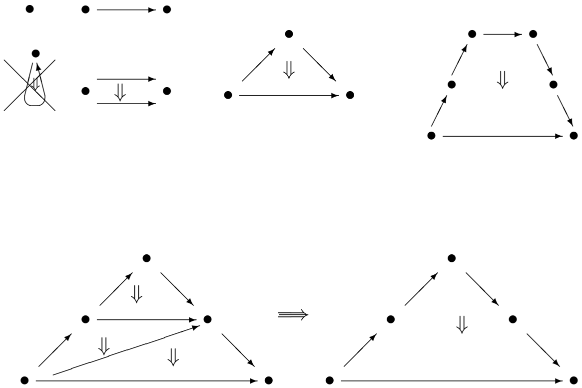
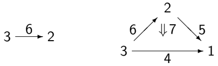
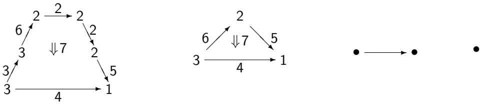


Morphisms of Opetopes - informal introduction

Morphisms of opetopes send faces to faces of the same dimension preserving domains and codomains (face maps only) **Ope**:



Contraction morphisms of opetopes send faces to faces of **at most** the same dimension preserving domains and codomains (some degeneracies allowed) $\mathbf{pOpe}_\iota$:



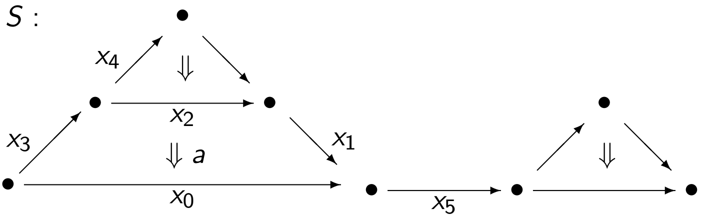
OpetopicSets	
Type of definition	*Authors*
Algebraic	Baez – Dolan 1997*, Hermida – Makkai – Power 2001, Cheng 2003 Szawiel – MZ 2015, Fiore – Saville 2017
Categorical	Burroni 1994, MZ 2009
Combinatorial	Palm 2003, Cheng 2003, MZ 2007
Category of Opetopes	
BD 1997*, HMP 2001, TP 2003, MZ 2007	
Sets of Opetopes	
Leinster 2003, Ch 2003, Kock – Joyal – Batanin – Mascari 2007	
Opetopic cardinals	
MZ 2007	

Positive Opetopes

Positive opetopes have easier combinatorics and from now on I will talk only about **positive** opetopes and positive opetopic sets, positive opetopic sets with contractions, and the like.

Positive Opetopic Cardinals
primitive notions

An example of a positive 2-dimensional opetopic cardinal



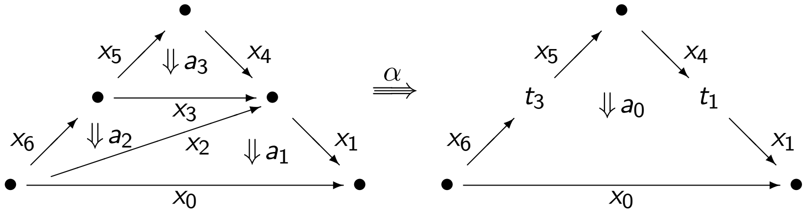
Primitive notions:

1. γ - the codomain operation: $\gamma(a) = x_0$
2. δ - the domain operation: $\delta(a) = \{x_1, x_2, x_3\}$ (positive = non-empty set)

Derived notions:

1. $<^-$ - lower order: $x_3 <^- x_2 <^- x_1 <^- x_5$; $(\gamma(x_3) \in \delta(x_2))$
2. $<^+$ - upper order: $x_4 <^+ x_2 <^+ x_0$; $(x_2 \in \delta(a)$ and $\gamma(a) = x_0)$

An example of a 3-dimensional positive opetope



$$\gamma(\alpha) = a_0, \quad \delta(\alpha) = \{a_1, a_2, a_3\}$$

$$\gamma\gamma(\alpha) = x_0, \qquad \delta\gamma(\alpha) = \{x_1, x_4, x_5, x_6\}$$

$$\gamma\delta(\alpha) = \{x_0, x_2, x_3\}, \qquad \delta\delta(\alpha) = \{x_1, x_2, x_3, x_4, x_5, x_6\}$$

Globularity. For any face α of dimension ≥ 2

$$\gamma\gamma(\alpha) = \gamma\delta(\alpha) - \delta\delta(\alpha), \qquad \delta\gamma(\alpha) = \delta\delta(\alpha) - \gamma\delta(\alpha).$$

A family of sets $S = \{S_k\}_{k\in\omega}$ (almost all empty), S_k - faces of dimension k, together with operations γ, δ and relations $<^+$ and $<^-$ is *a positive opetopic cardinal* iff it satisfies

Globularity. For any face $\alpha \in S_{\geq 2}$

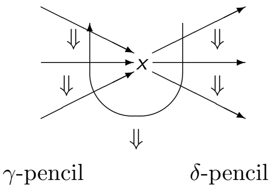
$$\gamma\gamma(\alpha) = \gamma\delta(\alpha) - \delta\delta(\alpha), \qquad \delta\gamma(\alpha) = \delta\delta(\alpha) - \gamma\delta(\alpha).$$

Strictness. The relation $<^+$ on each set S_k is a strict order (transitive irreflexive). The relation $<^+$ on S_0 is a linear order.

Disjointness. $<^+ \cap <^- = \emptyset$.

Pencil linearity. The sets of cells with common codomain (γ-*pencil*) and the sets of cells that have the same distinguished cell in the domain (δ-*pencil*) are linearly ordered by $<^+$.

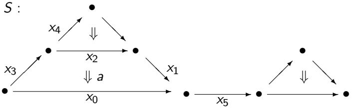
Pencil order $\prec_x$ over face x



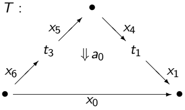
The *size of a positive opetopic cardinal* S is defined as an infinite sequence of natural numbers

$$size(S) = \{size(S)_k\}_{k\in\omega} = \{|S_k - \delta(S_{k+1})|\}_{k\in\omega}$$

(almost all equal 0).
A positive opetopic cardinal S is a *positive opetope* iff $size(S)_k \leq 1$ for $k \in \omega$.
Example.



$size(S) = (1, 3, 3, 0, \ldots)$,



$size(T) = (1, 1, 1, 0, \ldots)$.

A *morphism of positive opetopic cardinals* $f : T \to S$ is a family of functions $f_k : T_k \to S_k$, for $k \in \omega$, that commutes with γ's and δ's, i.e. for any $a \in T_{\geq 1}$,

$$\gamma(f(a)) = f(\gamma(a)) \text{ and } f_a : \delta(a), \longrightarrow \delta(f(a))$$

is a bijection, where f_a is the restriction of f to $\delta(a)$.

Thus we have a category **pOpe** of positive opetopes and the above morphism (face maps).

We have an embedding functor

$$(-)^* : \mathbf{pOpe} \longrightarrow \omega\mathbf{Cat}$$

$$S \mapsto S^*$$

k-cells in S^* are sub-opetopic cardinals T of S with $dim(T) \leq k$.

This embedding is full on isomorphisms and hence we can think of morphisms in **pOpe** as some ω-functors, those that send generators to generators.

The category $\mathbf{pOpe}_\iota$ of positive opetopes with contractions is the category whose objects are positive opetopes and whose morphisms are ω-functors that send generators to either generators or (iterated) identities on generators.

Theorem

The category $\mathbf{pOpe}_\iota$ *of positive opetopes with contractions is a test category.*

Since $\mathbf{pOpe}_\iota$ has terminal object it suffice to show that the one dimensional opetope I:

$$- \xrightarrow{a} +$$

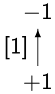
is locally aspherical. It is enough to show that for any positive opetope P the product $I \times P$ is aspherical (see G. Maltsiniotis book, La théorie de l'homotopie de Grothendieck, for much more). I will show that

$$I \times P = \bigcup_{\vec{x} \in \mathbf{Flags}(P)} P^{\vec{x}}$$

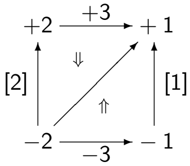
where $\mathbf{Flags}(P)$ is the set of flags in P and $P^{\vec{x}}$ is an opetope.

Some examples of products

$P = 1$:



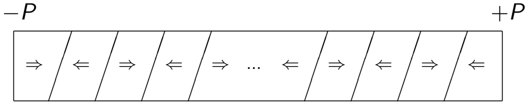
$P = 2 \xrightarrow{3} 1$:



Product $I \times P$

informal description

General picture of $I \times P$*:*



Product $I \times P$
informal description

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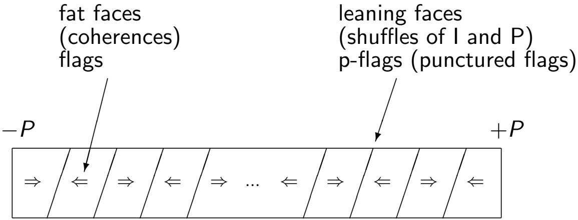

Product $I \times P$
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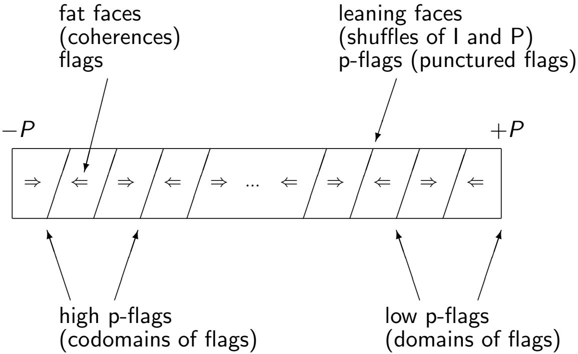
General picture of $I \times P$:



Formal definition of $I \times P$

cylinder in $\widehat{\textbf{pOpe}}$

We have an embedding $\kappa : \textbf{pOpe} \longrightarrow \textbf{pOpe}_\iota$ and hence a left Kan extension

$$\kappa_! : \widehat{\textbf{pOpe}} \longrightarrow \widehat{\textbf{pOpe}_\iota}$$

We shall describe the object $\textbf{Cyl}(P)$ in $\widehat{\textbf{pOpe}}$ that is aspherical already in $\widehat{\textbf{pOpe}}$ so that $\kappa_!(\textbf{Cyl}(P))$ is the product of I and P in $\widehat{\textbf{pOpe}_\iota}$.

The notion of a flag is due to T. Palm (2003).
A *flag* in P is a sequence of faces in P

$$\vec{x} = \begin{bmatrix} x_k \\ x_{k-1} \\ \dots \\ x_0 \end{bmatrix}$$

so that x_i is a face of dimension i and $x_i \in \delta(x_{i+1}) \cup \gamma(x_{i+1})$, $i = 0, \dots, k-1$. A flag is *maximal* if x_k is the top face of P, i.e. $dim(P) = k$.
Sign of flag

$$\mathbf{sgn}(x_k, x_{k-1}, \dots, x_1) \; = \begin{cases} 1 & \text{if } k = 1 \\ \mathbf{sgn}(x_{k-1}, \dots, x_0) & \text{if } x_{k-1} = \gamma(x_k) \\ (-1) \cdot \mathbf{sgn}(x_{k-1}, \dots, x_0) & \text{if } x_{k-1} \in \delta(x_k) \end{cases}$$

Formal definition of $\mathbf{Cyl}(P)$
order on flags

$\mathbf{Flags}_P$ - is the set of maximal flags in P.

Flag order $\lhd$ on $\mathbf{Flags}_P$, the set of maximal flags in P. Let $\vec{x}, \vec{y} \in \mathbf{Flags}_P$ be two different flags. Let $k = \min\{j | x_j \neq y_j\}$. We put $\vec{x} \lhd \vec{y}$ iff

1. $k = 0$ and $y_0 <^+ x_0$,
2. or $k > 0$, $\mathbf{sgn}(x_{k-1}, \ldots, x_l) = 1$ and $x_k \prec_{x_{k-1}} y_k$
3. or $k > 0$, $\mathbf{sgn}(x_{k-1}, \ldots, x_l) = -1$ and and $y_k \prec^{op}_{x_{k-1}} x_k$.

Clearly, the flag order is a strict and linear. Its reflexive closure will be denoted by $\unlhd$.

Formal definition of $\mathbf{Cyl}(P)$

p-flags

Lemma

Two consecutive flags differ by exactly one face.

'Intersection' of two consecutive flags is a p-flag (plug 0 - a dummy face) in place flags differ. They are of form

high p-flag $(\vec{x}_{high})$

$$\begin{bmatrix} x_k \\ 0 \\ x_{k-2} \\ \dots \\ x_0 \end{bmatrix}$$

low p-flag $(\vec{x}_{low})$

$$\begin{bmatrix} x_k \\ \gamma(x_k) \\ \dots \\ \gamma^{(l+2)}(x_k) \\ t \\ 0 \\ x_{l-1} \\ \dots \\ x_0 \end{bmatrix}$$

where $t \in \delta\gamma^{(l+2)}(x_k)$.

Formal definition of $\mathbf{Cyl}(P)$

faces of the cylinder

Faces in the opetopic set $\mathbf{Cyl}(P)$ in $\widehat{\mathbf{pOpe}}$:

1. Flat faces $\{-\} \times P$ and $\{+\} \times P$;
2. All flags of all faces of P;
3. All p-flags of all faces in P.

The dimension of a face $-p$ or $+p$ is the dimension of p. The dimension of a flag or p-flag is the number of non-zero faces in the sequence. $\mathbf{Cyl}(P)_k$ is the set of all faces of $\mathbf{Cyl}(P)$ of dimension k.

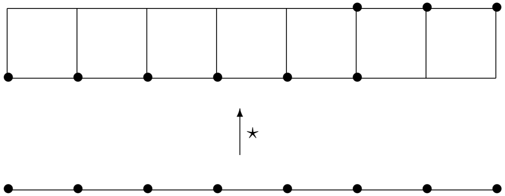
Formal definition of $\mathbf{Cyl}(P)$

projection

We have a ‘projection’ function $\pi_P : \mathbf{Cyl}(P) \longrightarrow P$ such that for a face $\varphi \in \mathbf{Cyl}(P)$

$$\pi_P(\varphi) \; = \left\{ \begin{array}{ll} x & \text{if } \varphi \in \{-x, +x\}, \\ x_k & \text{if } \varphi = x_k, \ldots, x_0 \end{array} \right.$$

Intuitions from symplicial sets $\triangle_1 \times \triangle_7$:



Formal definition of **Cyl**(P)
star operation

We define an 'inverse' operation to projection on flags

$$\star : P \times \mathbf{Flags}_P \to \mathbf{Cyl}(P)$$

so that, for $\vec{x} = [x_n, \ldots, x_0] \in \mathbf{Flags}_P$ and $p \in P_k$, we have

$$p \star \vec{x} = \begin{cases} \vec{x}\lceil_k & \text{if } p = x_k; \\ & \text{(flag)} \\ [p, 0, \vec{x}\lceil_{k-2}] & \text{otherwise, if } k > 0 \text{ and } x_k <^+ p; \\ & \text{(high p-flag)} \\ [p, t, 0, \vec{x}\lceil_{k-2}] & \text{otherwise, if } k > 1,\ x_{k-1} \leq^+ t \in \delta(p); \\ & \text{(low p-flag)} \\ [p, \gamma(p) \star \vec{x}] & \text{otherwise, if } k > 2,\ \gamma(p) \star \vec{x} \text{ is a p-flag}; \\ & \text{(induction, low p-flag)} \\ -p & \text{otherwise, if } \gamma^{(0)}(p) \leq^+ x_0; \\ & \text{(bottom flat face)} \\ +p & \text{otherwise, if } x_0 <^+ \gamma^{(0)}(p). \\ & \text{(top flat face)} \end{cases}$$

Formal definition of **Cyl**(P)
domains and codomains of flags

Let $\vec{x} = x_k, \ldots, x_0$ be a flag in $\mathbf{Cyl}(P)$. We put

$$\gamma(\vec{x}) = \vec{x}_{high}$$

and

$$\delta(\vec{x}) = \{\vec{x}_{\lceil k-1}, \vec{x}_{low}\}$$

$$\delta(\vec{x}) = \begin{cases} \{\vec{x}_{\lceil k-1}, -x_k\} & \text{if } \vec{x} \text{ is the first flag,} \\ \{\vec{x}_{\lceil k-1}, +x_k\} & \text{if } \vec{x} \text{ is the last flag,} \\ \{\vec{x}_{\lceil k-1}, \vec{x}_{low}\} & \text{otherwise.} \end{cases}$$

Let $\vec{x} = x_k, \ldots, \widehat{x_i}, \ldots, x_0$ be a p-falg in P. We put

$$\gamma(\vec{x}) = \gamma(x_k) \star \vec{x} = \begin{cases} \begin{bmatrix} \gamma(x_k) \\ 0 \\ x_{n-3} \\ \ldots \\ x_0 \end{bmatrix} & \text{if } i = k-1, k-2, \\ \begin{bmatrix} x_{k-1} \\ x_{k-2} \\ \ldots \\ \widehat{x_i} \\ \ldots \\ x_0 \end{bmatrix} & \text{otherwise.} \end{cases}$$

$$\delta(\vec{x}) = \begin{cases} \{p \star \vec{x} \mid p \in \delta(x_k)\} & \text{if } \vec{x} \text{ is a low p-flag,} \\ \{p \star \vec{x} \mid p \in \delta(x_k)\} \cup \{\vec{x}_{\lceil k-2}\} & \text{if } \vec{x} \text{ is a high p-flag.} \end{cases}$$

Formal definition of **Cyl**(P)
domains and codomains of flat faces

For $p \in P_{\geq 1}$ we have

$$\gamma(-p) = -\gamma(p), \quad \gamma(+p) = +\gamma(p)$$

and

$$\delta(-p) = \{-q : q \in \delta(p)\}, \quad \delta(+p) = \{+q : q \in \delta(p)\}.$$

Let $\vec{x}$ be a flag or p-flag in P.
By $P^{\vec{x}}$ we denote the least subset of faces of $\mathbf{Cyl}(P)$ containing the face $\vec{x}$ and closed under γ's and δ's.

Lemma

$P^{\vec{x}}$ is an opetope.

Theorem

1. *Let $\vec{x}' \lhd \vec{x}$ be two consecutive flags. Then, in $\widehat{\mathbf{pOpe}}$, we have*

$$\Big(\bigcup_{\vec{y} \unlhd \vec{x}'} P^{\vec{y}}\Big) \cap P^{\vec{x}} = P^{\vec{x}' \cap \vec{x}};$$

2. $\mathbf{Cyl}(P) = \bigcup_{\vec{x} \in \mathbf{Flags}_P} P^{\vec{x}}$ *in* $\widehat{\mathbf{pOpe}}$*;*
3. $I \times P = \mathbf{Cyl}_\iota(P) := \kappa_!(\mathbf{Cyl}(P))$ *in* $\widehat{\mathbf{pOpe}_\iota}$*;*
4. $I \times P$ *is aspherical in* $\widehat{\mathbf{pOpe}_\iota}$*.*

Combinatorial definition of contractions

Let P and Q be positive opetopes. A *contraction morphism of opetopes* (or *contraction* for short) $h : Q \longrightarrow P$ is function $h : |Q| \longrightarrow |P|$ between faces of opetopes such that

1. $dim(q) \geq dim(h(q))$ for $q \in Q$.
2. (preservation of codomains) $h(\gamma^{(k)}(q)) = \gamma^{(k)}(h(q))$, for $k \geq 0$ and $q \in Q_{k+1}$,
3. (preservation of domains)
   1. if $dim(h(q)) \geq dim(q) - 1$, then h restricts to a bijection

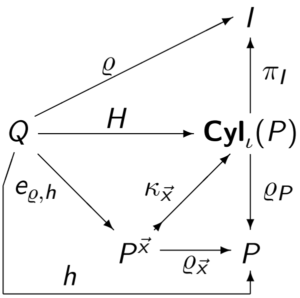
      $$(\delta^{(k)}(q) - \ker(h)) \xrightarrow{h} \delta^{(k)}(h(q))$$

      for $k \geq 0$ and $q \in Q_{k+1}$; where *the kernel of h* is defined as

      $$\ker(h) = \{q \in Q | dim(q) > dim(h(q))\},$$

   2. if $dim(h(q)) < dim(q) - 1$, then $\delta^{(k)}(q) \subseteq \ker(h)$.

Universal property of product $I \times P$
projections



Universal property of product $I \times P$

splitting faces

We say that the face $q \in Q_1$ $\langle \varrho, h \rangle$*-splits* face $p \in P_0$ iff $h(q) = 1_p$ and $\varrho(q) = \mathbf{a}$.

We say that the face $q \in Q_{k+1}$ $\langle \rho, h \rangle$*-splits* face $p \in P_k$ with $k > 0$ iff

1. $h(q) = p$ and $\rho(q) = \mathbf{a}$;
2. There is a face $q' \in \delta(q)$ such that q' $\langle \varrho, h \rangle$-splits $h(q') \in P_{k-1}$.

If q is a splitting face, then $p = h(q)$ is the *splitted face*.

Universal property of product $I \times P$

threshold faces

We say that the face $q \in Q_1$ is a $\langle \varrho, h \rangle$-*threshold face* iff $h(q) =\in P_1$ and $\varrho(q) = \mathbf{a}$.

We say that the face $q \in Q_{k+1}$ is a $\langle \rho, h \rangle$-*threshold face* iff $h(q) =\in P_{k+1}$ and there is a $\langle \rho, h \rangle$-splitting faces in $\delta(q)$.

Universal property of product $I \times P$

splitting faces and threshold faces in a cone over product

$$
\begin{array}{cccccccccccccccc}
s^1_0 & a^2_1 & \dots & a^2_{u_1} & s^1_{u_1} & s^2_0 & t^1_0 & b^2_1 & \dots & b^2_{v_1} & t^1_{v_1} & & & & & \\
 & & & & & & & & & & \Vert & & & & & \\
 & & & & & a^3_1 & & & & & \gamma^{(1)}(q) & & & & & \\
 & & & & & s^2_1 & & & & & & & & & & \\
 & & & & & \dots & & & & & & & & & & \\
 & & & & & a^3_2 & & & & & & & & & & \\
 & & & & & s^2_{u_2} & & & & & & & & & & \\
 & & & & & s^3_0 & a^4_1 & \dots & a^4_{u_3} & s^3_{u_3} & t^4_0 & t^3_0 & b^4_1 & \dots & b^4_{v_3} & t^3_{v_3} \\
 & & & & & & & & & & & & & & & \Vert \\
 & & & & & t^2_0 & & & & & b^5_0 & & & & & \gamma^{(2)}(q) \\
 & & & & & b^3_1 & & & & & t^4_1 & & & & & \\
 & & & & & \dots & & & & & \dots & & & & & \\
 & & & & & b^3_{v_2} & & & & & b^5_{v_4} & & & & & \\
 & & & & & t^2_{v_2} & & & & & t^4_{v_4} & & & & & \\
 & & & & & \Vert & & & & & \Vert & & & & & \\
 & & & & & \gamma^{(2)}(q) & & & & & \gamma^{(4)}(q) & & & & &
\end{array}
$$

with

$$s^1_0 \in \delta\gamma^{(2)}(q), \quad s^2_0 \in \delta\gamma^{(3)}(q), \quad s^3_0 \in \delta\gamma^{(3)}(q).$$

Universal property of product $I \times P$
morphism into product

Given ι-maps h and ϱ as above, we define a ι-map

$$H : Q \longrightarrow \mathbf{Cyl}_\iota(P)$$

as follows. Let $q \in Q$.
If $\varrho(q) \in \{-,+\}$, then we put

$$H(q) = \begin{cases} -h(q) & \text{if } \varrho(q) = - \\ +h(q) & \text{if } \varrho(q) = + \end{cases}$$

If defined, $\sigma(q)$, $\tau(q)$ are splitting and threshold faces in $\delta(q)$, respectively. If $\varrho(q) = \mathbf{a}$, we put

$$H(q) = \begin{cases} [h(q)] & \text{if } q \in Q_1 \text{ is a splitting face, i.e. } h(q) \in P_0 \\ & (H(q) \text{ is a flag of length } 1); \\ [h(q), 0] & \text{if } q \in Q_1 \text{ is not a splitting face, i.e. } h(q) \in P_1 \\ & (H(q) \text{ is a p-flag of length } 2); \\ [h(q), H(\sigma(q))] & \text{if } q \in Q_{\geq 2} \text{ is a splitting face} \\ & (H(q) \text{ is a flag }); \\ [h(q), 0, H(\sigma(q))] & \text{if } q \in Q_{\geq 2} - \ker(h) \text{ and } \sigma(q) \text{ is defined} \\ & (H(q) \text{ is a high p-flag}); \\ [h(q), H(\tau(q))] & \text{if } q \in Q_{\geq 2} - \ker(h) \text{ and } \tau(q) \text{ is defined} \\ & (H(q) \text{ is a low p-flag of codim } 2); \\ [h(q), H(\gamma(q))] & \text{otherwise, if } q \in Q_{\geq 2} - \ker(h) \\ & (H(q) \text{ is a low p-flag of codim } > 2); \\ H(\gamma(q)) & \text{otherwise } (H(q) \text{ is an identity on a face}). \end{cases}$$

Thank You for Your Attention!